MATH 210 PRACTICE FINAL ANSWERS

Semester I, 2007-2008 *Lecture 1*

1. (7 points) All the diamonds are removed from a deck of cards, so it has 13 red cards and 26 black cards. Then, a five card hand is dealt at random from this deck. What is the probability that the hand contains more black cards than red cards? Here, you may leave your answer as an arithmetical expression, without evaluating it.

The probability of getting x black cards (and $5 - x$ red cards) is:

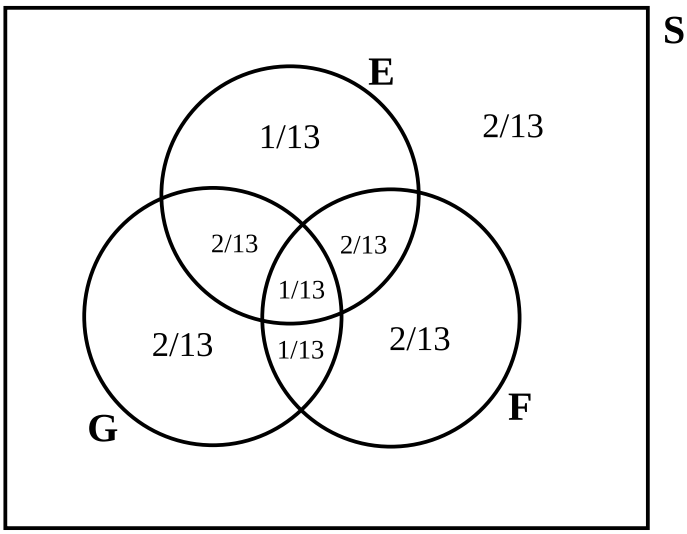
$$\frac{n(E)}{n(S)} = \frac{C(26, x)\, C(13, 5 - x)}{C(39, 5)} \quad .$$

So, the probability of getting 5 or 4 or 3 black cards is:

$$\frac{n(E)}{n(S)} = \frac{C(26, 5)\, C(13, 0) + C(26, 4)\, C(13, 1) + C(26, 3)\, C(13, 2)}{C(39, 5)} \quad .$$

You could omit the "$C(13, 0)$" (since it's 1), and you could have "13" instead of "$C(13, 1)$".

2. (9 points) Let Pr be a probability measure on S with $E, F, G \subset S$. Assume that $\Pr[E'] = \frac{7}{13}$, $\Pr[F] = \frac{6}{13}$, $\Pr[G'] = \frac{7}{13}$, $\Pr[E' \cap F] = \frac{3}{13}$, $\Pr[E' \cap G'] = \frac{4}{13}$, $\Pr[F \cap G'] = \frac{4}{13}$, and $\Pr[E' \cap F \cap G'] = \frac{2}{13}$. Find each of $\Pr[E \cup F \cup G]$ and $\Pr[E' \cup F \cup G]$.



This Venn diagram was filled out in order:

$$E' \cap F \cap G',\ E' \cap F \cap G,\ E' \cap F' \cap G',\ E \cap F \cap G',$$
$$E' \cap F' \cap G,\ E \cap F \cap G,\ E \cap F' \cap G',\ E \cap F' \cap G$$

From the Venn diagram, we see that
$\Pr[E \cup F \cup G] = \frac{11}{13}$ and $\Pr[E' \cup F \cup G] = \frac{12}{13}$.

3. (7 points) Consider the following game: You roll a fair die once, and then get paid the *square* of the number you roll. So, if you roll a 2, you get \$4; if you roll a 5, you get \$25.

Find the expected value of your payoff.

Number rolled	Value of X	Probability	Product
1	1	1/6	1/6
2	4	1/6	4/6
3	9	1/6	9/6
4	16	1/6	16/6
5	25	1/6	25/6
6	36	1/6	36/6
			$E(X) = 91/6$

4. (10 points) Consider the following game: I have a wad of six bills: three \$1 bills, two \$2 bills, and one \$5. Out of this wad, you choose two of them at random, and keep them (and not the other four).

Find the expected value, the variance, and the standard deviation of the amount of money you get to keep. You may leave the result for $\sigma(X)$ in terms of a $\sqrt{}$.

There are $C(6,2) = 15$ possible outcomes. Considering the various possibilities of choosing bills, we get the first three columns of the following table, and we then extend the table to figure out $\mu = E(X)$, $Var(X)$, and $\sigma(X)$.

Chosen Bills	Value of X	Probability	Product	$X - \mu$	$(X-\mu)^2$	$\times$Prob
$1, 1$	2	3/15	6/15	-2	4	12/15
$1, 2$	3	6/15	18/15	-1	1	6/15
$1, 5$	6	3/15	18/15	2	4	12/15
$2, 2$	4	1/15	4/15	0	0	0/15
$2, 5$	7	2/15	14/15	3	9	18/15
	$\mu = E(X) = 60/15 = 4$			$Var(X) = 48/15$		

Then,

$$\sigma(X) = \sqrt{\frac{48}{15}} = \sqrt{\frac{16}{5}} = \sqrt{3.2} \quad ;$$

any of these three answers is OK.

5. (10 points) Solve the system of equations:

$$\begin{array}{rcrcrcl} x & + & 3y & - & 13z & = & 7 \\ 4x & - & 8y & + & 4z & = & 4 \\ x & - & 3y & + & 4z & = & 6 \end{array}$$

There is only one solution.

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 1 & 3 & -13 & 7 \\ 1 & -3 & 4 & 6 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 5 & -14 & 6 \\ 0 & -1 & 3 & 5 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -5 & -9 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & 1 & 31 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 146 \\ 0 & 1 & 0 & 88 \\ 0 & 0 & 1 & 31 \end{array}\right)$$

So, $x = 146$; $y = 88$; $z = 31$.

6. (10 points) A Markov chain has three states: State 1, State 2, State 3. It has the transition matrix P shown below. Find the vector W of stable probabilities (so, $WP = W$). There is a unique solution here.

$$P = \begin{bmatrix} .6 & .2 & .2 \\ .3 & .6 & .1 \\ .1 & .3 & .6 \end{bmatrix}$$

$$10(P - I) = \begin{bmatrix} -4 & 2 & 2 \\ 3 & -4 & 1 \\ 1 & 3 & -4 \end{bmatrix}$$

Solving $w_1 + w_2 + w_3 = 1$ plus $W(P - I) = 0$ gives us:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ -4 & 3 & 1 & 0 \\ 2 & -4 & 3 & 0 \\ 2 & 1 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 7 & 5 & 4 \\ 0 & -6 & 1 & -2 \\ 0 & -1 & -6 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 6 & 2 \\ 0 & 7 & 5 & 4 \\ 0 & -6 & 1 & -2 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & -1 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & -37 & -10 \\ 0 & 0 & 37 & 10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -5 & -1 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & \frac{10}{37} \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{13}{37} \\ 0 & 1 & 0 & \frac{14}{37} \\ 0 & 0 & 1 & \frac{10}{37} \end{array}\right]$$

So, $W = [\frac{13}{37} \quad \frac{14}{37} \quad \frac{10}{37}]$.

7. (8 points) You plant 1,000,000 corn seeds. Each seed has a germination rate of 90% (that is, it has a probability of .9 of sprouting). Use the normal approximation to the binomial to estimate the probabilities of the following happening:

a. The number of seeds that sprout is between 900,000 and 900,450.

b. The number of seeds that sprout is at least 900,030.

$n = 1,000,000$, $p = .9$, $q = .1$, so
$\mu = np = 900,000$ and $\sigma = \sqrt{npq} = \sqrt{90000} = 300$.

a. $\Pr[\mu \leq X \leq \mu + 1.5\,\sigma] = \Pr[0 \leq Z \leq 1.5] = .4332$

b. $\Pr[\mu + .1\,\sigma \leq X] = \Pr[.1 \leq Z] = .5000 - .0398 = .4602$

8. (8 points) Let X be a random variable with probability density function f, where

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 0.1 & \text{if } 0 \leq x < 6 \\ 0.4 & \text{if } 6 \leq x < 7 \\ 0 & \text{if } 7 \leq x \end{cases}$$

Find $\Pr[X \geq 3 \mid X \leq 5]$ and $\Pr[X \leq 5 \mid X \geq 3]$.

If E denotes the event that $X \geq 3$ and F denotes the event that $X \leq 5$, then finding the areas on the graph we see that

$$\Pr[E] = .7 \qquad \Pr[F] = .5 \qquad \Pr[E \cap F] = .2$$

So

$$\Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]} = \frac{2}{5} \qquad\qquad \Pr[F|E] = \frac{\Pr[E \cap F]}{\Pr[E]} = \frac{2}{7}$$

9. (7 points) A fair coin is tossed 8 times. Find the probability that your sequence of 8 tosses comes up heads exactly 5 times.

This is a Bernoulli process with $p = q = \frac{1}{2}$, so

$$\Pr[5 \text{ heads}] = C(8,5)\left(\frac{1}{2}\right)^8 = C(8,3)\frac{1}{256} = \frac{8 \cdot 7 \cdot 6}{6}\frac{1}{256} = \frac{56}{256}$$

10. (7 points) Suppose that, starting now, you put \$50 into a bank account every month for the next twenty years. You start with a balance of zero. The bank pays 12% interest (annual rate), and the interest is compounded monthly. How much money will you have at the end of the twenty years? You may leave your answer as an arithmetical expression, without evaluating it.

This is a sinking fund, with $k = .12/12 = .01$, $Y = 50$, and $n = 12 \cdot 20 = 240$, so

$$\text{amount} = \frac{Y}{k}\left[(1+k)^n - 1\right] = \frac{50}{.01}\left[(1.01)^{240} - 1\right]$$

11. (10 points) Find the maximum and minimum values of $x + y$ subject to the constraints:

$$-x + 3y \leq 6 \qquad 2x + y \leq 4 \qquad -2x + y \leq 4$$

If the maximum and/or minimum doesn't exist, say so.

The three half-planes are bounded by the lines:

$$① \ -x + 3y = 6 \qquad ② \ 2x + y = 4 \qquad ③ \ -2x + y = 4$$

Plotting S, you see that S is an unbounded region, with corner points A, on the intersection of ① , ③ ; and B, on the intersection of ① , ② .

Solving ① , ③ , you get $A = (-\frac{6}{5}, \frac{8}{5})$. Solving ① , ② , you get $B = (\frac{6}{7}, \frac{16}{7})$. Then $f(A) = \frac{2}{5}$, $f(B) = \frac{22}{7}$, so $f(A)$ is the smaller value and $f(B)$ is the larger value.

Test for A being a minimum: The function is increasing towards B on line ① but decreasing towards $(-2, 0)$ on line ③ , since $f(-2, 0) = -2 < f(A)$. Thus, there is no minimum value of f on S.

Test for B being a maximum: The function is decreasing towards A on line ① and decreasing towards $(2, 0)$ on line ② , since $f(2, 0) = 2 < f(B)$. Thus, the value at B is the maximum.

12. (7 points) Suppose that you buy a ten year annuity with \$20,000. The annuity pays you monthly over the next ten years; after that, your money is completely used up. During the ten years, you continue to earn interest on the funds not yet paid out, at a 6% annual rate, compounded monthly. Find the monthly payments of this annuity. You may leave your answer as an arithmetical expression, without evaluating it.

This is an annuity, so

$$\text{amount} = \frac{Y}{k}\left[1-(1+k)^{-n}\right] \quad .$$

here, $k = .06/12 = .005$, $n = 12 \cdot 10 = 120$, and Y (the monthly payment) is unknown, so

$$20000 = \frac{Y}{.005}\left[1-(1.005)^{-120}\right] \quad .$$

so,

$$Y = \frac{20000 \cdot .005}{1-(1.005)^{-120}}$$