MATH 210 **PRACTICE** EXAM 3

Semester I, 2006-2007 *Lecture 4*

Name: ______________________________ **Section:** _______________

NO CALCULATORS, NOTES, BOOKS, ETC. ALLOWED.
EXPLAIN YOUR WORK.
ANSWERS WITHOUT EXPLANATION WILL RECEIVE 0 CREDIT.

Unless you are instructed otherwise, your answer should be computed completely (e.g., as a whole number, or a simple fraction, or a decimal).

Number	MAX	Grade
1	15	
2	15	
3	10	
4	10	
5	15	
6	10	
7	15	
8	10	
SUM	100	

1. (15 points) Solve the system of equations:

$$\begin{array}{rcrcrcr} 2x & + & 3y & + & 5z & = & 7 \\ 3x & + & 6y & + & 6z & = & 9 \\ 3x & + & 2y & + & 9z & = & 7 \end{array}$$

There is a unique solution here.

$$\left(\begin{array}{ccc|c} 2 & 3 & 5 & 7 \\ 3 & 6 & 6 & 9 \\ 3 & 2 & 9 & 7 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 2 & 3 & 5 & 7 \\ 3 & 2 & 9 & 7 \end{array}\right) \sim \left(\begin{array}{rrr|r} 1 & 2 & 2 & 3 \\ 0 & -1 & 1 & 1 \\ 0 & -4 & 3 & -2 \end{array}\right) \sim$$
$$\left(\begin{array}{rrr|r} 1 & 0 & 4 & 5 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & -6 \end{array}\right) \sim \left(\begin{array}{rrr|r} 1 & 0 & 0 & -19 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 6 \end{array}\right)$$

So, $x = -19, \quad y = 5, \quad z = 6.$

2. (15 points) Solve the system of equations:

$$\begin{array}{ccccccccccc} 2x & + & 4y & + & 2z & + & 4t & - & 2u & = & 4 \\ 3x & + & 7y & - & 2z & + & 4t & + & 3u & = & 1 \end{array}$$

Here, three of the variables can be arbitrary, with the other two expressed in terms of them.

$$\left(\begin{array}{ccccc|c} 2 & 4 & 2 & 4 & -2 & 4 \\ 3 & 7 & -2 & 4 & 3 & 1 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & -1 & 2 \\ 3 & 7 & -2 & 4 & 3 & 1 \end{array}\right) \sim$$
$$\left(\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & -1 & 2 \\ 0 & 1 & -5 & -2 & 6 & -5 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 0 & 11 & 6 & -13 & 12 \\ 0 & 1 & -5 & -2 & 6 & -5 \end{array}\right)$$

So, z, t, u are arbitrary, and $x = -11z - 6t + 13u + 12$ and $y = 5z + 2t - 6u - 5$.

3. (10 points) Formulate the following as a linear optimization problem. Say explicitly what the variables stand for, what the constraints are, and what the objective function is. Don't solve the problem.

You're selling bouquets of flowers. There are four popular bouquets. Bouquet 1 contains 8 roses, 4 daisies, and 4 tulips. Bouquet 2 contains 4 roses, 8 daisies, and 4 tulips. Bouquet 3 contains 4 roses, 4 daisies, and 8 tulips. Bouquet 4 contains 6 roses, 6 daisies, and 6 tulips. You have on hand 75 roses, 80 daisies, and 50 tulips. Your profits are \$1 for each bouquet of type 1, \$2 for each bouquet of type 2, \$3 for each bouquet of type 3, and \$4 for each bouquet of type 4. Assume that you know you will sell all the bouquets you make up. How many of each bouquet should you make to maximize your profits?

Let x, y, z, t be the number of bouquets of types 1,2,3,4 respectively. Then the objective function is $f(x, y, z, t) = x + 2y + 3z + 4t$ and the problem is to maximize $f(x, y, z, t)$ subject to the constraints:

$8x + 4y + 4z + 6t \leq 75$ (from the roses)

$4x + 8y + 4z + 6t \leq 80$ (from the daisies)

$4x + 4y + 8z + 6t \leq 50$ (from the tulips)

$x \geq 0 \quad y \geq 0 \quad z \geq 0 \quad t \geq 0$ (values can't be negative)

4. (10 points) Let S be the set of points in the plane which satisfy the following inequalities:

$$3x + 4y \geq -4 \qquad x + 5y \leq 8 \qquad x - 2y \geq -4$$

Find all the corner points of S, and say whether S is bounded or unbounded.

The three half-planes are bounded by the lines:

$$① \ 3x + 4y = -4 \qquad ② \ x + 5y = 8 \qquad ③ \ x - 2y = -4$$

Plotting S, you see that S is unbounded, with corner points A, on the intersection of ① , ③ , and B, on the intersection of ② , ③ . Solving ① , ③ , you get $A = (-\frac{12}{5}, \frac{4}{5})$. Solving ② , ③ , you get $B = (-\frac{4}{7}, \frac{12}{7})$.

5. (15 points) Find the maximum and minimum values of $x+y$ subject to the constraints:

$$2x + 5y \leq 8 \qquad -2x + y \leq 4 \qquad 2x + y \leq 4$$

If the maximum and/or minimum doesn't exist, say so.

The three half-planes are bounded by the lines:

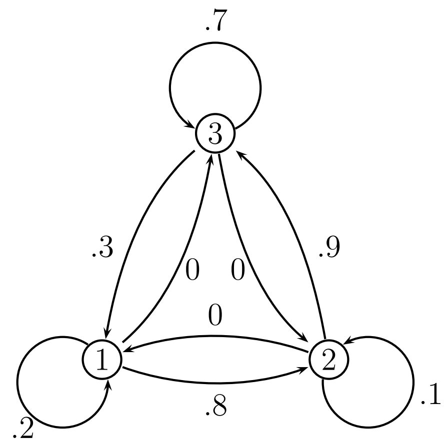
$$① \ 2x + 5y = 8 \qquad ② \ -2x + y = 4 \qquad ③ \ 2x + y = 4$$

Plotting S, you see that S is unbounded, with corner points A, on the intersection of ① , ② , and B, on the intersection of ① , ③ . Solving ① , ② , you get $A = (-1, 2)$. Solving ① , ③ , you get $B = (\frac{3}{2}, 1)$. Then $f(A) = 1$ and $f(B) = \frac{5}{2}$.

The smaller value is $f(A) = 1$, but f is decreasing along line ② out of A, since $f(-2, 0) = -2 < 1$, so there is no minimum value.

The larger value is $f(B) = \frac{5}{2}$. Also, f is decreasing along line ① out of B (because $f(A) < f(B)$), and f is decreasing along line ③ out of B (because $f(2, 0) = 2 < f(B)$). Thus, $f(B) = \frac{5}{2}$ is the maximum value of f on the set S.

6. (10 points) A Markov chain has three states: State 1, State 2, State 3. Assume that it has the transition probabilities given by the diagram below and that it is known to be in State 2 on the initial observation. Find the probability that it is in each of the States 1,2,3 after two transitions.



The transition matrix is

$$P = \begin{bmatrix} .2 & .8 & 0 \\ 0 & .1 & .9 \\ .3 & 0 & .7 \end{bmatrix}$$

We're beginning with state $X_0 = [0 \quad 1 \quad 0]$. After 1 transition we have

$$X_1 = [0 \quad 1 \quad 0] \begin{bmatrix} .2 & .8 & 0 \\ 0 & .1 & .9 \\ .3 & 0 & .7 \end{bmatrix} = [0 \quad .1 \quad .9]$$

After 2 transitions we have

$$X_2 = [0 \quad .1 \quad .9] \begin{bmatrix} .2 & .8 & 0 \\ 0 & .1 & .9 \\ .3 & 0 & .7 \end{bmatrix} = [.27 \quad .01 \quad .72]$$

So, the given system has probabilities .27, .01, .72, respectively, of being in State 1, State 2, State 3.

7. (15 points) A Markov chain has three states: State 1, State 2, State 3. It has the transition matrix P shown below. Find the vector W of stable probabilities (so, $WP = W$). There is a unique solution here.

$$P = \begin{bmatrix} .7 & .1 & .2 \\ .1 & .6 & .3 \\ .2 & .1 & .7 \end{bmatrix}$$

The equation $W(P - I) = 0$ gives us

$$[w_1 \quad w_2 \quad w_3] \begin{bmatrix} -.3 & .1 & .2 \\ .1 & -.4 & .3 \\ .2 & .1 & -.3 \end{bmatrix} = [0 \quad 0 \quad 0]$$

Multiplying this by 10 and adding $w_1 + w_2 + w_3$:

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ -3 & 1 & 2 & 0 \\ 1 & -4 & 1 & 0 \\ 2 & 3 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 0 & 4 & 5 & 3 \\ 0 & -5 & 0 & -1 \\ 0 & 1 & -5 & -2 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 0 & 1 & -5 & -2 \\ 0 & 4 & 5 & 3 \\ 0 & -5 & 0 & -1 \end{array}\right] \sim$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 6 & 3 \\ 0 & 1 & -5 & -2 \\ 0 & 0 & 25 & 11 \\ 0 & 0 & -25 & -11 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 6 & 3 \\ 0 & 1 & -5 & -2 \\ 0 & 0 & 1 & \frac{11}{25} \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 0 & \frac{9}{25} \\ 0 & 1 & 0 & \frac{5}{25} \\ 0 & 0 & 1 & \frac{11}{25} \end{array}\right]$$

So, $W = [\frac{9}{25} \quad \frac{5}{25} \quad \frac{11}{25}]$.

8. (10 points) Say which of the following transition matrices is regular, and explain why.

$$P = \begin{bmatrix} 0 & 0 & .26 & .74 \\ 0 & 0 & .13 & .87 \\ .49 & .51 & 0 & 0 \\ .41 & .59 & 0 & 0 \end{bmatrix} \qquad Q = \begin{bmatrix} .37 & .63 & 0 & 0 \\ 0 & .45 & .55 & 0 \\ 0 & 0 & .81 & .19 \\ .27 & 0 & 0 & .73 \end{bmatrix}$$

Since the actual values are irrelevant, we compute:

$$P^2 = \begin{bmatrix} 0 & 0 & + & + \\ 0 & 0 & + & + \\ + & + & 0 & 0 \\ + & + & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & + & + \\ 0 & 0 & + & + \\ + & + & 0 & 0 \\ + & + & 0 & 0 \end{bmatrix} = \begin{bmatrix} + & + & 0 & 0 \\ + & + & 0 & 0 \\ 0 & 0 & + & + \\ 0 & 0 & + & + \end{bmatrix}$$

and then $P^3 = P$. Since the pattern will now keep repeating, P is not regular.

$$Q^2 = \begin{bmatrix} + & + & 0 & 0 \\ 0 & + & + & 0 \\ 0 & 0 & + & + \\ + & 0 & 0 & + \end{bmatrix} \begin{bmatrix} + & + & 0 & 0 \\ 0 & + & + & 0 \\ 0 & 0 & + & + \\ + & 0 & 0 & + \end{bmatrix} = \begin{bmatrix} + & + & + & 0 \\ 0 & + & + & + \\ + & 0 & + & + \\ + & + & 0 & + \end{bmatrix}$$

and then

$$Q^3 = \begin{bmatrix} + & + & 0 & 0 \\ 0 & + & + & 0 \\ 0 & 0 & + & + \\ + & 0 & 0 & + \end{bmatrix} \begin{bmatrix} + & + & + & 0 \\ 0 & + & + & + \\ + & 0 & + & + \\ + & + & 0 & + \end{bmatrix} = \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix}$$

So, Q is regular.

SCRAP PAPER

MORE SCRAP PAPER