Counting processes, stochastic equations, and asymptotics for stochastic models

- Poisson processes and Watanabe's theorem
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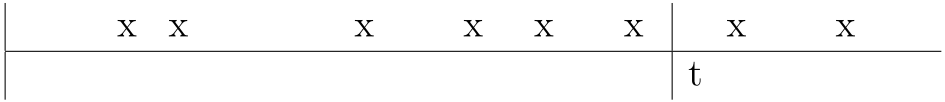
Poisson processes

A Poisson process is a model for a series of random observations occurring in time.



Let $Y(t)$ denote the number of observations by time t. Note that for $t < s$, $Y(s) - Y(t)$ is the number of observations in the time interval $(t, s]$. We assume the following:

1) Observations occur one at a time.

2) Numbers of observations in disjoint time intervals are independent random variables, i.e., if $t_0 < t_1 < \cdots < t_m$, then $Y(t_k) - Y(t_{k-1})$, $k = 1, \ldots, m$ are independent random variables.

3) The distribution of $Y(t + a) - Y(t)$ does not depend on t.

Characterization of a Poisson process

Theorem 1 *Under assumptions 1), 2), and 3), there is a constant* $\lambda > 0$ *such that, for* $t < s$, $Y(s) - Y(t)$ *is Poisson distributed with parameter* $\lambda(s-t)$, *that is,*

$$P\{Y(s) - Y(t) = k\} = \frac{(\lambda(s-t))^k}{k!} e^{-\lambda(s-t)}.$$

If $\lambda = 1$, then Y is a *unit* (or rate one) Poisson process. If Y is a unit Poisson process and $Y_\lambda(t) \equiv Y(\lambda t)$, then Y_λ is a Poisson process with parameter λ.

Note that

$$P\{Y_\lambda(t + \Delta t) - Y_\lambda(t) \geq 1 | \mathcal{F}_t^{Y_\lambda}\} = 1 - e^{-\lambda \Delta t} \approx \lambda \Delta t$$

Watanabe's theorem

N is a *counting process* if $N(0) = 0$ and N is constant except for jumps of $+1$.

Theorem 2 *Suppose that N is a counting process, $\lambda > 0$, and that M defined by*

$$M(t) = N(t) - \lambda t$$

is a martingale. Then N is a Poisson process with intensity λ.

Conditional intensities for counting processes

Assume N is adapted to $\{\mathcal{F}_t\}$.

$\lambda \geq 0$ is the $\{\mathcal{F}_t\}$-conditional intensity if (intuitively)

$$P\{N(t+\Delta t) > N(t)|\mathcal{F}_t\} \approx \lambda(t)\Delta t$$

or (precisely)

$$M(t) \equiv N(t) - \int_0^t \lambda(s)ds$$

is an $\{\mathcal{F}_t\}$-local martingale, that is, if τ_k is the kth jump time of N,

$$E[M((t+s)\wedge\tau_k)|\mathcal{F}_t] = M(t\wedge\tau_k)$$

for all $s, t \geq 0$ and all k.

Time-change representation

Lemma 3 *If* N *has* $\{\mathcal{F}_t\}$*-intensity* λ*, then there exists a unit Poisson process (may need to enlarge the sample space) such that*

$$N(t) = Y(\int_0^t \lambda(s)ds)$$

Modeling with counting processes

Specify $\lambda(t) = \gamma(t, N)$, where γ is nonanticipating in the sense that $\gamma(t, N) = \gamma(t, N(\cdot \wedge t))$.

Martingale problem. Require

$$N(t) - \int_0^t \gamma(s, N)ds \tag{1}$$

to be a local martingale.

Time-change equation. Require

$$N(t) = Y(\int_0^t \gamma(s, N)ds). \tag{2}$$

These formulations are equivalent in the sense that the solutions have the same distribution.

Systems of counting processes

Lemma 4 *(Meyer (1971), Kurtz (1980)) Assume* $N = (N_1, \ldots, N_m)$ *is a vector of counting processes with no common jumps and* λ_k *is the* $\{\mathcal{F}_t\}$*-intensity for* N_k*. Then there exist independent unit Poisson processes* $Y_1, \ldots, Y_m$ *(may need to enlarge the sample space) such that*

$$N_k(t) = Y_k\left(\int_0^t \lambda_k(s)ds\right)$$

Specifying nonanticipating intensities $\lambda_k(t) = \gamma_k(t, N)$:

$$N_k(t) = Y_k\left(\int_0^t \gamma_k(s, N)ds\right)$$

Representing continuous time Markov chains

If

$$P\{X(t+\Delta t)-X(t)=l|\mathcal{F}_t^X\}\approx\beta_l(X(t))\Delta t,\quad l\in\mathbb{Z}^d.$$

then we can write

$$N_l(t)=Y_l(\int_0^t\beta_l(X(s))ds),$$

where the Y_l are independent, unit Poisson processes. Consequently,

$$\begin{aligned}X(t) &= X(0)+\sum_l lN_l(t)\\ &= X(0)+\sum_l lY_l(\int_0^t\beta_l(X(s))ds).\end{aligned}$$

Poisson random measures

Let ν be a σ-finite measure on (U, d_U), a complete, separable metric space. Let $\mathcal{N}(U)$ denote the collection of counting measures on U.

Definition 5 *A* Poisson random measure *with mean measure* ν *is a random counting measure* ξ *(a* $\mathcal{N}(U)$*-valued random variable) such that*

a) For $A \in \mathcal{B}(U)$, $\xi(A)$ *has a Poisson distribution with expectation* $\nu(A)$

b) $\xi(A)$ *and* $\xi(B)$ *are independent if* $A \cap B = \emptyset$.

For $f \in M(U)$, $f \geq 0$, define

$$\psi_\xi(f) = E[\exp\{-\int_U f(u)\xi(du)\}] = \exp\{-\int (1 - e^{-f}) d\nu\}$$

(Verify the second equality by approximating f by simple functions.)

Space-time Poisson random measures

Let ξ be a Poisson random measure on $U \times [0,\infty)$ with mean measure $\nu \times \ell$ (where ℓ denotes Lebesgue measure).

$\xi(A,t) \equiv \xi(A \times [0,t])$ is a Poisson process with parameter $\nu(A)$.

$\widetilde{\xi}(A,t) \equiv \xi(A \times [0,t]) - \nu(A)t$ is a martingale.

Definition 6 *ξ is $\{\mathcal{F}_t\}$-compatible, if for each $A \in \mathcal{B}(U)$, $\xi(A,\cdot)$ is $\{\mathcal{F}_t\}$-adapted and for all $t,s \geq 0$, $\xi(A \times (t,t+s])$ is independent of $\mathcal{F}_t$.*

Predictable processes

Definition 7 *The* σ*-algebra* $\mathcal{P}_U$ *of* $\{\mathcal{F}_t\}$-predictable *sets in* $U \times [0,\infty) \times \Omega$ *is the smallest* σ*-algebra containing all sets of the form*

$$A \times (t, t+s] \times H$$

where $t, s \geq 0$, $A \in \mathcal{B}(U)$, $H \in \mathcal{F}_t$.

A stochastic process Z *taking values in* $M(U)$ *is* $\{\mathcal{F}_t\}$-predictable *if the mapping* $(u, t, \omega) \in U \times [0,\infty) \times \Omega \to Z(u, t, \omega)$ *is* $\mathcal{P}_U$*-measurable.*

Properties of integrals

Lemma 8 *If* ξ *is* $\{\mathcal{F}_t\}$*-compatible and* Z *is* $\{\mathcal{F}_t\}$*-predictable and satisfies*

$$E[\int_{U\times[0,t]}|Z(u,s)|\nu(du)ds]<\infty,\quad t>0,$$

then $\int_{U\times[0,t]}Z(u,s)\xi(du\times ds)$ *exists almost surely,*

$$M(t)=\int_{U\times[0,t]}Z(u,s)\xi(du\times ds)-\int_{U\times[0,t]}Z(u,s)\nu(du)ds$$

is a martingale, and

$$E[\int_{U\times[0,t]}Z(u,s)\xi(du\times ds)]=E[\int_{U\times[0,t]}Z(u,s)\nu(du)ds]$$

Centered Poisson random measures

If Z is predictable (e.g., left continuous and adapted) and

$$E[\int_{U\times[0,t]}|Z(u,s)|^2\nu(du)ds]<\infty,\quad t>0,$$

then

$$M(t)=\int_{U\times[0,t]}Z(u,s)\widetilde{\xi}(du\times ds)$$

is a square integrable martingale satisfying

$$[M]_t=\int_{U\times[0,t]}Z(u,s)^2\xi(du\times ds)$$

and

$$E[M(t)^2]=E[[M]_t]=E[\int_{U\times[0,t]}Z(u,s)^2\nu(du)ds]$$

Obtaining counting processes from space-time Poisson processes

Let ξ be a space-time Poisson process on $[0,\infty)\times[0,\infty)$ with Lebesgue mean measure, that is for Borel $A\subset[0,\infty)\times[0,\infty)$, $\xi(A)$ is Poisson distributed with mean $|A|$. (It follows that if A_1 and A_2 are disjoint, then $\xi(A_1)$ and $\xi(A_2)$ are independent.)

Then the solution of

$$\begin{aligned} N(t) &= \xi(\{(u,s): u\leq \gamma(s-,N), s\leq t\}) \\ &= \int_{[0,\infty)\times[0,t]} \mathbf{1}_{[0,\gamma(s-,N)]}(u)\xi(du\times ds) \end{aligned}$$

has the same distribution as (1) and (2). This representation goes back at least to Kerstan (1964) and Grigelionis (1971).

The multidimensional version also works.

Comparison of stochastic equations

γ_k nonanticipating, Y_k independent unit Poisson processes, ξ_k independent Poisson random measures with Lebesgue mean measure on $[0,\infty)\times[0,\infty)$.

$$N_k(t) = Y_k\Big(\int_0^t \gamma_k(s,N)ds\Big) \qquad \widehat{N}_k(t) = \int_{[0,\infty)\times[0,t]} \mathbf{1}_{[0,\gamma_k(s-,\widehat{N})]}(u)\xi_k(du\times ds)$$

Properties

- N and $\widehat{N}$ have the same distribution.
- Equation for N is more intuitive.
- $\widehat{N}$ adapted to

$$\mathcal{F}_t = \sigma(\xi_k(A\times[0,s]) : s \le t, A \in \mathcal{B}([0,\infty), k = 1,2,\ldots\}$$

regardless of choice of γ.

Couplings

Given intensities γ and $\widetilde{\gamma}$,

$$
\begin{aligned}
E[|N(t)-\widetilde{N}(t)|] &= E[|Y(\int_0^t \gamma(s,N)ds) - Y(\int_0^t \widetilde{\gamma}(s,\widetilde{N})ds)|] \\
&= E[|\int_0^t \gamma(s,N)ds - \int_0^t \widetilde{\gamma}(s,\widetilde{N})ds|]
\end{aligned}
$$

or

$$
\begin{aligned}
E[|N(t)-\widetilde{N}(t)|] &= E[|\int_{[0,\infty)\times[0,t]} (\mathbf{1}_{[0,\gamma(s-,N)]}(u) - \mathbf{1}_{[0,\widetilde{\gamma}(s-,\widetilde{N})]}(u))\xi(du\times ds)|] \\
&\leq E[\int_0^t |\gamma(s,N) - \widetilde{\gamma}(s,\widetilde{N})|ds]
\end{aligned}
$$

Alternative coupling

Note that using the time-change representation, we can write

$$
\begin{aligned}
N(t) &= Y_1\left(\int_0^t \gamma(s,N) \wedge \widetilde{\gamma}(s,\widetilde{N})ds\right) \\
&\qquad + Y_2\left(\int_0^t (\gamma(s,N) - \gamma(s,N) \wedge \widetilde{\gamma}(s,\widetilde{N}))ds\right) \\
\widetilde{N}(t) &= Y_1\left(\int_0^t \gamma(s,N) \wedge \widetilde{\gamma}(s,\widetilde{N})ds\right) \\
&\qquad + Y_3\left(\int_0^t (\widetilde{\gamma}(s,N) - \gamma(s,N) \wedge \widetilde{\gamma}(s,\widetilde{N}))ds\right)
\end{aligned}
$$

Example: Model with two time-scales

$$
\begin{aligned}
X_1^n(t) &= X_1^n(0) + Y_1(n\lambda t) - Y_2(n\int_0^t \mu X_1^n(s)ds) \\
X_2^n(t) &= X_2^n(0) + n^{-1}Y_3(n\int_0^t \alpha X_1^n(s)ds) - n^{-1}Y_4(n\int_0^t \beta X_2^n(s))ds)
\end{aligned}
$$

Then

$$\int_0^t X_1^n(s)ds \to \frac{\lambda}{\mu}t$$

and X_2^n converges to the solution of

$$x_2(t) = x_2(0) + \frac{\alpha\lambda}{\mu}t - \int_0^t \beta x_2(s)ds$$

Central limit theorem

Setting $V_2^n(t) = \sqrt{n}(X_2^n(t) - x_2(t))$

$$
\begin{aligned}
V_2^n(t) &= V_2^n(0) + \frac{1}{\sqrt{n}}\widetilde{Y}_3(n\int_0^t \alpha X_1^n(s)ds) - \frac{1}{\sqrt{n}}\widetilde{Y}_4(n\int_0^t \beta X_2^n(s))ds) \\
&\quad +\sqrt{n}\int_0^t (\alpha X_1^n(s) - \frac{\alpha\lambda}{\mu})ds - \int_0^t \beta V_2^n(s)ds \\
&= V_2^n(0) + \frac{1}{\sqrt{n}}\widetilde{Y}_3(n\int_0^t \alpha X_1^n(s)ds) - \frac{1}{\sqrt{n}}\widetilde{Y}_4(n\int_0^t \beta X_2^n(s))ds) \\
&\quad -\frac{1}{\sqrt{n}}\frac{\alpha}{\mu}(\widetilde{Y}_1(n\lambda t) - \widetilde{Y}_2(n\int_0^t \mu X_1^n(s)ds)) - \int_0^t \beta V_2^n(s)ds \\
&\quad +\frac{X_1^n(0) - X_1^n(t)}{\sqrt{n}}
\end{aligned}
$$

Limiting process

$$\begin{aligned} V_2(t) &= V_2(0) + W_3(\frac{\alpha\lambda}{\mu}t) - W_4(\int_0^t \beta x_2(s)ds) - \frac{\alpha}{\mu}(W_1(\lambda t) - W_2(\lambda t)) \\ &\quad - \int_0^t \beta V_2(s)ds \end{aligned}$$

Example: Nonlinear Hawkes model Brémaud and Massoulié (1996)

Let

$$\gamma(s,N) = \phi(\int_{(-\infty,t]} h(t-s)N(ds))$$

Assume $|\phi(x) - \phi(y)| \leq \alpha|x-y|$ and

$$\alpha \int_0^\infty |h(s)|ds < 1.$$

ξ a Poisson random measure on $[0,\infty) \times (-\infty,\infty)$ with Lebesgue mean measure.

$$N(r,t] = \int_{[0,\infty)\times(r,t]} \mathbf{1}_{[0,\gamma(s-,N))]}(u)\xi(du \times ds)$$

Recursion

Let $N^{(0)}(r,t] = \int_{[0,\infty)\times(r,t]} \mathbf{1}_{[0,\lambda]}(u)\xi(du\times ds)$ and

$$N^{(k+1)}(r,t] = \int_{[0,\infty)\times(r,t]} \mathbf{1}_{[0,\gamma(s-,N^{(k)})]}(u)\xi(du\times ds)$$

Then $N^{(k)}$ has stationary increments.

$$N^{(k+1)}\triangle N^{(k)}(r,t] = \int_{[0,\infty)\times(r,t]} |\mathbf{1}_{[0,\gamma(s-,N^{(k)})]}(u)-\mathbf{1}_{[0,\gamma(s-,N^{(k-1)})]}(u)|\xi(du\times ds)$$

and setting $E[N^{(k+1)}\triangle N^{(k)}(r,t]] = \rho^{(k)}(t-r)$

$$\begin{aligned}
\rho^{(k)} &= E[\int_0^1 |\gamma(r,N^{(k)}) - \gamma(r,N^{(k-1)})|dr] \\
&\leq \alpha E[|\int_{-\infty}^0 h(-s)N^{(k)}(ds) - \int_{-\infty}^0 h(-s)N^{(k-1)}(ds)|] \\
&\leq \alpha\rho^{(k-1)}\int_0^\infty |h(s)|ds
\end{aligned}$$

Birth and death processes

Let $\nu \in \mathcal{P}([0,\infty))$, and let ξ be a space-time Poisson random measure on $[0,\infty) \times [0,\infty)$ with mean measure $\nu \times \ell$. Let X be given by

$$X(t) = \xi(B(t)) - \xi(D(t))$$

where

$$\begin{aligned} B(t) &= \{(y,u) : u \leq \int_0^t \lambda(X(s))ds\} \\ D(t) &= \{(y,u) : y \leq t, u \leq \int_0^{t-y} \lambda(X(s))ds\} \end{aligned}$$

Law of large numbers for Poisson random measures

Lemma 9 *For* $n = 1, 2, \ldots$, *let* ξ_n *be a Poisson random measure with mean measure* $n\nu \times \ell$. *Then for* $A \in \mathcal{B}([0,\infty) \times [0,\infty))$ *with* $\nu \times \ell(A) < \infty$,

$$n^{-1}\xi_n(A) \to \nu \times \ell(A)$$

Definition 10 A *is a* lower layer *if* $(y_1, u_1) \in A$ *and* $y_2 \le y_1$, $u_2 \le u_1$ *implies* $(y_2, u_2) \in A$.

For $\alpha > 0$, let $\mathcal{L}_\alpha$ be the collection of lower layers contained in $[0,\infty) \times [0,\alpha]$.

Theorem 11 *Stute (1976) For each* $\alpha > 0$,

$$\sup_{A \in \mathcal{L}_\alpha} |n^{-1}\xi_n(A) - \nu \times \ell(A)| \to 0.$$

A fluid limit

Let

$$X_n(t) = n^{-1}\xi_n(B_n(t)) - n^{-1}\xi_n(D_n(t))$$

where

$$\begin{aligned} B_n(t) &= \{(y,u) : u \le \int_0^t \lambda(X_n(s))ds\} \\ D_n(t) &= \{(y,u) : y \le t, u \le \int_0^{t-y} \lambda(X_n(s))ds\} \end{aligned}$$

so

$$X_n(t) = \int_{[0,\infty)\times[0,\infty)} (\mathbf{1}_{B_n(t)}(y,u) - \mathbf{1}_{D_n(t)}(y,u))\xi_n(dy \times du)$$

For each t, $B_n(t)$ and $D_n(t)$ are predictable.

$\xi_n(B_n(t))$ is a counting process with intensity $n\lambda(X_n(s))$ and

$$\nu \times \ell(B_n(t) - D_n(t)) = \int_0^t \lambda(X_n(s))\nu(t-s,\infty)ds$$

Theorem 12 *Let* $\tau_n^\alpha = \inf\{t : \int_0^t \lambda(X_n(s))ds \geq \alpha\}$. *Then*

$$\sup_{t\leq\tau_n^\alpha} |X_n(t) - \int_0^t \lambda(X_n(s))\nu(t-s,\infty)ds| \to 0$$

and if λ *is Lipschitz, for each* $T > 0$, $\sup_{t\leq T} |X_n(t) - x(t)| \to 0$ *where* x *is the unique solution of*

$$x(t) = \int_0^t \lambda(x(s))\nu(t-s,\infty)ds$$

Wang (1975, 1977); Kurtz (1983); Foley (1986); Solomon (1987); Fricker and Jaïbi (2007); Decreusefond and Moyal (2008)

Gaussian approximation

For each $A \in \mathcal{B}([0,\infty) \times [0,\infty))$,

$$\frac{\xi_n(A) - n\nu \times \ell(A)}{\sqrt{n}} \Rightarrow W(A),$$

where W is space-time Gaussian white noise with $E[W(A_1)W(A_2)] = \nu \times \ell(A_1 \times A_2)$, but the convergence is *not* uniform over lower layers.

Convergence of stochastic integrals

For $\varphi \in L_2(\nu)$, define

$$W_n(\varphi, r) = \frac{1}{\sqrt{n}} \int_{[0,\infty)\times[0,r]} \varphi(y) \widetilde{\xi}_n(dy \times du)$$

and observe that for $\{\varphi_i\} \subset L_2(\nu)$

$$(W_n(\varphi_1, \cdot), \ldots, W_n(\varphi_m, \cdot)) \Rightarrow (W(\varphi_1, \cdot), \ldots, W(\varphi_m, \cdot)).$$

Assume that ξ_n is $\{\mathcal{F}_t^n\}$-compatible.

Theorem 13 *Suppose Z_n is a cadlag, $L_2(\nu)$-valued, $\{\mathcal{F}_t^n\}$-adapted process and that for each $\{\varphi_k\} \subset L_2(\nu)$,*

$$(Z_n, W_n(\varphi_1, \cdot), \ldots, W_n(\varphi_m, \cdot)) \Rightarrow (Z, W(\varphi_1, \cdot), \ldots, W(\varphi_m, \cdot))$$

in $D_{L_2(\nu)\times\mathbb{R}^m}[0,\infty)$. Then

$$\int_{U\times[0,\cdot]} Z_n(y, u-) W_n(dy \times du) \Rightarrow \int_{U\times[0,\cdot]} Z(y, u-) W(dy \times du).$$

Proof. See Kurtz and Protter (1996). □

The result also holds for $\mathbb{R}^d$-valued Z_n, and under additional conditions, for function-valued Z_n.

Central limit theorem for X_n

Taking $Z_n(y,u-) = \mathbf{1}_{B_n(t)}(y,u) - \mathbf{1}_{D_n(t)}(y,u)$ and $u_0 >> \int_0^t \lambda(x(s))ds$

$$W_n(B_n(t)) - W_n(D_n(t)) = \int_{[0,\infty)\times[0,u_0]} Z_n(y,u-)W_n(dy\times du)$$

converges to

$$\int_{[0,\infty)\times[0,u_0]} (\mathbf{1}_{B(t)}(y,u) - \mathbf{1}_{D(t)}(y,u))W(dy\times du) = W(B(t)) - W(D(t))$$

and more generally, the finite dimensional distributions converge.

Limiting Gaussian process

Verifying tightness and setting $V_n(t) = \sqrt{n}(X_n(t) - x(t))$,

$$\begin{aligned} V_n(t) &= W_n(B_n(t)) - W_n(D_n(t)) \\ &\qquad + \sqrt{n}\int_0^t (\lambda(X_n(s)) - \lambda(x(s)))\nu(t-s,\infty)ds \end{aligned}$$

converges to the solution of

$$V(t) = W(B(t)) - W(A(t)) + \int_0^t \lambda'(x(s))V(s)\nu(t-s,\infty)ds$$

Some related work

Kurtz (1980) treats infinite systems of time-change equations.

Kurtz and Protter (1996) treats infinite systems of Poisson embedding equations.

Stochastic equations for spatial birth and birth and death processes are discussed in Massoulié (1998); Garcia and Kurtz (2006, 2008).

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Abstract

Counting processes, stochastic equations, and asymptotics for stochastic models

Many stochastic models can be represented as solutions of stochastic equations for which Poisson processes or Poisson random measures are the primary inputs. A variety of examples will be given illustrating these equations and how they can be used to study limit theorems for stochastic models.