Conditioning on values of random variables

If X has a discrete distribution with pmf p_X:

$$P(A) = \sum_{x\in\mathcal{R}(X)} P(A \cap \{X = x\}) = \sum_{x\in\mathcal{R}(X)} P(A|X = x)P\{X = x\} = \sum_{x\in\mathcal{R}(X)} P(A|X = x)p_X(x)$$

If X has a continuous distribution with pdf f_X:

$$P(A) = \int_{-\infty}^{\infty} P(A|X = x)f_X(x)dx$$

$$P(A \cap \{a \le X \le b\}) = \int_a^b P(A|X = x)f_X(x)dx \tag{1}$$

If X has a continuous distribution and A is an event with $P(A) > 0$, then define $f_X(x|A)$ to be the function satisfying

$$P(a \le X \le b|A) = \int_a^b f_X(x|A)dx$$

By (1), we must have

$$f_X(x|A) = \frac{P(A|X = x)f_X(x)}{P(A)}$$

If X and Y have joint pdf $f_{XY}(x, y)$, then

$$P\{a \le X \le b, c \le Y \le d\} = \int_a^b \int_c^d f_{XY}(x, y)dydx = \int_a^b \int_c^d \frac{f_{XY}(x, y)}{f_X(x)} dy f_X(x)dx$$

and hence

$$P(c \le Y \le d|X = x) = \int_c^d \frac{f_{XY}(x, y)}{f_X(x)} dy$$

and the conditional density for Y given the value of X is defined to be

$$f_{Y|X}(y|x) = \frac{f_{XY}(x, y)}{f_X(x)}.$$

Note that

$$f_{Y|X}(y|x) = \frac{\partial}{\partial y} P(Y \le y|X = x).$$

If X and Y are independent, then

$$f_{Y|X}(y|x) = f_Y(y)$$

Problems

1. Suppose that X has density function

$$f_X(x) = \begin{cases} \frac{3}{4}x(2-x) & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

and that Y is uniformly distributed on $[0, X]$, that is,

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{x} & 0 \le y \le x \\ 0 & \text{otherwise} \end{cases}$$

   (a) What is the joint density for X and Y?

   (b) What is the probability density function for Y?

   (c) Given that $Y = 1$, what is the probability that $X > 1.5$?

   (d) What is the conditional density function for X given Y?

2. Suppose that Y is uniformly distributed on [0,2] and that, given the value of Y, X is exponentially distributed with parameter Y, that is

$$f_{X|Y}(x|y) = \begin{cases} ye^{-yx} & x \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

   (a) What is the joint density for X and Y?

   (b) What is the density for X?

   (c) What is the conditional density for Y given X?

3. A coin is selected from a very large collection of lopsided coins and flipped three times. Suppose that the probability that the selected coin comes up heads on a flip is a random variable Y that is uniformly distributed on $[0, 1]$. Let A_k be the event that the kth flip is heads.

   (a) What is $P(A_k)$?

   (b) Let $B = A_1 \cap A_2^c \cap A_3 = \{\text{first flip heads, second tails, third heads}\}$. What is $P(B)$.

   (c) What is the conditional probability that Y is less than .5 given B?

   (d) What is $f_Y(y|B)$?

   (e) Sketch the graph of $f_Y(y|B)$. For what value of y is this conditional density at its maximum?

   (f) Suppose the selected coin is flipped n times. Let B_k be the event that exactly k heads occur in the n flips. For what value of y is $f_Y(y|B_k)$ at its maximum? (Note that you do not need to calculate any integrals to answer this question.) What does this calculation say about the "most likely" value of Y given the observation B_k?

4. Suppose that Y is exponentially distributed with parameter 4 and that

$$P\{X = k|Y = y\} = e^{-y}\frac{y^k}{k!}, \; k = 0, 1, \ldots$$

   (a) What is $P\{X = 0\}$?

   (b) Find the general formula for $P\{X = k\}$. (You may want to consult a table of integrals.)

5. Suppose that Y has density

$$f_Y(y) = \begin{cases} ye^{-y} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and that X is uniformly distributed on $[0, Y]$. What is the probability density function for X?

6. Consider a sequence of flips of a button with probability of heads p. Let N_1 be the number of the flip on which the first head appears. Let N_2 be the number of the flip on which the second head appears.

   (a) What is the conditional distribution of N_2 given N_1?

   (b) What is the conditional distribution of $N_2 - N_1$ given N_1?

7. Let

$$f_{XY}(x, y) = \begin{cases} 9e^{-3x-3y} & x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

   (a) What is the conditional density of Y given X?

   (b) What is the conditional density of X given $X + Y$?

8. Suppose that X is uniformly distributed on $[0, 1]$ and that Y is uniformly distributed on $[0, X]$. What is the joint density for X and Y?

9. Doctors are developing a simple new test to indicate high blood sugar levels. If a patient's blood sugar level is x, then the test is positive with probability $\frac{x}{100+x}$. The blood sugar level of a randomly selected patient is assumed to be uniformly distributed over $[0, 200]$.

   (a) What is the probability that the test is positive for a randomly selected patient?

   (b) If the test is positive, what is the probability that the patient's blood sugar level is over 150?

10. During the day, Sam keeps the temperature in his back yard workshop at 68 degrees, but he shuts the heat off at night. If the overnight low temperature outside falls to y degrees, then the overnight low temperature inside the workshop is uniformly distributed over the interval $[y, 68]$. Suppose that the overnight low outside temperature Y has density

$$f_Y(y) = \begin{cases} \frac{y}{1600} & 20 \leq y \leq 60 \\ 0 & \text{otherwise.} \end{cases}$$

What is the probability that the inside low temperature falls below freezing (32 degrees)?

11. Jane plans to leave for work each morning at 7:00 AM. Because the traffic is worse the later she leaves, she figures that if she leaves x minutes late, then her driving time will be uniformly distributed over $[x + 15, x + 30]$. Suppose that she leaves for work X minutes late, where X has density function

$$f_X(x) = \begin{cases} \frac{30-x}{450} & 0 \leq x \leq 30 \\ 0 & \text{otherwise.} \end{cases}$$

She is supposed to start work at 7:45. What is the probability that she will be late? [Give an exact integral expression for the probability, with the correct limits and the correct integrand. You do not need to compute the integral.]

12. An electronic device attached to a telescope is supposed to record the times that a meteor is sighted. There is a delay in the device so that if a meteor enters the field of view of the telescope at time t, then the time recorded by the device is uniformly distributed on $[t, t+10]$. (Time is measured in seconds since the last meteor sighting.) If the actual arrival time is exponentially distributed with parameter .0025, what is the conditional distribution of the actual arrival time given the recorded time? (This problem has a messy solution.)

13. A particular chemical is unstable in solution. If a solution with concentration x moles/liter is produced, then one day later, the concentration in the solution falls to a level that can be modeled as a random variable, uniformly distributed over the interval $[0, x]$. Suppose that the initial concentration is uniformly distributed over $[0, 4]$. What is the probability density for the concentration one day later?