MATH 210 PRACTICE EXAM 3 ANSWERS

Semester I, 2007-2008   *Lecture 1*

1. (13 points) Let S be the set of points in the plane which satisfy the following inequalities:

$$3x + 2y \geq -6 \qquad 3x - 2y \leq 6 \qquad x - 4y \leq 5$$

Find all the corner points of S, and say whether S is bounded or unbounded.

The three half-planes are bounded by the lines:

$$① \ 3x + 2y = -6 \qquad ② \ 3x - 2y = 6 \qquad ③ \ x - 4y = 5$$

Plotting S, you see that S is unbounded, with corner points B, on the intersection of ① , ③ , and C, on the intersection of ② , ③ .

Solving ① , ③ , you get $B = (-1, -\frac{3}{2})$. Solving ② , ③ , you get $C = (\frac{7}{5}, -\frac{9}{10})$.

2. (15 points) Find the maximum and minimum values of $x+y$ subject to the constraints:

$$2x - y \geq -1 \qquad -x + 2y \geq -1 \qquad 4x + 3y \leq 12$$

If the maximum and/or minimum doesn't exist, say so.

The three half-planes are bounded by the lines:

$$① \; 2x - y = -1 \qquad ② \; -x + 2y = -1 \qquad ③ \; 4x + 3y = 12$$

Plotting S, you see that S is a triangle, with corner points: A, on the intersection of ① , ③ ; B, on the intersection of ② , ③ ; C, on the intersection of ① , ② .

Solving ① , ③ , you get $A = (\frac{9}{10},\ 2\frac{4}{5})$.
Solving ② , ③ , you get $B = (2\frac{5}{11},\ \frac{8}{11})$.
Solving ① , ② , you get $C = (-1, -1)$.
Then $f(A) = 3\frac{7}{10}$, $f(B) = 3\frac{2}{11}$, $f(C) = -2$.

So, $f(C) = -2$ is the minimum value of f on the set S, and $f(A) = 3\frac{7}{10}$, is the maximum value of f on the set S.

3. (15 points) Solve the system of equations:

$$\begin{array}{ccccccc} 3x & + & 4y & + & 7z & = & 12 \\ 2x & + & 4y & + & 6z & = & 8 \\ 4x & + & 2y & + & 5z & = & 6 \end{array}$$

There is a unique solution here.

$$\left(\begin{array}{ccc|c} 3 & 4 & 7 & 12 \\ 2 & 4 & 6 & 8 \\ 4 & 2 & 5 & 6 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 3 & 4 & 7 & 12 \\ 4 & 2 & 5 & 6 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -2 & -2 & 0 \\ 0 & -6 & -7 & -10 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & -6 & -7 & -10 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -10 \end{array}\right) \sim\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & 10 \end{array}\right)$$

So, $x = -6, \quad y = -10, \quad z = 10.$

4. (12 points) Solve the system of equations:

$$\begin{array}{rcrcrcrcl} 4x & + & 11y & + & 6z & - & 2t & = & 4 \\ 3x & + & 6y & - & 3z & + & 9t & = & 9 \end{array}$$

Here, two of the variables can be arbitrary, with the other two expressed in terms of them.

$$\left(\begin{array}{rrrr|r} 4 & 11 & 6 & -2 & 4 \\ 3 & 6 & -3 & 9 & 9 \end{array}\right) \sim \left(\begin{array}{rrrr|r} 1 & 2 & -1 & 3 & 3 \\ 4 & 11 & 6 & -2 & 4 \end{array}\right) \sim$$
$$\left(\begin{array}{rrrr|r} 1 & 2 & -1 & 3 & 3 \\ 0 & 3 & 10 & -14 & -8 \end{array}\right) \sim \left(\begin{array}{rrrr|r} 1 & 2 & -1 & 3 & 3 \\ 0 & 1 & \frac{10}{3} & \frac{-14}{3} & \frac{-8}{3} \end{array}\right) \sim$$
$$\left(\begin{array}{rrrr|r} 1 & 0 & \frac{-23}{3} & \frac{37}{3} & \frac{25}{3} \\ 0 & 1 & \frac{10}{3} & \frac{-14}{3} & \frac{-8}{3} \end{array}\right)$$

So, z, t are arbitrary, and $x = \frac{23}{3}z - \frac{37}{3}t + \frac{25}{3}$ and $y = -\frac{10}{3}z + \frac{14}{3}t - \frac{8}{3}$.

5. (10 points) Formulate the following as a linear optimization problem. Say explicitly what the variables stand for, what the constraints are, and what the objective function is. Don't solve the problem.

You're selling 25 pound sacks of mixed bird seed. Mix 1 contains 30% sunflower seeds, 45% peanuts, and 25% white millet. Mix 2 contains 50% sunflower seeds, 20% peanuts, and 30% white millet. Mix 3 contains 40% sunflower seeds, 35% peanuts, and 25% white millet. Mix 4 contains 40% sunflower seeds, 25% peanuts, and 35% white millet. You have on hand 1 ton of sunflower seeds, 2 tons of peanuts, and 3 tons of white millet. Your profits are \$3 for each sack of mix 1, \$2 for each sack of mix 2, \$5 for each sack of mix 3, and \$4 for each sack of mix 4. Assume that you know you will sell all the sacks you make up. How many sacks of each mix should you prepare to maximize your profits?

1 ton = 2000 pounds.

Rewriting the above verbiage and expressing weights in pounds, we have:

	Sunfl's	P'nuts	Millet
Mix 1	30/4	45/4	25/4
Mix 2	50/4	20/4	30/4
Mix 3	40/4	35/4	25/4
Mix 4	40/4	25/4	35/4

Let x, y, z, t be the number of mixes of types 1,2,3,4 respectively. Then the objective function is

$$f(x, y, z, t) = 3x + 2y + 5z + 4t \quad ,$$

and the problem is to maximize $f(x, y, z, t)$ subject to the constraints:

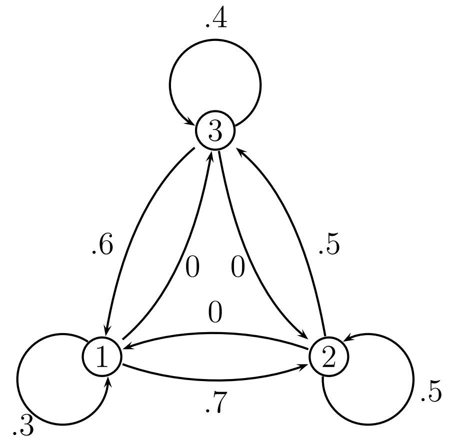
$\frac{30}{4}x + \frac{50}{4}y + \frac{40}{4}z + \frac{40}{4}t \leq 2000$ (from the sunflowers)

$\frac{45}{4}x + \frac{20}{4}y + \frac{35}{4}z + \frac{25}{4}t \leq 4000$ (from the peanuts)

$\frac{25}{4}x + \frac{30}{4}y + \frac{25}{4}z + \frac{35}{4}t \leq 6000$ (from the millet)

$x \geq 0 \quad y \geq 0 \quad z \geq 0 \quad t \geq 0$ (can't have a negative number of sacks)

6. (10 points) A Markov chain has three states: State 1, State 2, State 3. Assume that it has the transition probabilities given by the diagram below and that it is known to be in State 1 on the initial observation. Find the probability that it is in each of the States 1,2,3 after two transitions.



The transition matrix is

$$P = \begin{bmatrix} .3 & .7 & 0 \\ 0 & .5 & .5 \\ .6 & 0 & .4 \end{bmatrix}$$

We're beginning with state $X_0 = [1 \quad 0 \quad 0]$, and we need to find $X_2 = X_0 P^2$. It's quicker to find $X_1 = X_0 P$ and then $X_2 = X_1 P$ (rather than computing P^2), so:

$$X_1 = [1 \quad 0 \quad 0] \begin{bmatrix} .3 & .7 & 0 \\ 0 & .5 & .5 \\ .6 & 0 & .4 \end{bmatrix} = [.3 \quad .7 \quad 0]$$

and then

$$X_2 = [.3 \quad .7 \quad 0] \begin{bmatrix} .3 & .7 & 0 \\ 0 & .5 & .5 \\ .6 & 0 & .4 \end{bmatrix} = [.09 \quad .56 \quad .35]$$

So, the given system has probabilities .09, .56, .35, respectively, of being in State 1, State 2, State 3.

7. (15 points) A Markov chain has three states: State 1, State 2, State 3. It has the transition matrix P shown below. Find the vector W of stable probabilities (so, $WP = W$). There is a unique solution here.

$$P = \begin{bmatrix} .6 & .3 & .1 \\ .1 & .6 & .3 \\ .1 & .1 & .8 \end{bmatrix}$$

$$P - I = \frac{1}{10}\begin{bmatrix} -4 & 3 & 1 \\ 1 & -4 & 3 \\ 1 & 1 & -2 \end{bmatrix}$$

Solving $w_1 + w_2 + w_3 = 1$ plus $W(P - I) = 0$ gives us:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ -4 & 1 & 1 & 0 \\ 3 & -4 & 1 & 0 \\ 1 & 3 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 5 & 5 & 4 \\ 0 & -7 & -2 & -3 \\ 0 & 2 & -3 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & \frac{4}{5} \\ 0 & -7 & -2 & -3 \\ 0 & 2 & -3 & -1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{5} \\ 0 & 1 & 1 & \frac{4}{5} \\ 0 & 0 & 5 & \frac{13}{5} \\ 0 & 0 & -5 & -\frac{13}{5} \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{5} \\ 0 & 1 & 1 & \frac{4}{5} \\ 0 & 0 & 1 & \frac{13}{25} \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{5} \\ 0 & 1 & 0 & \frac{7}{25} \\ 0 & 0 & 1 & \frac{13}{25} \end{array}\right]$$

So, $W = [\frac{1}{5} \quad \frac{7}{25} \quad \frac{13}{25}]$.

Note: When we write down the first augmented matrix, the lower left 3×3 comes from the entries of $P - I$, but the entries get *transposed*; the rows become columns and the columns become rows.

8. (10 points) Say which of the following transition matrices is regular, and explain why; one is regular and one is not.

$$P = \begin{bmatrix} .12 & .11 & .77 & 0 \\ 0 & .77 & 0 & .23 \\ .11 & 0 & .12 & .77 \\ 0 & .25 & 0 & .75 \end{bmatrix} \qquad Q = \begin{bmatrix} .23 & 0 & .77 & 0 \\ 0 & .29 & 0 & .71 \\ 0 & .57 & .43 & 0 \\ .39 & 0 & 0 & .61 \end{bmatrix}$$

Since the numerical values are irrelevant, we compute:

$$P^2 = \begin{bmatrix} + & + & + & 0 \\ 0 & + & 0 & + \\ + & 0 & + & + \\ 0 & + & 0 & + \end{bmatrix} \begin{bmatrix} + & + & + & 0 \\ 0 & + & 0 & + \\ + & 0 & + & + \\ 0 & + & 0 & + \end{bmatrix} = \begin{bmatrix} + & + & + & + \\ 0 & + & 0 & + \\ + & + & + & + \\ 0 & + & 0 & + \end{bmatrix}$$

and then $P^4 = P^2$. Since the pattern will now keep repeating, P is not regular.

$$Q^2 = \begin{bmatrix} + & 0 & + & 0 \\ 0 & + & 0 & + \\ 0 & + & + & 0 \\ + & 0 & 0 & + \end{bmatrix} \begin{bmatrix} + & 0 & + & 0 \\ 0 & + & 0 & + \\ 0 & + & + & 0 \\ + & 0 & 0 & + \end{bmatrix} = \begin{bmatrix} + & + & + & 0 \\ + & + & 0 & + \\ 0 & + & + & + \\ + & 0 & + & + \end{bmatrix}$$

and then

$$Q^4 = \begin{bmatrix} + & + & + & 0 \\ + & + & 0 & + \\ 0 & + & + & + \\ + & 0 & + & + \end{bmatrix} \begin{bmatrix} + & + & + & 0 \\ + & + & 0 & + \\ 0 & + & + & + \\ + & 0 & + & + \end{bmatrix} = \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix}$$

So, Q is regular.