MATH 210 **PRACTICE** FINAL

Semester I, 2006-2007 *Lecture 4*

Name: ______________________________ **Section:** ______________

NO CALCULATORS, NOTES, BOOKS, ETC. ALLOWED.
EXPLAIN YOUR WORK.
ANSWERS WITHOUT EXPLANATION WILL RECEIVE 0 CREDIT.

Unless you are instructed otherwise, your answer should be computed completely (e.g., as a whole number, or a simple fraction, or a decimal).

A table of the areas under the standard normal curve is attached to this exam.

Number	MAX	Grade
1	7	
2	9	
3	9	
4	8	
5	8	
6	7	
7	8	
8	10	
9	10	
10	10	
11	7	
12	7	
SUM	100	

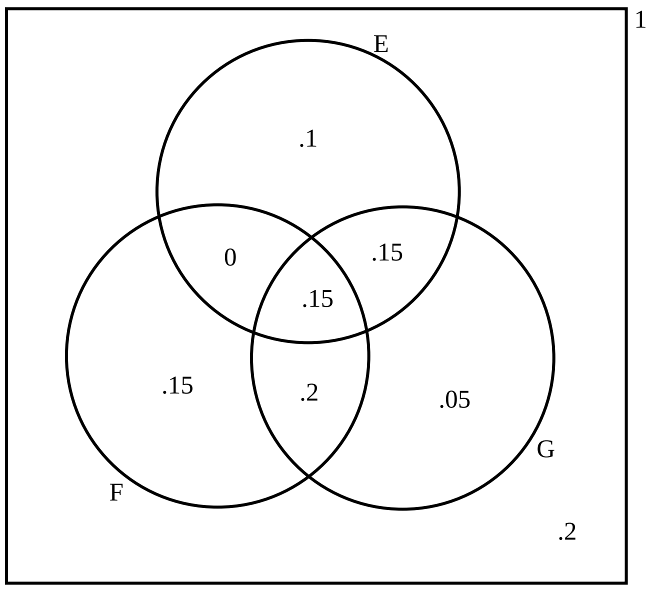
1. (7 points) A five card poker hand is dealt at random from a standard deck of 52 cards. What is the probability that the hand will contain at least one card from each of the four suits? Here, you may leave your answer as an arithmetical expression, without evaluating it.

$n(S) = C(52, 5)$. E is the event that your hand will contain 2 cards from one suit and 1 card from each of the other three. There's 4 possibilities for the suit which gets two cards chosen, so $n(E) = 4 \cdot 13^3 \cdot C(13, 2)$. Then

$$\Pr[E] = \frac{4 \cdot 13^3 \cdot C(13,2)}{C(52,5)}$$

2. (9 points) Let Pr be a probability measure on S with $E, F, G \subset S$. Assume that $\Pr[E'] = 0.6$, $\Pr[F'] = 0.5$, $\Pr[G] = 0.55$, $\Pr[E' \cap F'] = 0.25$, $\Pr[E' \cap G] = 0.25$, $\Pr[F' \cap G] = 0.2$, and $\Pr[E' \cap F' \cap G] = 0.05$. Find each of $\Pr[E \cup F \cup G]$ and $\Pr[E' \cup F \cup G]$.

First, fill out the Venn diagram:



Then, we see that $\Pr[E \cup F \cup G] = .8$ and $\Pr[E' \cup F \cup G] = .9$.

3. (9 points) Consider the following game: Make up a 12–card deck using just the twos, threes, and fours; your deck has 6 red cards and 6 black cards. Then, you get dealt a hand of four cards from this deck. If you get four red cards, you get paid \$990. If you get exactly three red cards, you get paid \$495. If you get two or fewer red cards, you get paid nothing.

Find the expected value of your payoff.

$E(\text{payoff}) = 990 \cdot \Pr(4 \text{ reds}) + 495 \cdot \Pr(3 \text{ reds})$.

$n(S) = C(12, 4) = 495$. Then,

$\Pr(4 \text{ reds}) = C(6, 4)/495 = 15/495$ and $\Pr(3 \text{ reds}) = C(6, 3) \cdot 6/495 = 120/495$, so

$$E(\text{payoff}) = 990 \cdot 15/495 + 495 \cdot 120/495 = \$150.$$

4. (8 points) A random variable X has the density function shown below. Find the expected value, variance, and standard deviation of X.

Value of X	Probability	product	$(X-\mu)^2$	$\cdot$ probability
-10	0.1	-1	100	10.0
-4	0.1	$-.4$	16	1.6
-2	0.2	$-.4$	4	.8
-1	0.1	$-.1$	1	.1
1	0.1	.1	1	.1
2	0.2	.4	4	.8
4	0.1	.4	16	1.6
10	0.1	1	100	10.0
		$\mu = 0$		$Var(X) = 25$

Then, $\sigma(X) = \sqrt{25} = 5$.

5. (8 points) Let X be a random variable with probability density function f, where

$$f(x) = \begin{cases} 0 & \text{if } x < -4 \\ 0.1 & \text{if } -4 \leq x < 0 \\ 0.3 & \text{if } 0 \leq x < 2 \\ 0 & \text{if } 2 \leq x \end{cases}$$

Find $\Pr[X \geq -3 \mid X \leq 1]$ and $\Pr[X \leq 1 \mid X \geq -3]$.

If E is the event "$X \geq -3$" and F is the event "$X \leq 1$", then $E \cap F$ is the event "$-3 \leq X \leq 1$". Looking at the graph and computing areas, we see that $\Pr[E] = .9$, $\Pr[F] = .7$, and $\Pr[E \cap F] = .6$. Then

$$\Pr[E \mid F] = \frac{\Pr[E \cap F]}{\Pr[F]} = \frac{6}{7} \qquad \Pr[F \mid E] = \frac{\Pr[E \cap F]}{\Pr[E]} = \frac{6}{9}$$

6. (7 points) A fair die is rolled 9 times. Assume that the results of the rolls are independent. Find the probability that your sequence of 9 rolls comes up 6 at least 7 times. Here, you may leave your answer as an arithmetical expression, without evaluating it.

This is a binomial distribution with $n = 9$, $p = 1/6$, and $q = 5/6$, so the probability of getting 7 or 8 or 9 sixes is

$$C(9,7)\left(\frac{1}{6}\right)^7\left(\frac{5}{6}\right)^2 + C(9,8)\left(\frac{1}{6}\right)^8\left(\frac{5}{6}\right)^1 + C(9,9)\left(\frac{1}{6}\right)^9\left(\frac{5}{6}\right)^0$$

7. (8 points) An unfair coin has a probability of 0.4 of coming up heads and 0.6 of coming up tails. It is tossed 150 times. Use the normal approximation to the binomial to estimate the probabilities of the following happening:

a. The number of heads is between 54 and 66.

b. The number of heads is at least 48.

There is a table of the areas under the standard normal curve at the end of this exam.

$n = 150$, $p = .4$, $q = .6$, so $\mu = np = 60$ and $\sigma = \sqrt{npq} = \sqrt{36} = 6$.

If X is the number of heads and Z is the standard normal random variable, then

$$Z = \frac{X - \mu}{\sigma} = \frac{X - 60}{6}$$

a. $\Pr[54 \le X \le 66] = \Pr[-1 \le X \le 1] = 2 \times .3413 = .6826$
b. $\Pr[48 \le X] = \Pr[-2 \le X] = .5 + .4772 = .9772$.

8. (10 points) Solve the system of equations:

$$\begin{array}{rcrcrcl} x & + & 2y & + & 3z & = & 7 \\ 5x & + & 5y & - & 10z & = & 5 \\ 2x & + & y & + & z & = & 6 \end{array}$$

There is only one solution.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 5 & 5 & -10 & 5 \\ 2 & 1 & 1 & 6 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 1 & 2 & 3 & 7 \\ 2 & 1 & 1 & 6 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & 5 & 6 \\ 0 & -1 & 5 & 4 \end{array}\right) \sim$$
$$\left(\begin{array}{ccc|c} 1 & 0 & -7 & -5 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 10 & 10 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

So, $x = 2, \quad y = 1, \quad z = 1.$

9. (10 points) A Markov chain has three states: State 1, State 2, State 3. It has the transition matrix P shown below. Find the vector W of stable probabilities (so, $WP = W$). There is a unique solution here.

$$P = \begin{bmatrix} .6 & .2 & .2 \\ .2 & .7 & .1 \\ .2 & .2 & .6 \end{bmatrix}$$

The equation $W(P - I) = 0$ gives us

$$[w_1 \quad w_2 \quad w_3] \begin{bmatrix} -.4 & .2 & .2 \\ .2 & -.3 & .1 \\ .2 & .2 & -.4 \end{bmatrix} = [0 \quad 0 \quad 0]$$

Multiplying this by 10 and adding "$w_1 + w_2 + w_3 = 1$":

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ -4 & 2 & 2 & 0 \\ 2 & -3 & 2 & 0 \\ 2 & 1 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 0 & 6 & 6 & 4 \\ 0 & -5 & 0 & -2 \\ 0 & -1 & -6 & -2 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 0 & 1 & 6 & 2 \\ 0 & 6 & 6 & 4 \\ 0 & -5 & 0 & -2 \end{array}\right] \sim$$

$$\left[\begin{array}{rrr|r} 1 & 0 & -5 & -1 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & -30 & -8 \\ 0 & 0 & 30 & 8 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & -5 & -1 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & \frac{4}{15} \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 0 & \frac{5}{15} \\ 0 & 1 & 0 & \frac{6}{15} \\ 0 & 0 & 1 & \frac{4}{15} \end{array}\right]$$

So, $W = [\frac{5}{15} \quad \frac{6}{15} \quad \frac{4}{15}]$.

10. (10 points) Find the maximum and minimum values of $x+y$ subject to the constraints:

$$x + 5y \leq 6 \qquad 3x - y \leq 2$$
$$x - 3y \leq 2 \qquad 5x + y \geq -6$$

If the maximum and/or minimum doesn't exist, say so.

The four half-planes are bounded by the lines:

① $x + 5y = 6$ ② $x - 3y = 2$ ③ $3x - y = 2$ ④ $5x + y = -6$

Plotting S, you see that S is a bounded four-sided figure, with corner points A, on the intersection of ① , ③ , B, on the intersection of ① , ④ , C, on the intersection of ② , ④ , D, on the intersection of ② , ③ .

Solving the pairs of equations, you get $A = (1,1)$, $B = (-3/2, 3/2)$, $C = (-1,-1)$, $D = (1/2, -1/2)$. So, $f(A) = 2$, $f(B) = 0$, $f(C) = -2$, $f(D) = 0$. Thus, $2 = f(A)$ is the maximum value on f on the set S and $-2 = f(C)$ is the minimum value on f on the set S.

11. (7 points) Find the monthly payments that are necessary to amortize a loan of \$10,000 over 5 years with interest at 5% annual rate. Here, you may leave your answer as an arithmetical expression, without evaluating it.

If L is the loan value, then

$$L = \frac{Y}{k}\left[1 - (1+k)^{-n}\right]$$

Here, $L = 10000$, $n = 60$, $k = .05/12$, and you have to solve for Y, so

$$Y = \frac{10000 \times \frac{.05}{12}}{1 - \left(1 + \frac{.05}{12}\right)^{-60}}$$

12. (7 points) You need to have some money ten years from now, so you put money in a sinking fund, depositing monthly into an account which pays 6% interest (annual rate), compounded monthly. Each month you deposit \$100. How much money will you have at the end of the ten years? Here, you may leave your answer as an arithmetical expression, without evaluating it.

If S is the value of the sinking fund at the end of the ten years, then

$$S = \frac{Y}{k}\left[(1+k)^n - 1\right]$$

Here, $Y = 100$, $n = 120$, $k = .005$, so

$$S = \frac{100}{.005}\left[(1.005)^{120} - 1\right]$$

SCRAP PAPER

MORE SCRAP PAPER

Area under the Standard Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990