

MATH 210 PRACTICE EXAM 1 ANSWERS  
 Semester I, 2007-2008      Lecture 1

Name: \_\_\_\_\_

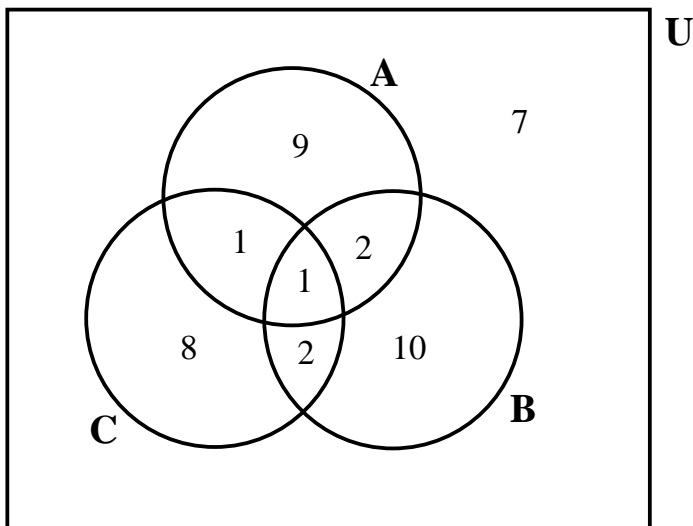
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**NO CALCULATORS, NOTES, BOOKS, ETC. ALLOWED.  
 EXPLAIN YOUR WORK.  
 ANSWERS WITHOUT EXPLANATION WILL RECEIVE 0  
 CREDIT.**

1. (13 points) Sets  $A, B, C$  are subsets of a universal set  $U$ , and they satisfy:

$$\begin{aligned} n(A) &= 13 & n(B) &= 15 & n(C) &= 12 & n(A \cap B \cap C) &= 1 \\ n(A \cup B) &= 25 & n(A \cup C) &= 23 & n(B \cup C) &= 24 & n(U) &= 40 \end{aligned}$$

Find  $n(A' \cap B' \cap C')$ .



Fill in the numbers in this Venn diagram, in the order

$$\begin{aligned} &n(A \cap B \cap C), \\ &n(A \cap B \cap C'), n(A \cap B' \cap C), n(A' \cap B \cap C), \\ &n(A \cap B' \cap C'), n(A' \cap B \cap C'), n(A' \cap B' \cap C), \\ &n(A' \cap B' \cap C'). \end{aligned}$$

For  $n(A \cap B \cap C')$ , use the fact that  $n(A \cap B) = n(A) + n(B) - n(A \cup B)$ ; then  $n(A \cap B' \cap C)$  and  $n(A' \cap B \cap C)$  are similar. The rest are easily done by subtraction. So,  $n(A' \cap B' \cap C') = 7$ .

**2.** (12 points) Let  $\Pr$  be a probability measure on  $S$  with  $E, F \subset S$ . Assume that  $\Pr[E \mid F] = \frac{1}{3}$ ,  $\Pr[F \mid E] = \frac{1}{2}$ , and  $\Pr[E' \cap F'] = \frac{3}{5}$ . Find  $\Pr[E]$ ,  $\Pr[F]$ , and  $\Pr[E \cap F]$ .

Let  $x = \Pr[E \cap F]$ . Then  $\frac{1}{3} = \Pr[E \mid F] = x / \Pr[F]$ , so  $\Pr[F] = 3x$ . Likewise,  $\frac{1}{2} = \Pr[F \mid E] = x / \Pr[E]$ , so  $\Pr[E] = 2x$ . Also,  $E \cup F = (E' \cap F')'$ , so  $\Pr[E \cup F] = 1 - \frac{3}{5} = \frac{2}{5}$ . Then

$$\frac{2}{5} = \Pr[E \cup F] = \Pr[E] + \Pr[F] - \Pr[E \cap F] = 2x + 3x - x \quad ,$$

so  $\Pr[E \cap F] = x = \frac{1}{10}$  ,  $\Pr[E] = 2x = \frac{2}{10}$  , ,  $\Pr[F] = 3x = \frac{3}{10}$ .

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3. (13 points) Let  $\Pr$  be a probability measure on  $S$  with  $E, F, G \subset S$ . Assume that they satisfy:

$$\Pr[E] = 0.55$$

$$\Pr[F] = 0.5$$

$$\Pr[G'] = 0.45$$

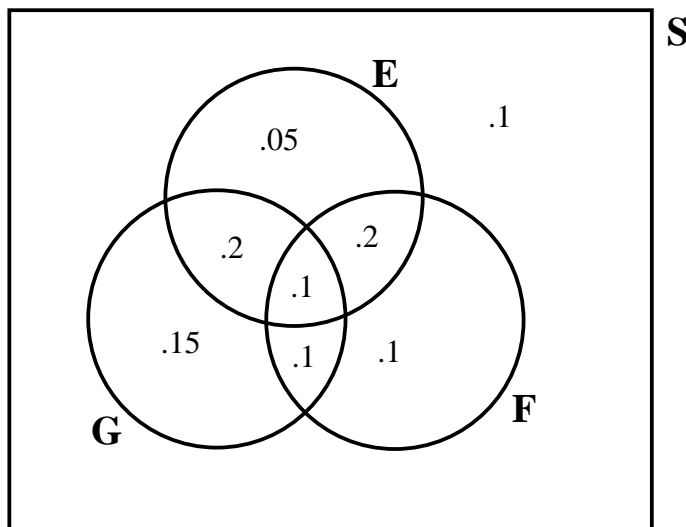
$$\Pr[E \cap F] = 0.3$$

$$\Pr[E \cap G'] = 0.25$$

$$\Pr[F \cap G'] = 0.3$$

$$\Pr[E \cap F \cap G'] = 0.2$$

Find  $\Pr[E' \cap F' \cap G]$ .



Fill in the numbers in this Venn diagram, in the order

$$\Pr[E \cap F \cap G'], \Pr[E \cap F \cap G], \Pr[E \cap F' \cap G'], \Pr[E \cap F' \cap G],$$

$$\Pr[E' \cap F \cap G'], \Pr[E' \cap F \cap G], \Pr[E' \cap F' \cap G], \Pr[E' \cap F' \cap G']$$

Of course, you can stop when you get to  $\Pr[E' \cap F' \cap G] = .15$ .

4. (12 points) Two fair dice are rolled; so the sum of the two numbers is between 2 and 12. Let  $E$  be the event that the sum of the numbers is 11 or 12. Let  $F$  be the event that the sum of the numbers is 2, 3, or 11. Find  $\Pr[E \mid F]$  and  $\Pr[F \mid E]$ .

$$n(S) = 6 \cdot 6 = 36.$$

$$E = \{(5, 6), (6, 5), (6, 6)\}, \text{ so } n(E) = 3.$$

$$F = \{(1, 1), (1, 2), (2, 1), (5, 6), (6, 5)\}, \text{ so } n(F) = 5.$$

$$E \cap F = \{(5, 6), (6, 5)\}, \text{ so } n(E \cap F) = 2.$$

Then:  $\Pr[E] = 3/36$ ,  $\Pr[F] = 5/36$ ,  $\Pr[E \cap F] = 2/36$ .

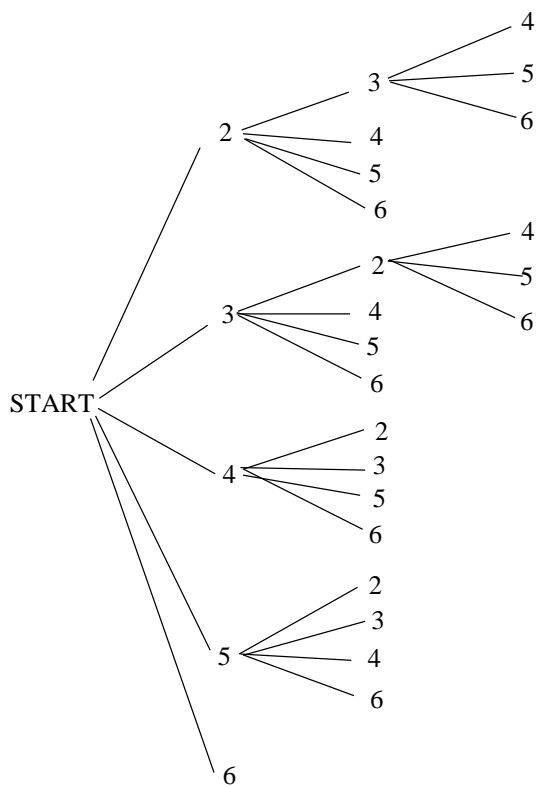
Finally,  $\Pr[E \mid F] = \Pr[E \cap F] / \Pr[F] = 2/5$  and

$$\Pr[F \mid E] = \Pr[E \cap F] / \Pr[E] = 2/3.$$

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5. (12 points) Consider the following experiment: You start with a deck of 5 cards,  $\{\diamond 2, \diamond 3, \diamond 4, \diamond 5, \diamond 6\}$ . Now, shuffle the deck and deal them out on the table, one at a time; STOP when the sum of the numbers is 6 or greater. Find the sample space of this experiment. You can either draw a tree here, or you can just list all the possible outcomes. Note that you never deal the same card twice.



It's enough to draw the tree here, since the sample space is obvious from the tree. You could also just list  $S$ , without the tree:

$$\begin{aligned}
 S = \{ & (2, 3, 4), (2, 3, 5), (2, 3, 6), (2, 4), (2, 5), (2, 6), \\
 & (3, 2, 4), (3, 2, 5), (3, 2, 6), (3, 4), (3, 5), (3, 6), \\
 & (4, 2), (4, 3), (4, 5), (4, 6), (5, 2), (5, 3), (5, 4), (5, 6), (6) \}
 \end{aligned}$$

**6.** (13 points) An unfair die has probabilities  $w_1, w_2, w_3, w_4, w_5, w_6$  of coming up 1, 2, 3, 4, 5, 6, respectively. Assume that 1 and 6 are equally likely, 2 and 5 are equally likely, and 3 and 4 are equally likely, but 2 is twice as likely as 1, and 3 is twice as likely as 2. Find  $w_1, w_2, w_3, w_4, w_5, w_6$ . Your answers should be a simple fraction (of form  $\frac{m}{n}$ ).

Let  $x = w_1 = w_6$ . Then  $w_2 = w_5 = 2x$  and  $w_3 = w_4 = 4x$ . The weights add up to 1, so  $x + x + 2x + 2x + 4x + 4x = 1$ , so  $x = \frac{1}{14}$ . Then  $w_1 = w_6 = \frac{1}{14}$ ,  $w_2 = w_5 = \frac{2}{14} = \frac{1}{7}$ , and  $w_3 = w_4 = \frac{4}{14} = \frac{2}{7}$ .

*Remark.* On exams, you are not required to reduce fractions to simplest form, so either  $\frac{2}{14}$  or  $\frac{1}{7}$  would be OK as a value for  $w_2 = w_5$ .

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7. (13 points) Let  $S$  be the set of ten numbers,  $\{1, 2, 3, \dots, 10\}$ . How many subsets of  $S$  contain the same number of even numbers as odd numbers? Your answer should be a whole number. Examples of such sets are:

$\{2, 3\}$ ,  $\{2, 4, 8, 1, 7, 9\}$ ,  $\emptyset$ ,  $S$

There are 5 even numbers and 5 odd numbers in  $S$ . So, there are  $C(5, r) \cdot C(5, r)$  ways to choose  $r$  even numbers and  $r$  odd numbers. Here, we have to add up these values for  $r = 0, 1, 2, 3, 4, 5$ , getting:

$$\begin{aligned} & (C(5, 0))^2 + (C(5, 1))^2 + (C(5, 2))^2 + (C(5, 3))^2 + (C(5, 4))^2 + (C(5, 5))^2 = \\ & 1^2 + 5^2 + 10^2 + 10^2 + 5^2 + 1^2 = \\ & 1 + 25 + 100 + 100 + 25 + 1 = 252 \quad . \end{aligned}$$

8. (12 points) Start with a stack of 50 bills: 10 each of \$1 bills, \$5 bills, \$10 bills, \$20 bills, and \$50 bills. You randomly choose 10 of them. What is the probability of choosing exactly 2 of each kind? Your answer should be of the form  $\frac{\text{a product of whole numbers}}{\text{another product of whole numbers}}$ ; you don't have to simplify it.

Let  $S$  be the set of all 10-element subsets of the 50 bills; so  $n(S) = C(50, 10)$ . Let  $E \subset S$  be the set of those 10-element subsets which contain exactly 2 of each kind of bill. Since there are 10 bills of each kind,  $n(E) = C(10, 2)^5$ . Since  $C(10, 2) = (10 \cdot 9)/2 = 45$ ,

$$\Pr[E] = \frac{n(E)}{n(S)} = \frac{(45)^5 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44 \cdot 43 \cdot 42 \cdot 41} .$$

*Remark.* You would lose a few points if you just answered

$$\Pr[E] = \frac{n(E)}{n(S)} = \frac{C(10, 2)^5}{C(50, 10)} ,$$

since you should show how to express the answer in terms of simple arithmetic, which you could then work out exactly using elementary school math if you had lots of time.