Part I. Do all problems in the space provided. Clearly define all random variables and other notation that you use, and clearly specify what you are calculating at each step in a calculation. Justify all answers. Clearly mark your answers and explicitly state what it is you have computed. (Don't just put a number in a box. If you are computing $E[X]$ and the answer is 4.5, write $E[X] = 4.5$, not just 4.5.)

1. (6 pts)Suppose that X and Y satisfy $E[X] = 1$, $E[Y] = 2$, $Var(X) = 1$, $Var(Y) = 2$, $Cov(X, Y) = -1$. Let $U = X + Y$. Calculate $E[U]$ and $Var(U)$.

   Solution: $E[U] = E[X] + E[Y] = 3$, $Var(U) = Var(X) + 2Cov(X, Y) + Var)Y) = 1$

2. (9 pts) From past experience, a statistics professor knows that it takes an average of 15 minutes to grade an exam with a standard deviation of 5 minutes. If 60 students take the exam, what (approximately) is the probability that it will take more than 16 hours to grade?

   Solution: X_k = time it takes to grade kth student's exam (in minutes)

$$\begin{aligned} P\{\sum_{k=1}^{60} X_k > 960\} &= P\{\frac{\sum_{k=1}^{60} X_k - 60\mu}{\sqrt{60}\sigma} > \frac{960 - 900}{\sqrt{60} \times 5}\} \\ &\approx 1 - \Phi(1.549) = 1 - .9394 = .0606 \end{aligned}$$

3. (6 pts) Let $Y_1, \ldots, Y_6$ be random variables such that $E[Y_k] = (k - 1)$. Let $Z = \sum_{k=1}^{6} Y_k$. Compute $E[Z]$.

   Solution: $E[Z] = \sum_{k=1}^{6} E[X_k] = 15$

4. (9 pts) Suppose the the joint pmf for X and Y is

x \ y	0	1	2	3
0	.1	.05	.05	.05
1	.05	.1	.05	.05
2	.05	.05	.1	.05
3	.05	.05	.05	.1

   (a) Calculate $P\{Y > X\}$. $P\{Y > X\} = .3$

   (b) Calculate $E[XY]$. $.1(1 + 4 + 9) + .05(2 + 3 + 6 + 6 + 3 + 2) = 1.4 + 1.1 = 2.5$

   (c) Calculate $P\{Y > X|X = 2\}$. $\frac{.05}{.25} = \frac{1}{5}$

Part II.

1. Suppose that X and Y are random variables with joint density function

$$f_{XY}(x,y) = \begin{cases} \frac{1}{4}(x+y) & 0 \le x \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Calculate $P\{Y < 2X\}$?

$$P\{Y \ge 2X\} = \int_0^2 \int_0^{\frac{1}{2}y} \frac{1}{4}(x+y)dx\, dy = \int_0^2 \left(\frac{1}{8}x^2 + yx \Big|_0^{y/2} \right) dy = \int_0^2 \frac{5}{32} y^2 dy = \frac{5}{12}$$

   so

$$P\{Y < 2X\} = 1 - \frac{5}{12} = \frac{7}{12}.$$

   (b) Calculate $f_Y(y)$? For $0 \le y \le 2$

$$f_Y(y) = \int_0^y \frac{1}{4}(x+y)dx = \frac{3}{8}y^2$$

   so

$$f_Y(y) = \begin{cases} \frac{3}{8}y^2 & 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

   (c) If $Y = 1$, what is the probability that X is less than .5?

$$f_{X|Y}(x|y) = \begin{cases} \frac{2}{3}\frac{x+y}{y^2} & 0 \le x \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

   so

$$\int_0^{.5} \frac{2}{3}(x+1)dx = \frac{2}{3}(\frac{1}{2}x^2 + x)\Big|_0^{.5} = \frac{5}{12}$$

2. On average a can of paint will cover 500 square feet with a standard deviation of 50 square feet. Sam has 40 cans of green paint. How large an area can he contract to paint and be able to fulfill the contract with probability at least .97 without buying any more paint?

   Solution: Let X_k be the coverage by the kth gallon and let a be the area contracted. We want

$$P\{\sum_{k=1}^{40} X_k \ge a\} \ge .97$$

$$\begin{aligned} P\{\sum_{k=1}^{40} X_k \ge a\} &= P\{\frac{\sum_{k=1}^{40} X_k - 40\mu}{\sqrt{40}\sigma} \ge \frac{a - 40 \times 500}{\sqrt{40} \times 50}\} \\ &\approx 1 - \Phi(\frac{a - 40 \times 500}{\sqrt{40} \times 50}) = .97 \end{aligned}$$

   so

$$\frac{a - 40 \times 500}{\sqrt{40} \times 50} = -1.88$$

   and

$$a = 20,000 - 188 \times \sqrt{10} = 19405.5$$

3. A group of 30 men and 20 women are seated randomly in a row of 50 chairs. Calculate the expected number of men who have a woman sitting next to them.

Solution: Let $N =$ be the number of men who have a woman sitting next to them. Let $X_k = 1$ if the kth man sits next to a women and $X_k = 0$ otherwise. Let $B_k = \{k\text{th man sits on end}\}$

$$P\{X_k = 1\} = P(X_k = 1|B_k)P(B_k) + P(X_k = 1|B_k^c)P(B_k^c) = \frac{20}{49} \times \frac{2}{50} + (1 - \frac{29 \times 28}{49 \times 48}) \times \frac{48}{50}$$

so

$$E[N] = \sum_{k=1}^{30} E[X_k] = 30 \times .645 = 19.3$$

Alternatively, let $Y_k = 1$ if the kth seat is occupied by a man and there is a woman sitting in at least one of the adjoining seats, and let $Y_k = 0$ otherwise. Let $A_k = \{k\text{th seat is occupied by a man}\}$ Then

$$P\{Y_1 = 1\} = P(A_1 \cap A_2^c) = P\{Y_{50} = 1\} = P(A_{50} \cap A_{49}^c) = \frac{30 \times 20}{50 \times 49}$$

and for $1 < k < 50$,

$$\begin{aligned} P\{Y_k = 1\} &= P(A_k \cap (A_{k-1}^c \cup A_{k+1}^c)) \\ &= P(A_{k-1}^c \cup A_{k+1}^c | A_k)P(A_k) \\ &= (P(A_{k-1}^c|A_k) + P(A_{k+1}^c|A_k) - P(A_{k-1}^c \cap A_{k+1}^c|A_k))P(A_k) \\ &= (\frac{20}{49} + \frac{20}{49} - \frac{20 \times 19}{49 \times 48})\frac{30}{50} \end{aligned}$$

4. The length of time the battery in a smoke detector lasts (measured in days) is exponentially distributed with parameter $\frac{1}{400}$. Once the battery is dead, there is a delay before someone notices and the battery is replaced. Suppose that the length of the delay (in days) until someone notices is exponentially distributed with parameter $\frac{1}{20}$.

   (a) What is the distribution of the total time after a new battery is installed that a dead battery is discovered? (Assume that the lifetime of the battery is independent of the delay in discovery.)

   Solution: Let $X =$ lifetime of battery, $Y =$ length of time after batteries dies that someone notices $Z = X + Y =$ time that the dead battery is discovered.

$$f_{XY}(x, y) = \begin{cases} \frac{1}{8000}e^{-\frac{1}{400}x - \frac{1}{20}y} & x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

   For $z \geq 0$,

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(z - y, y)dy = \frac{1}{8000}e^{-\frac{1}{400}z}\int_0^z e^{-\frac{19}{400}y}dy$$

   so

$$f_Z(z) = \begin{cases} \frac{1}{380}e^{-\frac{1}{400}z}(1 - e^{-\frac{19}{400}z}) & z \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

   Alternatively, you can compute the cdf for Z directly from f_{XY}. $\mathcal{R}(Z) = [0, \infty)$ and for $z \geq 0$,

$$\begin{aligned} F_Z(z) &= P\{Z \leq z\} = P\{X + Y \leq z\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{z-y} f_{XY}(x, y)dxdy \end{aligned}$$

$$
\begin{aligned}
&= \int_0^z \int_0^{z-y} \frac{1}{8000} e^{-\frac{1}{400}x-\frac{1}{20}y} dxdy \\
&= \int_0^z \left(-\frac{1}{20} e^{-\frac{1}{400}x-\frac{1}{20}y} \Big|_{x=0}^{x=z-y} \right) dy \\
&= \int_0^z \frac{1}{20} \left(e^{-\frac{1}{20}y} - e^{-\frac{1}{400}z-\frac{19}{400}y} \right) dy \\
&= -e^{-\frac{1}{20}y} \Big|_{y=0}^{y=z} + \frac{20}{19} e^{-\frac{1}{400}z-\frac{19}{400}y} \Big|_{y=0}^{y=z} \\
&= 1 + \frac{1}{19} e^{-\frac{1}{20}z} - \frac{20}{19} e^{-\frac{1}{400}z}
\end{aligned}
$$

Consequently,

$$
F_Z(z) = \begin{cases} 1 + \frac{1}{19} e^{-\frac{1}{20}z} - \frac{20}{19} e^{-\frac{1}{400}z} & z \geq 0 \\ 0 & z < 0 \end{cases}
$$

(b) 500 days after the battery is installed, someone notices that the battery is dead. What is the probability that the battery has been dead for more than 30 days?

Solution: Since for $0 \leq y \leq z$

$$
P\{Z \leq z, Y \leq y\} = P\{X + Y \leq z, Y \leq y\} = \int_0^y \int_0^{z-u} f_{XY}(x, u) dx du,
$$

$$
f_{Z,Y}(z, y) = \begin{cases} \frac{1}{400} e^{-\frac{1}{400}(z-y)} \frac{1}{20} e^{-\frac{1}{20}y} & 0 \leq y \leq z \\ 0 & \text{otherwise} \end{cases}
$$

and

$$
f_{Y|Z}(y|z) = \begin{cases} \frac{19}{400} \frac{e^{-\frac{19}{400}y}}{1-e^{-\frac{19}{400}z}} & 0 \leq y \leq z \\ 0 & \text{otherwise} \end{cases}
$$

$$
P\{Y > 30 | Z = 500\} = \frac{e^{-\frac{19}{400}30} - e^{-\frac{19}{400}500}}{1 - e^{-\frac{19}{20}500}}
$$

5. Let N be a Poisson process with parameter $\lambda = 0.5$. Calculate the following:

(a)

$$
\begin{aligned}
P\{S_1 \leq 2.2, S_3 > 3.0\} &= P\{N(2.2) \geq 1, N(3) \leq 2\} \\
&= P\{N(2.2) = 1, N(3) - N(2.2) = 0\} \\
&\quad + P\{N(2.2) = 1, N(3) - N(2.2) = 1\} \\
&\quad + P\{N(2.2) = 2, N(3) - N(2.2) = 0\} \\
&= e^{-1.1} 1.1 \times e^{-.4} + e^{-1.1} 1.1 \times e^{-.4} .4 + e^{-1.1} \frac{1.1^2}{2} \times e^{-.4} \\
&= e^{-1.5}(1.1 + .44 + .605) = 2.145 e^{-1.5}
\end{aligned}
$$

(b)

$$
E[N(1.0) + N(4.0)] = .5 + 4 \times .5 = 2.5
$$

(c)

$$
\begin{aligned}
P\{N(1.0) = 2, N(4.0) = 4\} &= P\{N(1.0) = 2, N(4.0) - N(1.0) = 2\} \\
&= e^{-2} \frac{.5^2}{2} \times \frac{1.5^2}{2} = \frac{9}{64} e^{-2}
\end{aligned}
$$