1. (12 pts) Suppose that the joint pmf for X and Y is

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>(\frac{1}{16})</td>
<td>(\frac{1}{16})</td>
<td>(\frac{1}{16})</td>
<td>(\frac{1}{16})</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>(\frac{7}{12})</td>
<td>(\frac{7}{12})</td>
<td>(\frac{1}{12})</td>
<td>(\frac{1}{12})</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>(\frac{1}{8})</td>
<td>(\frac{1}{8})</td>
<td>(\frac{1}{8})</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(\frac{1}{3})</td>
<td>(\frac{1}{3})</td>
</tr>
</tbody>
</table>

(a) Calculate the probability mass functions for X and for Y.

\[ \mathcal{R}(X) = \{0, 1, 2, 3\} \]

\[
\begin{array}{c|cccc}
   x & 0 & 1 & 2 & 3 \\
\hline
   p_X(x) & \frac{1}{6} & \frac{1}{3} & \frac{1}{6} & \frac{1}{3} \\
\end{array}
\]

\[ \mathcal{R}(Y) = \{0, 1, 2, 3\} \]

\[
\begin{array}{c|cccc}
   y & 0 & 1 & 2 & 3 \\
\hline
   p_Y(y) & \frac{1}{16} & \frac{1}{32} & \frac{1}{16} & \frac{1}{32} \\
\end{array}
\]

(b) Calculate \(P\{Y > X | Y \geq 2\}\).

\[
P\{Y > X | Y \geq 2\} = \frac{P\{Y > X, Y \geq 2\}}{P\{Y \geq 2\}} = \frac{5/12}{19/24} = \frac{10}{19}
\]

2. (10 pts) Suppose that \(X_1, \ldots, X_{144}\) are independent and identically distributed with \(E[X_k] = 2\) and \(Var(X_k) = 9\). Use the central limit theorem to approximate \(P\{\sum_{k=1}^{144} X_k \leq 360\}\).

\[
P\left\{ \sum_{k=1}^{144} X_k \leq 360 \right\} = P\left\{ \frac{\sum_{k=1}^{144} X_k - 144\mu}{\sqrt{144\sigma}} \leq \frac{360 - 288}{12 \times 3} \right\} 
\approx \Phi(2) = .9772
3. (10 pts) Let $X$ have a Gamma density

$$f_X(x) = \begin{cases} 
4x^2e^{-2x} & x \geq 0 \\
0 & \text{otherwise},
\end{cases}$$

and let the conditional density of $Y$ given $X$ be

$$f_{Y|X}(y|x) = \begin{cases} 
\frac{2y}{x^2} & 0 \leq y \leq x \\
0 & \text{otherwise}
\end{cases}$$

Calculate the pdf for $Y$.

$$f_{XY}(x, y) = f_{Y|X}(y|x)f_X(x) = \begin{cases} 
8ye^{-2x} & 0 \leq y \leq x \\
0 & \text{otherwise}
\end{cases}$$

For $y \geq 0$,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)dx = \int_{y}^{\infty} 8ye^{-2x} dx = 4ye^{-2y}$$

so

$$f_Y(y) = \begin{cases} 
4ye^{-2y} & y \geq 0 \\
0 & \text{otherwise}
\end{cases}$$
Name: ____________________________

4. (6 pts) Let $Y_1, \ldots, Y_5$ be random variables with $E[Y_k] = (k - 1)$. Let $Z = \sum_{k=1}^{5} Y_k$. Compute $E[Z]$.

$$E[Z] = \sum_{k=1}^{5} E[Y_k] = \sum_{k=1}^{5} (k - 1) = 10$$

5. (10pts) Let $N(t)$ be a Poisson process with parameter 4, and let $S_1, S_2, \ldots$ be the corresponding arrival times.

(a) Give a statement in terms of $N$ that is equivalent to the statement $S_3 < 2.5 \leq S_4$.

(SHOULD HAVE BEEN $S_3 \leq 2.5 < S_4$.)

$$\{S_3 \leq 2.5 < S_4\} = \{N(2.5) = 3\}$$

(b) Calculate $P\{N(0.5) = 3, N(1.5) = 5\}$. (Simplify the answers as far as you can.)

$$P\{N(0.5) = 3, N(1.5) = 5\} = P\{N(0.5) = 3, N(1.5) - N(0.5) = 2\} = P\{N(0.5) = 3\} P\{N(1.5) - N(0.5) = 2\} = e^{-0.5 \times 4} \left(0.5 \times 4\right)^3 3! e^{-1 \times 4} \left(1 \times 4\right)^2 2! = \frac{32}{3} e^{-6}$$
6. (12 pts) Jane drops Sam at the Coliseum box office to stand in line for tickets to a rock concert. There are 99 people in line ahead of Sam. If the amount of time it takes the box office to serve a customer averages 3 minutes with a standard deviation of 2, how long should Jane wait before coming back to get Sam if she wants the probability that he is ready to leave (that is, he has been served) to be at least .96?

\[ X_k = \text{time to serve the } k\text{th customer} \]
\[ T = \text{time Jane should wait before coming back} \]

\[
P\{\sum_{k=1}^{100} X_k \leq T\} = P\left\{ \frac{\sum_{k=1}^{100} X_k - 100\mu}{\sqrt{100}\sigma} \leq \frac{T - 300}{10 \times 2} \right\} \\
\approx \Phi\left( \frac{T - 300}{20} \right) = .96
\]

Therefore

\[
\frac{T - 300}{20} \approx 1.75
\]

and \( T = 335 \).
7. (8 pts) Suppose $E[X] = 3$, $E[Y] = 1$, $Var(X) = 5$, $Var(Y) = 2$, and $Cov(X,Y) = -1$. Let $Z = 3X - 2Y$. Calculate $E[Z]$ and $Var(Z)$.

$$V(Z) = 3^2 Var(X) + 2 \times 3 \times (-2) Cov(X,Y) + (-2)^2 Var(Y) = 65$$

8. (12 pts) 6 first graders, 6 second graders, and 6 third graders are seated randomly in a row. Let $N$ be the number of first graders who are seated next to both a second grader and a third grader. What is $E[N]$?

Let $X_k = \begin{cases} 1 & \text{kth first grader seated next to a 2nd and a 3rd grader} \\ 0 & \text{otherwise} \end{cases}$

Select seat for 1st grader, then select the person to his/her left, then select the person to his/her right:

$$P\{X_k = 1\} = \frac{16 \times 12 \times 6}{18 \times 17 \times 16} = \frac{4}{17}$$

so

$$E[N] = \frac{24}{17}$$

Alternatively

$$Y_i = \begin{cases} 1 & \text{if the i-th seat is occupied by a first grader} \\ 0 & \text{otherwise} \end{cases}$$

for $i = 2, \ldots, 17$.

$$P\{Y_i = 1\} = \frac{6 \times \binom{6}{1} \binom{6}{1}}{18 \times 17 \times 16/2} = \frac{3}{34}$$

so

$$E[N] = 16 \times \frac{3}{34} = \frac{24}{17}$$