1. A coin is flipped three times. Let

\[ A = \{ \text{first flip is heads} \} \]
\[ B = \{ \text{second flip is heads} \} \]
\[ C = \{ \text{third flip is heads} \} \]

Using unions, intersections, complements, etc., express each of the following events in terms of A, B, and C.

\[ D = A^c \cap B^c \cap C^c \]
\[ E = \{ \text{there are more heads than tails} \} = (A \cap B) \cup (A \cap C) \cup (B \cap C) \]

2. Suppose \( P(A) = .4 \), \( P(B) = .3 \), \( P(A \cup B) = .5 \). Calculate \( P(A \cap B) \).

\[ P(A \cap B) = P(A) + P(B) - P(A \cup B) = .2 \]
1. A coin is flipped three times. Let

\[ A = \{ \text{first flip is heads} \} \]
\[ B = \{ \text{second flip is heads} \} \]
\[ C = \{ \text{third flip is heads} \} \]

Using unions, intersections, complements, etc., express each of the following events in terms of A, B, and C.

\[ D = \{ \text{all flips are heads} \} \]
\[ E = \{ \text{the second flip is different from the other two} \} \]

2. Suppose \( P(A) = .6, P(B) = .2, P(A \cup B) = .7 \). Calculate \( P(A \cap B) \).
An urn contains 4 red balls, 2 white balls, and 1 black ball. The balls are drawn one at a time until all are drawn.

1. What is the probability that the last ball drawn is white?
   \[ P\{\text{last ball drawn is white}\} = \frac{2}{7} \]

2. What is the probability that the first three balls drawn are red?
   \[ P\{\text{first three balls drawn are red}\} = \frac{4 \times 3 \times 2}{7 \times 6 \times 5} = \frac{4}{35} \]

3. What is the probability that a white ball is drawn before a red ball?
   \[ A = \{\text{white ball drawn before a red ball}\} \]
   \[ B = \{\text{first ball drawn is white}\} \]
   \[ C = \{\text{first ball drawn is black and second is white}\} \]
   Then
   \[ P(A) = P(B \cup C) = P(B) + P(C) = \frac{2}{7} + \frac{1 \times 2}{7 \times 6} = \frac{1}{3} \]
An urn contains 2 red balls, 3 white balls, and 1 black ball. The balls are drawn one at a time until all are drawn.

1. What is the probability that the last ball drawn is white?

2. What is the probability that the first three balls drawn are white?

3. What is the probability that a white ball is drawn before a red ball?
1. Suppose $P(C) = .4$ and $P(D) = .2$.

   (a) If $C$ and $D$ are disjoint, what is $P(C \cup D)$?
   
   $P(C \cup D) = P(C) + P(D) = .6$

   (b) If $C$ and $D$ are independent, what is $P(C \cup D)$?
   
   $P(C \cup D) = P(C) + P(D) - P(C \cap D) = .6 - .08 = .52$

2. A particular kind of tomato plant is imported from outside the country. 10% of the imported plants carry a certain fungus. The imported tomato plants are held in quarantine for 10 days. If a plant carries the fungus, it will be detectable within the quarantine period with probability .95. If the plant does not carry the fungus, then no fungus will be detected. (Clearly define all events that you use and interpret all numbers that you use in terms of the events that you define.)

   $A = \{\text{plant carries fungus}\}$
   $B = \{\text{plant passes inspection}\}$

   $P(A) = .1 \quad P(A^c) = .9 \quad P(B|A) = .05 \quad P(B|A^c) = 1$

   (a) What is the probability that a plant passes the inspection (that is, no fungus is detected during the quarantine period)?
   
   $P(B) = P(B \cap A) + P(B \cap A^c) = P(B|A)P(A) + P(B|A^c)P(A^c) = .05 \times .1 + .9 = .905$

   (b) If the fungus is not detected within the quarantine period, what is the probability that the plant carries the fungus?

   $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{.005}{.905} = \frac{1}{181}$. 
1. Suppose $P(A) = .3$ and $P(B) = .5$.

   (a) If $A$ and $B$ are disjoint, what is $P(A \cup B)$?

   (b) If $A$ and $B$ are independent, what is $P(A \cup B)$?

2. A particular kind of tomato plant is imported from outside the country. 20% of the imported plants carry a certain fungus. The imported tomato plants are held in quarantine for 10 days. If a plant carries the fungus, it will be detectable within the quarantine period with probability .99. 10% of plants that do not carry the fungus still exhibit symptoms that cause them to fail the inspection. (Clearly define all events that you use and interpret all numbers that you use in terms of the events that you define.)

   (a) What is the probability that a plant passes the inspection?

   (b) If the plant passes the inspection, what is the probability that the plant carries the fungus?
Let $X$ have probability mass function

<table>
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<th>4</th>
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</thead>
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<td>$p_X(x)$</td>
<td>2/5</td>
<td>1/5</td>
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</tbody>
</table>

1. What is $\mathcal{R}(X)$?
   $\mathcal{R}(X) = \{1, 2, 3, 4\}$

2. Calculate $P\{X \geq 2\}$.
   
   $P\{X \geq 2\} = \frac{3}{5}$

   
   $E[X] = 1 \times \frac{2}{5} + 2 \times \frac{1}{5} + 3 \times \frac{1}{5} + 4 \times \frac{1}{5} = \frac{11}{5}$

   
   $E[(X - 2)^2] = 1 \times \frac{2}{5} + 0 \times \frac{1}{5} + 1 \times \frac{1}{5} + 4 \times \frac{1}{5} = \frac{7}{5}$

5. Calculate $P\{X \leq 3|X \geq 2\}$.
   
   $P\{X \leq 3|X \geq 2\} = \frac{P\{2 \leq X \leq 3\}}{P\{X \geq 2\}} = \frac{2}{3}$. 
Miniquiz, February 17, 2005

Let $Z$ have probability mass function

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<td>$\frac{1}{5}$</td>
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</tbody>
</table>

1. What is $\mathcal{R}(Z)$?

2. Calculate $P\{Z \geq 2\}$.


5. Calculate $P\{Z \leq 2 | Z \geq 2\}$.
Suppose $X$ is a random variable with probability density function

$$f_X(x) = \begin{cases} \frac{1}{9}x^2 & 0 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

1. What is the range of $X$?

$\mathcal{R}(X) = [0, 3]$

2. Calculate $P\{1 < X < 2\}$.

$$P\{1 < X < 2\} = \int_1^2 \frac{1}{9}x^2 \, dx = \left[\frac{1}{27}x^3\right]_1^2 = \frac{8}{27} - \frac{1}{27} = \frac{7}{27}$$


$$E[X] = \int_0^3 \frac{1}{9}x^2 \, dx = \left[\frac{1}{36}x^4\right]_0^3 = \frac{9}{4}$$


$$E[X^3] = \int_0^3 \frac{1}{9}x^5 \, dx = \left[\frac{1}{54}x^6\right]_0^3 = \frac{27}{2}$$

5. Calculate $P\{1 < X < 2|X > 1\}$.

$$P\{1 < X < 2|X > 1\} = \frac{P\{1 < X < 2\}}{P\{X > 1\}} = \frac{7/27}{26/27} = \frac{7}{27}$$
Miniquiz, February 24, 2005

Suppose $Z$ is a random variable with probability density function

$$f_Z(z) = \begin{cases} \frac{1}{4}z^3 & 0 \leq z \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

1. What is the range of $Z$?

2. Calculate $P\{0.5 < Z \leq 1\}$.


5. Calculate $P\{0.5 < Z \leq 1|Z \leq 1\}$.
Suppose $Z$ is a random variable with probability density function

$$f_Z(z) = \begin{cases} \frac{1}{4}z^3 & 0 \leq z \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

1. Calculated the cumulative distribution function for $Z$?

$$F_Z(z) = \begin{cases} 0 & z < 2 \\ \int_0^z \frac{1}{4}x^3 \, dx & 0 \leq z \leq 2 \\ 1 & z > 2 \end{cases} = \begin{cases} 0 & z < 2 \\ \frac{1}{16}z^4 & 0 \leq z \leq 2 \\ 1 & z > 2 \end{cases}$$

2. Let $Y = Z^2$. Calculate the probability density function for $Y$.

$\mathcal{R}(Y) = [0, 4]$

For $0 \leq y \leq 4$,

$$f_Y(y) = P\{Y \leq y\} = P\{Z^2 \leq y\} = P\{Z \leq \sqrt{y}\} = \frac{1}{16}y^2$$

$$f_Y(y) = \begin{cases} \frac{1}{8}y & 0 \leq y \leq 4 \\ 0 & \text{otherwise} \end{cases}$$
Suppose $X$ is a random variable with probability density function

$$f_X(x) = \begin{cases} \frac{1}{5}x^2 & 0 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

1. Calculate the cumulative distribution function for $X$.

2. Let $Y = X^3$. Calculate the probability density function for $Y$. 
If the cross sectional area of a buried metallic object is $y$ cm$^2$, then the reading on a metal detector is distributed according to the conditional density

$$f_{X|Y}(x|y) = \begin{cases} \frac{2x}{y^2} & 0 \leq x \leq y \\ 0 & \text{otherwise} \end{cases}$$

An archeologist is seeking a buried urn. Depending on the orientation of the urn, the cross sectional area is uniformly distributed between 80 cm$^2$ and 100 cm$^2$. In order to avoid digging too many unnecessary holes, the archeologist only digs if the reading on the detector is greater than 60. What is the probability that the archeologist finds the urn?

**Solution:** Let $Y$ be the cross sectional area of the urn and $X$ the reading on the metal detector. The intention of the problem was to compute $P\{X > 60\}$. Since

$$f_{X,Y}(x,y) = f_{X|Y}(x|y)f_Y(y) = \begin{cases} \frac{x}{10y^2} & 0 \leq x \leq y, \ 80 \leq y \leq 100 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$P\{X > 60\} = \int_{80}^{100} \int_{60}^{y} \frac{x}{10y^2} dx \ dy = \frac{11}{20}$$
If the cross sectional area of a buried metallic object is \( x \text{ cm}^2 \), then the reading on a metal detector is distributed according to the conditional density

\[
f_{Y|X}(y|x) = \begin{cases} \frac{3y^2}{x^3} & 0 \leq y \leq x \\ 0 & \text{otherwise} \end{cases}
\]

An archeologist is seeking a buried urn. Depending on the orientation of the urn, the cross sectional area is uniformly distributed between 80 cm\(^2\) and 100 cm\(^2\). In order to avoid digging too many unnecessary holes, the archeologist only digs if the reading on the detector is greater than 50. What is the probability that the archeologist finds the urn?

**Solution:** Let \( X \) be the cross sectional area of the urn and \( Y \) the reading on the metal detector. The intention of the problem was to compute \( P\{Y > 50\} \). Since

\[
f_{XY}(x,y) = f_{Y|X}(y|x)f_X(x) = \begin{cases} \frac{3y^2}{20x^3} & 0 \leq y \leq x, \ 80 \leq x \leq 100 \\ 0 & \text{otherwise} \end{cases}
\]

Then

\[
P\{Y > 50\} = \int_{80}^{100} \int_{50}^{x} \frac{3y^2}{20x^3} dy \ dx = \frac{211}{256} = .824
\]
30 balls numbered 1 through 30 are drawn one at a time. For $1 < k < 30$, we say that a *local maximum* occurs on draw $k$ if the number obtained on draw $k$ is larger than the numbers obtained on draws $k−1$ and $k+1$. A local maximum occurs at $k = 1$ if the number obtained on draw 1 is larger than the number on draw 2, and a local maximum occurs at $k = 30$ if the number obtained on draw 30 is larger than the number obtained on draw 29. Let $N$ be the number of local maxima in the thirty draws.

1. What is the range of $N$?
2. What is $E[N]$?

**Solution:**

$\mathcal{R}(N) = \{1, \ldots, 15\}$

Let $X_k = 1$ if there is a local maximum at $k$, $X_k = 0$ otherwise.

$P\{X_1 = 1\} = 1/2$, $P\{X_{30} = 1\} = 1/2$, $P\{X_k = 1\} = 1/3$ for $1 < k < 30$. Therefore

$$E[N] = \sum_{k=1}^{30} E[X_k] = 2 \times \frac{1}{2} + 28 \times \frac{1}{3} = \frac{31}{3}$$
Miniquiz, April 7, 2005

Cards are drawn from a standard deck (52 cards in 4 suits) one at a time without replacement. A match occurs if a card has the same value as the card drawn just before it. Let $N$ be the number of matches in the 52 draws.

1. What is the range of $N$?

2. What is $E[N]$?

$\mathcal{R}(N) = \{0, \ldots, 39\}$

Let $X_k = 1$ if there is a match on the $k$th draw and $X_k = 0$ otherwise. Then $P\{X_k = 1\} = \frac{3}{51}$ and

$$E[N] = \sum_{k=2}^{52} E[X_k] = 51 \times \frac{3}{51} = 3$$