MATH 210 PRACTICE EXAM 2 ANSWERS

Semester I, 2007-2008 *Lecture 1*

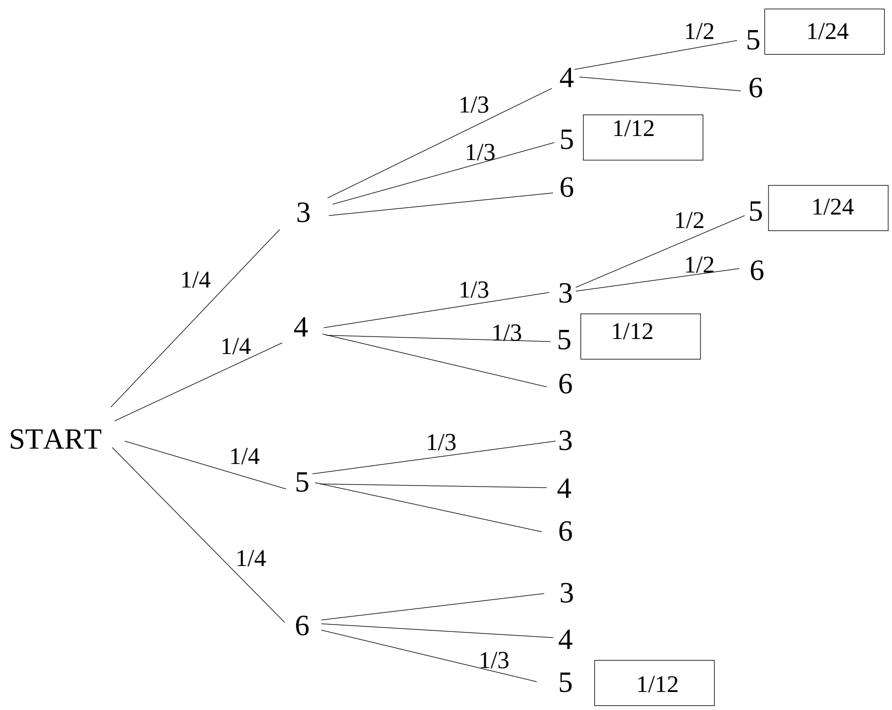
1. (10 points) A fair die is rolled 9 times. Assume that the results of the rolls are independent. Find the probability that your sequence of 9 rolls contains an even number of ones. For example, $3, 2, 1, 5, 6, 6, 1, 2, 3$ has 2 ones; $6, 2, 4, 5, 6, 6, 4, 2, 3$ has 0 ones. You may leave your answer as an arithmetical expression, without evaluating it.

The probability of getting exactly r ones in 9 rolls is $C(9, r)(\frac{1}{6})^r(\frac{5}{6})^{9-r}$, so you just add these up for the even values of r, getting

$$(\frac{5}{6})^9 + C(9,2)(\frac{1}{6})^2(\frac{5}{6})^7 + C(9,4)(\frac{1}{6})^4(\frac{5}{6})^5 + C(9,6)(\frac{1}{6})^6(\frac{5}{6})^3 + C(9,8)(\frac{1}{6})^8\frac{5}{6} \quad .$$

Note that using the "C" is OK here in an arithmetic expression. Also, you are not required to simplify the answer; for example, the first term in the sum could have been $C(9,0)(\frac{1}{6})^0(\frac{5}{6})^9$ instead of $(\frac{5}{6})^9$.

2. (15 points) Consider the following experiment: You start with a deck of 4 cards, $\{\diamondsuit 3, \diamondsuit 4, \diamondsuit 5, \diamondsuit 6\}$. Now, shuffle the deck and deal them out on the table, one at a time; STOP when the sum of the numbers is 8 or greater. Find the probability that the last card dealt is the $\diamondsuit 5$. Assume that the deal is a fair deal.



Adding up the probabilities for the outcomes where the last card is a $\diamondsuit 5$, you get $3\frac{1}{12} + 2\frac{1}{24} = \frac{4}{12} = \frac{1}{3}$.

You are not required to fill in the probabilities for the other outcomes, although if you have time, you could check your arithmetic by filling them all in and making sure that they add up to 1.

3. (10 points) A random variable X has the density function shown below. Find the expected value, $E(X)$.

Value of X	Probability	Product
-20	0.05	-1
-10	0.2	-2
0	0.3	0
10	0.1	1
20	0.15	3
30	0.2	6
		$E(X) = 7$

4. (15 points) Consider the following experiment: You start with a deck of 5 cards, $\{\diamondsuit 2, \diamondsuit 3, \diamondsuit 4, \diamondsuit 5, \diamondsuit 6\}$. Now, shuffle the deck and deal out exactly two. Let X be the sum of the two numbers you get. Find the expected value, variance, and standard deviation of X. Assume that the deal is a fair deal. You may leave the result for $\sigma(X)$ in terms of a $\sqrt{\quad}$.

There are $C(5,2) = 10$ possible outcomes. Listing them:

$$\{2,3\}\ \{2,4\}\ \{2,5\}\ \{2,6\}\ \{3,4\}\ \{3,5\}\ \{3,6\}\ \{4,5\}\ \{4,6\}\ \{5,6\}$$

you can make a table of the possible values of X and their probabilities, and then extend the table to figure out $\mu = E(X)$, $Var(X)$, and $\sigma(X)$.

Value of X	Probability	Product	$X-\mu$	$(X-\mu)^2$	$\times$Prob
5	0.1	0.5	-3	9	0.9
6	0.1	0.6	-2	4	0.4
7	0.2	1.4	-1	1	0.2
8	0.2	1.6	0	0	0.0
9	0.2	1.8	1	1	0.2
10	0.1	1.0	2	4	0.4
11	0.1	1.1	3	9	0.9
		$\mu = E(X) = 8.0$			$Var(X) = 3.0$

Then, $\sigma(X) = \sqrt{3}$.

5. (12 points) Let X be a normal random variable with $\mu = E(X) = 73$ and $\sigma(X) = 10$. Find $\Pr[70 \leq X \leq 80]$.

Converting to the standard normal random variable

$$Z = (X - \mu)/\sigma = (X - 73)/10 \quad ,$$

you have to find $\Pr[-0.3 \leq Z \leq 0.7]$, which is:

$$\Pr[0 \leq Z \leq 0.3] + \Pr[0 \leq Z \leq 0.7] = .1179 + .2580 = .3759$$

6. (15 points) An unfair coin is tossed 625 times. The coin has probability $\frac{1}{5}$ of coming up heads. Use the normal approximation to the binomial to estimate the probabilities of the following happening:

a. You get between 120 and 140 heads

b. You get 135 or fewer heads.

This is a Bernoulli process with $n = 625$, $p = \frac{1}{5}$ and $q = \frac{4}{5}$. If X denotes the number of heads, then $\mu = E(X) = np = 125$ and $\sigma = \sqrt{npq} = \sqrt{100} = 10$. So, the standard normal random variable $Z = (X - 125)/10$.

There are now two acceptable ways to work the problem.

The quicker way:

a. $\Pr[120 \leq X \leq 140] = \Pr[-.5 \leq Z \leq 1.5] = .1915 + .4332 = .6247$.

b. $\Pr[X \leq 135] = \Pr[Z \leq 1.0] = .5000 + .3413 = .8413$.

The following gives you a somewhat more accurate estimate:

a. $\Pr[119.5 \leq X \leq 140.5] = \Pr[-.55 \leq Z \leq 1.55] = .2088 + .4394 = .6482$.

b. $\Pr[X \leq 135.5] = \Pr[Z \leq 1.05] = .5000 + .3531 = .8531$.

7. (13 points) Let X be a random variable with probability density function f, where

$$f(x) = \begin{cases} 0 & \text{if } x < -2 \\ 0.3 & \text{if } -2 \le x < 0 \\ 0.1 & \text{if } 0 \le x < 4 \\ 0 & \text{if } 4 \le x \end{cases}$$

Find $\Pr[X \ge -1 \mid X \le 3]$ and $\Pr[X \le 3 \mid X \ge -1]$.

If E denotes the event that $X \ge -1$ and F denotes the event that $X \le 3$, then finding the areas on the graph we see that

$$\Pr[E] = .7 \qquad \Pr[F] = .9 \qquad \Pr[E \cap F] = .6$$

So

$$\Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]} = \frac{6}{9} \qquad\qquad \Pr[F|E] = \frac{\Pr[E \cap F]}{\Pr[E]} = \frac{6}{7}$$

8. (10 points) Solve the system of equations:

$$6x + 5y = 3 \qquad\qquad 2x + 3y = 5$$

There is a unique solution here.

$$\begin{array}{rcr} 6x + 5y & = & 3 \\ 6x + 9y & = & 15 \\ \hline -4y & = & -12 \end{array}$$

So, $y = 3$, and then $2x + 9 = 5$ implies $x = -2$.