

Part I. Do all problems.

1. (8pts) Let $P(A) = .3$ and $P(B) = .4$.

a) If A and B are disjoint, what is $P(A \cup B)$?

Sol:

$$P(A \cup B) = P(A) + P(B) = .7$$

b) If A and B are independent, what is $P(A \cup B)$?

Sol:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A)P(B) = .58$$

2. (8pts) Let $N(t)$ be a Poisson process with parameter 4. Calculate

$$P\{N(.5) = 2, N(1) = 4, N(2) = 4\}.$$

Sol:

$$\begin{aligned} P\{N(.5) = 2, N(1) = 4, N(2) = 4\} &= P\{N(.5) = 2, N(1) - N(.5) = 2, N(2) - N(1) = 0\} \\ &= P\{N(.5) = 2\}P\{N(1) - N(.5) = 2\}P\{N(2) - N(1) = 0\} \\ &= e^{-2} \frac{2^2}{2!} e^{-2} \frac{2^2}{2!} e^{-4} \\ &= e^{-8} 4 = .00134 \end{aligned}$$

3. (8pts) A shipment of 40 electronic components contains 3 defective items. If a radio receiver contains 5 of the components from the shipment (selected randomly), what is the probability that it contains none of the defective ones?

Sol: Let $A = \{\text{receiver contains no defectives}\}$

$$P(A) = \frac{\binom{37}{5}}{\binom{40}{5}} = \frac{35 \cdot 34 \cdot 33}{40 \cdot 39 \cdot 38} = .6624$$

4. (9pts) Suppose X has a discrete distribution with pmf given by

x	1	2	3	4	5	6
$p_X(x)$	1/3	1/6	1/12	1/12	1/6	1/6

a) What is $\mathcal{R}(X)$? **Sol:** $\mathcal{R}(X) = \{1, 2, 3, 4, 5, 6\}$

b) What is $P\{X \geq 4\}$? **Sol:** $P\{X \geq 4\} = \frac{1}{12} + \frac{1}{6} + \frac{1}{6} = \frac{5}{12}$

c) What is $E[X]$? **Sol:**

$$E[X] = 1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{12} + 4 \cdot \frac{1}{12} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{37}{12}$$

5. (9pts) Suppose Z has a continuous distribution with pdf given by

$$f_Z(z) = \begin{cases} \frac{2}{9}z(3-z) & 0 \leq z \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

a) What is $\mathcal{R}(Z)$? **Sol:** $\mathcal{R}(Z) = (0, 3)$

b) What is $P\{Z > 1\}$? **Sol:**

$$P\{Z > 1\} = \int_1^3 \frac{2}{9}z(3-z)dz = \left(\frac{1}{3}z^2 - \frac{2}{27}z^3\right)\Big|_1^3 = \frac{20}{27}$$

c) What is $E[Z^2]$? **Sol:**

$$E[Z^2] = \int_0^3 z^2 \frac{2}{9}z(3-z)dz = \left(\frac{2}{12}z^4 - \frac{2}{45}z^5\right)\Big|_0^3 = \frac{27}{10}$$

6. (8pts) Appropriately diluted and sprayed, on the average, two gallons of Killlem Herbicide will cover one acre of a corn field. From past experience, it is known that the standard deviation for the area covered by a gallon is .2 acre. What, approximately, is the probability that 170 gallons of Killlem will cover an 80 acre corn field?

Sol: Let X_i = area covered by the i th gallon. $E[X_i] = .5$, $\sigma_X = .2$.

$$\begin{aligned} P\{80 \leq \sum_{i=1}^{170} X_i\} &= P\left\{\frac{80 - 85}{.2\sqrt{170}} \leq \frac{\sum_{i=1}^{170} X_i - 170 \cdot .5}{.2\sqrt{170}}\right\} \\ &\approx 1 - \Phi(-1.917) = \Phi(1.917) = .9724 \end{aligned}$$

Part II. Each of the following problems will be graded on the basis of 20 points. Your score on this part of the exam will be the sum of your 4 highest scores plus one half of the points you receive on the other two problems. Show all of your work in the bluebook. Make certain that you do four problems well!

1. Suppose that the lifetime, measured in hours, of an electronic component on board a ship is exponentially distributed with parameter .1. How many spare components should the ship carry to ensure that the components will operate for 350 hours with probability at least .9?

Sol: n = number of components to be carried, X_i lifetime of the i th component, $E[X_i] = 10$, $Var(X_i) = 100$.

$$\begin{aligned} .9 &\approx P\{350 \leq \sum_{i=1}^n X_i\} \\ &= P\left\{\frac{350 - n10}{10\sqrt{n}} \leq \frac{\sum_{i=1}^n X_i - nE[X]}{\sqrt{Var(X)}\sqrt{n}}\right\} \\ &\approx 1 - \Phi\left(\frac{350 - n10}{10\sqrt{n}}\right) = \Phi\left(\frac{n10 - 350}{10\sqrt{n}}\right) \end{aligned}$$

Therefore we want

$$\frac{n10 - 350}{10\sqrt{n}} \approx 1.28$$

Solving for \sqrt{n} gives $\sqrt{n} = 6.59$, so $n = 44$.

2. An insurance company classifies policy holders as good risks, average risks, or bad risks, with 20% of the population being good risks, 50% being average risks, and 30% being bad risks. Experience shows that 5% of good risk policy holders, 15% of average risk policy holders, and 30% of bad risk policy holders are involved in at least one accident over a one year time period.

Sol: $A_1 = \{\text{policy holder is a good risk}\}$, $A_2 = \{\text{policy holder is an average risk}\}$, $A_3 = \{\text{policy holder is a bad risk}\}$. $B = \{\text{policy holder has an accident}\}$.

$$P(A_1) = .2, \quad P(A_2) = .5, \quad P(A_3) = .3, \quad P(B|A_1) = .05, \quad P(B|A_2) = .15, \quad P(B|A_3) = .3$$

- a) What fraction of the population is involved in at least one accident in a one year time period? **Sol:**

$$\begin{aligned} P(B) &= P(B \cap A_1) + P(B \cap A_2) + P(B \cap A_3) \\ &= P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + P(B|A_3)P(A_3) \\ &= .175 \end{aligned}$$

- b) Suppose that a randomly selected policy holder had no accident in 1990. What is the probability that he or she is a good risk? **Sol:** $P(B^c) = 1 - P(B) = .825$

$$P(A_1|B^c) = \frac{P(B^c|A_1)P(A_1)}{P(B^c)} = \frac{.95 \times .2}{.825} = \frac{190}{825} = \frac{38}{165}$$

3. Suppose that the sides of a rectangle X and Y are random variables with joint density function

$$f_{XY}(x, y) = \begin{cases} \frac{1}{3}(x + y) & 1 \leq x \leq 2, 1 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

- a) What is the area of the rectangle? **Sol:** Area = length \times width = XY .

- b) What is the probability that the area of the rectangle is greater than 2? **Sol:**

$$P\{XY > 2\} = \int_1^2 \int_{\frac{2}{x}}^2 \frac{1}{3}(x + y) dy dx = \frac{2}{3}$$

4. A bucket contains 2 green balls and 3 red balls. Suppose that the balls are drawn one at a time without replacement. Let X_1 be the draw on which a green ball is first chosen and let X_2 be the draw on which the remaining green ball is chosen.

a) What is $\mathcal{R}(X_1)$? **Sol:** $\mathcal{R}(X_1) = \{1, 2, 3, 4\}$

b) What is $\mathcal{R}(X_2)$? **Sol:** $\mathcal{R}(X_2) = \{2, 3, 4, 5\}$

c) What is the pmf for X_1 ? **Sol:** $\#S = \binom{5}{2} = 10$ (select the two draws on which green balls will be obtained)

x	1	2	3	4
$p_{X_1}(x)$	$\frac{4}{10}$	$\frac{3}{10}$	$\frac{2}{10}$	$\frac{1}{10}$

d) What is the joint pmf for X_1 and X_2 ?

Sol: For $1 \leq k < l \leq 5$,

$$P\{X_1 = k, X_2 = l\} = \frac{1}{\binom{5}{2}}$$

5. A nondestructive testing method for detecting internal cracks in a casting gives a positive response with probability $\frac{x}{10}$ where x is the total length of the internal cracks in centimeters. Suppose that the total length of the cracks in a casting is uniformly distributed on the interval $[0, 10]$ and that if a casting of known total crack length is tested twice, the results of the tests are independent.

Sol: Let X be the total length of the cracks in the casting. $A_k = \{k\text{th test is positive}\}$

$$f_X(x) = \begin{cases} \frac{1}{10} & 0 \leq x \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

$$P(A_k|X = x) = \frac{x}{10}$$

a) If a casting is selected at random and tested, what is the probability that the test will give a positive response? **Sol:**

$$P(A_1) = \int_{-\infty}^{\infty} P(A_1|X = x) f_X(x) dx = \int_0^{10} \frac{x}{10} \frac{1}{10} dx = \frac{1}{2}$$

b) If a casting is tested once and gives a positive response, what is the probability that the casting has total crack length of over 5? **Sol:**

$$P(X > 5|A_1) = \frac{P(\{X > 5\} \cap A_1)}{P(A_1)} = \frac{\int_5^{10} \frac{x}{10} \frac{1}{10} dx}{\frac{1}{2}} = \frac{3}{4}$$

c) If a casting is tested twice and gives a positive response both times, what is the probability that the casting has total crack length of over 5?

Sol: Since $P(A_1 \cap A_2|X = x) = \frac{x}{10} \frac{x}{10}$,

$$P(X > 5|A_1 \cap A_2) = \frac{P(\{X > 5\} \cap A_1 \cap A_2)}{P(A_1 \cap A_2)} = \frac{\int_5^{10} \frac{x}{10} \frac{x}{10} \frac{1}{10} dx}{\int_0^{10} \frac{x}{10} \frac{x}{10} \frac{1}{10} dx} = \frac{7}{8}$$

6. The number of defects in a computer chip has the distribution

x	0	1	2	3
$p_X(x)$	1/2	1/4	1/8	1/8

The probability that a defect can be repaired is $2/3$. (Assume that success of repair of different defects is independent.) A chip is discarded unless all defects are repaired. What is the probability that a chip will be discarded?

Sol: Let $D = \{\text{chip is discarded}\}$, and let $X = \text{number of defects in the chip}$. Note that

$$P(D^c|X = k) = \left(\frac{2}{3}\right)^k$$

so

$$P(D^c) = \sum_{k=0}^3 P(D^c|X = k)P(X = k) = \frac{1}{2} + \frac{2}{3} \times \frac{1}{4} + \left(\frac{2}{3}\right)^2 \times \frac{1}{8} + \left(\frac{2}{3}\right)^3 \times \frac{1}{8} = \frac{41}{54}$$

and hence

$$P(D) = 1 - P(D^c) = \frac{13}{54}$$