Part I. Do all problems in the space provided. Clearly define all random variables and other notation that you use, and clearly specify what you are calculating at each step in a calculation. Justify all answers. In particular, make certain that it is clear how your calculations follow from the material of the course. Clearly mark all answers.

1. (8 pts) Three roommates take a test. Let

\[ A = \{ \text{Lou passes the test} \} \]
\[ B = \{ \text{Mary passes the test} \} \]
\[ C = \{ \text{Bob passes the test} \} \]

Using unions, intersections, complements, etc., express each of the following events in terms of A, B, and C.

(a) \( D = \{ \text{All three students pass the test} \} \)

\[ D = A \cap B \cap C \]

(b) \( E = \{ \text{Lou passes the test but Mary does not.} \} \)

\[ E = A \cap B^c \]

2. (12 pts) Let \( P(A) = .3 \) and \( P(B) = .5 \).

(a) If \( A \) and \( B \) are disjoint, what is \( P(A|B) \)?

\[ P(A|B) = 0 \]

(b) If \( A \) and \( B \) are independent, what is \( P(A|B) \)?

\[ P(A|B) = .3 \]

(c) If \( A \subset B \), what is \( P(A|B) \)?

\[ P(A|B) = \frac{3}{5} \]
3. (10 pts) Let $X$ have probability density function

$$f_X(x) = \begin{cases} \frac{1}{9}(3-x)^2 & 0 \leq x \leq 3 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Calculate the cdf for $X$? $\mathcal{R}(X) = [0, 3]$, so

$$F_X(x) = P\{X \leq x\} = \begin{cases} 0 & x \leq 0 \\ \int_0^x \frac{1}{9}(3-y)^2dy = 1 - \frac{1}{27}(3-x)^3 & 0 < x < 3 \\ 1 & x \geq 3 \end{cases}$$

(b) Calculate $P\{1 < X < 2\}$?

$$P\{1 < X < 2\} = F_X(2) - F_X(1) = \frac{1}{27}(8-1) = \frac{7}{27}$$

4. (10 pts) Suppose $Y$ has a discrete distribution with pmf given by

<table>
<thead>
<tr>
<th>$y$</th>
<th>$-1$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_Y(y)$</td>
<td>$1/2$</td>
<td>$1/8$</td>
<td>$1/8$</td>
<td>$1/4$</td>
</tr>
</tbody>
</table>

(a) What is $\mathcal{R}(Y)$?

$$\mathcal{R}(Y) = \{-1, 0, 1, 2\}$$

(b) Calculate $P(Y \leq 1|Y > 0)$?

$$P\{Y \leq 1|Y > 0\} = \frac{P\{Y = 1\}}{P\{Y > 0\}} = \frac{1/8}{3/8} = \frac{1}{3}$$

(c) Calculate $E[Y^4 - 1]$?

$$E[Y^4 - 1] = -1 \times \frac{1}{8} + 15 \times \frac{1}{4} = \frac{29}{8}$$
Part II. Each of the following problems will be graded on the basis of 15 points. Your score on this part of the exam will be the sum of your 3 highest scores plus one half of the points you receive on the other two problems. Clearly define all random variables and other notation that you use. Clearly mark your answers.

1. A multiple choice exam contains 6 questions, each with four possible answers. The answers are to be marked on a Scantron sheet that has six columns (one for each problem) containing four ovals. Only one answer is correct in each column.

(a) If a student randomly marks one oval in each column, what is the probability that the student gets exactly two answers correct? No more than two answers correct?

\[ P\{S_6 = 2\} = \binom{6}{2} \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^4 = \frac{15 \times 3^4}{4^6} = \frac{5 \times 3^5}{4^6} \]

\[ P\{S_6 \leq 2\} = P\{S_6 = 0\} + P\{S_6 = 1\} + P\{S_6 = 2\} = \frac{3^6 + 6 \times 3^5 + 15 \times 3^4}{4^6} = 14 \times \frac{3^5}{4^6} \]

(b) If the student randomly marks six ovals out of the 24 with out regard to the columns, what is the probability that the student marks two of the ovals that correspond to correct answers?

\[ Z = \text{number of correct ovals marked} \]

\[ P\{Z = 2\} = \binom{6}{2} \binom{18}{4} = \frac{15 \times 18 \times 17 \times 16 \times 15}{4 \times 3 \times 2 \times 1} \times \frac{24 \times 23 \times 22 \times 21 \times 20 \times 19}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{15 \times 3 \times 17 \times 15}{23 \times 11 \times 7 \times 19} \]
2. A cabinet contains three drawers. One drawer contains 6 black balls, one contains 6 red balls, and one contains 1 black ball and 3 red balls. A drawer is selected at random and the balls in that drawer are drawn one at a time.

(a) What is the probability that the first ball drawn is black?

Let $B_i = \{i\text{th ball drawn is black}\}$, $A_j = \{j\text{th drawer selected}\}$

$$P(B_1) = \sum_{j=1}^{3} P(B_1|A_j)P(A_j) = 1 \times \frac{1}{3} + \frac{1}{4} \times \frac{1}{3} = \frac{5}{12}$$

(b) If the first ball drawn is black, what is the probability that the second ball drawn is black?

Note that $A_1 \cap A_2 = B_1$

$$P(A_2|A_1) = \frac{P(A_1 \cap A_2)}{P(A_1)} = \frac{1/3}{5/12} = \frac{4}{5}$$
3. A bucket contains six balls numbered 1 through 5. Three balls are drawn at random. Let $X$ be the maximum of the three numbers and $Y$ be the sum.

(a) Calculate the joint probability mass function for $X$ and $Y$. Display the answer as a table.

<table>
<thead>
<tr>
<th>$x$</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$\frac{1}{10}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>$\frac{1}{10}$</td>
<td>$\frac{1}{10}$</td>
<td>$\frac{1}{10}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>$\frac{1}{10}$</td>
<td>$\frac{1}{10}$</td>
<td>$\frac{2}{10}$</td>
<td>$\frac{1}{10}$</td>
<td>$\frac{1}{10}$</td>
</tr>
</tbody>
</table>

(b) Calculate $P\{Y = 10|X = 5\}$.

$$P\{Y = 10|X = 5\} = \frac{2/10}{6/10} = \frac{1}{3}$$

(c) Calculate $E[Y - X]$.

$$E[Y - X] = E[Y] - E[X] = 9 - (3 \times \frac{1}{10} + 4 \times \frac{3}{10} + 5 \times \frac{6}{10}) = \frac{45}{10} = 4.5$$
4. Suppose that $U$ and $V$ are independent, exponentially distributed random variables, each with parameter 3.

(a) Calculate $P\{U < 2V\}$.

$$P\{U < 2V\} = \int_0^\infty \int_0^{2v} 9e^{-3u-3v}dudv = \int_0^\infty 3(e^{-3v} - e^{-6v-3v})du$$

$$\int_0^\infty 3(e^{-3v} - e^{-9v})du = 1 + \frac{1}{3} e^{-9v}\bigg|_0^\infty = 1 - \frac{1}{3} = \frac{2}{3}$$

(b) Let $Y = U - V$. Calculate the pdf $f_Y$ for $Y$.

For $y > 0$,

$$P\{Y \leq y\} = 1 - \int_y^\infty \int_0^{u-y} 9e^{-3u-3v}dvdu$$

$$= 1 - \int_y^\infty 3(e^{-3u} - e^{-6u+3y})du$$

$$= 1 - e^{-3y} + \frac{1}{2}e^{-3y} = 1 - \frac{1}{2}e^{-3y}$$

For $y < 0$,

$$P\{Y \leq y\} = \int_0^\infty \int_u^{\infty} 9e^{-3u-3v}dvdu$$

$$= \int_0^\infty 3e^{-6v+3y}du$$

$$= \frac{1}{2}e^{3y}$$

$$f_Y(y) = \begin{cases} \frac{3}{2}e^{-3y} & y \geq 0 \\ \frac{3}{2}e^{3y} & y < 0 \end{cases}$$
5. The demand for ice cream during the county fair is highly variable due to the temperature. Suppose that the demand $X$ in hundreds of gallons has density

$$f_X(x) = \begin{cases} 
\frac{4}{x^5} & x \geq 1 \\
0 & x < 1 
\end{cases}$$

If the ice cream costs $3 per gallon and will be sold at $10 per gallon, how much ice cream should the vendor buy to maximize EXPECTED profit? (Assume that leftover ice cream must be discarded.)

$q =$ quantity purchased (in hundreds of gallons)

$Y =$ revenue

$$E[Y] = \int_1^q 1000x \frac{4}{x^5} dx + \int_q^\infty 1000q \frac{4}{x^5} dx = 1000\left(\frac{4}{3}(1 - q^{-3}) + q^{-3}\right)$$

Profit $= 1000\left(\frac{4}{3} - \frac{1}{3}q^{-3}\right) - 300q \equiv H(q)$

$$H'(q) = 1000q^{-4} - 300$$

The vendor should buy $q = \left(\frac{10}{3}\right)^{1/4}$