CHAPTER 4

# Large deviations for stochastic processes

We focus on stochastic processes with cadlag trajectories. Throughout, $(E, r)$ will be a complete, separable metric space, $D_E[0,\infty)$ will be the space of cadlag, $E$-valued functions with the Skorohod $(J_1)$ topology, and $C_E[0,\infty)$ will be the space of continuous, $E$-valued functions. Theorems 4.1 and 4.4 in Section 4.1 characterize exponential tightness for these processes. These results are direct analogs of standard tightness theorems in the weak convergence theory. The effects of changes of time-scale on large deviation results are discussed in Section 4.2.

Exponential tightness is usually easier to verify if the state space is compact. Frequently, results can be obtained by first compactifying the state space, verifying the large deviation principle in the compact space, and then inferring the large deviations principle in the original space from the result in the compact space. Section 4.3 gives results supporting this approach.

Restricted to $C_E[0,\infty)$, the Skorohod topology is just the compact uniform topology. If the sequence of processes is asymptotically continuous in a strong enough sense, then we see in Section 4.4 that the large deviation principle in the Skorohod topology implies the large deviation principle in the compact uniform topology.

The main exponential tightness results are developed further in Sections 4.5 and 4.6 in the context of martingale problems. Section 4.7 gives the large deviation analog of the fact that tightness in the Skorohod topology plus convergence of the finite dimensional distributions implies weak convergence of the processes. Here, exponential tightness plus the large deviation principle for the finite dimensional distributions implies the large deviation principle for the processes. The rate function is identified as the appropriate supremum of the finite dimensional rate functions. The results can be viewed as a variation of the projective limit method (Dawson and Gärtner [**22**], de Acosta [**24**]) adapted to the Skorohod topology.

## 4.1. Exponential tightness for processes

We now consider exponential tightness for a sequence of processes $\{X_n\}$ in $D_E[0,\infty)$. Let $X_n$ be adapted to a right-continuous, complete filtration $\{\mathcal{F}_t^n\}$. Let $S^n(T)$ be the collection of all $\{\mathcal{F}_t^n\}$-stopping times bounded by $T$, and let $S_0^n(T) \subset S^n(T)$ be the sub-collection of discrete stopping times. Recall that each $\tau \in S^n(T)$ can be approximated by a decreasing sequence in $S_0^n(T)$. The following theorem is the analogue for exponential tightness of Ethier and Kurtz [**36**], Theorem 3.8.6. The weak convergence version of condition (b) is due to Kurtz [**71**] and the weak convergence version of condition (c) is due to Aldous [**2**]. For $\delta > 0$ and $T > 0$, define the modulus of continuity in $D_E[0,\infty)$ by

$$w'(x,\delta,T) = \inf_{\{t_i\}} \max_i \sup_{s,t\in[t_{i-1},t_i)} r(x(s),x(t)),$$

where the infimum is over $\{t_i\}$ satisfying

$$0 = t_0 < t_1 < \cdots < t_{m-1} < T \leq t_m$$

and $\min_{1 \leq i \leq n}(t_i - t_{i-1}) > \delta$. See Ethier and Kurtz [**36**], Section 3.6, for a discussion of the properties of $w'$. We define $q(x, y) = 1 \wedge r(x, y)$. Note that $q$ is a metric equivalent to $r$.

Theorem 4.1. *Let $\mathcal{T}_0$ be a dense subset of $[0, \infty)$. Suppose that for each $t \in \mathcal{T}_0$, $\{X_n(t)\}$ is exponentially tight. Then the following are equivalent.*

a) *$\{X_n\}$ is exponentially tight in $D_E[0, \infty)$.*

b) *For each $T > 0$, there exist $\beta > 0$ and random variables $\gamma_n(\delta, \lambda, T)$, $\delta, \lambda > 0$, satisfying*

$$E[e^{n\lambda q^\beta(X_n(t+u), X_n(t)) \wedge q^\beta(X_n(t), X_n(t-v))} | \mathcal{F}_t^n] \leq E[e^{\gamma_n(\delta,\lambda,T)} | \mathcal{F}_t^n], \tag{4.1}$$

*for $0 \leq t \leq T$, $0 \leq u \leq \delta$, and $0 \leq v \leq t \wedge \delta$, such that for each $\lambda > 0$,*

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log E[e^{\gamma_n(\delta,\lambda,T)}] = 0 \tag{4.2}$$

*and*

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log E[e^{n\lambda q^\beta(X_n(\delta), X_n(0))}] = 0. \tag{4.3}$$

c) *Condition (4.3) holds, and for each $T > 0$, there exists $\beta > 0$ such that for each $\lambda > 0$*

$$C_n(\delta, \lambda, T) \equiv \sup_{\tau \in S_0^n(T)} \sup_{u \leq \delta} E[\sup_{v \leq \delta \wedge \tau} e^{n\lambda q^\beta(X_n(\tau+u), X_n(\tau)) \wedge q^\beta(X_n(\tau), X_n(\tau-v))}]$$

*satisfies*

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log C_n(\delta, \lambda, T) = 0. \tag{4.4}$$

d) *For each $\epsilon > 0$ and $T > 0$*

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log P\{w'(X_n, \delta, T) > \epsilon\} = -\infty. \tag{4.5}$$

Remark 4.2. Note that (4.1) is implied by the simpler inequality

$$E[e^{n\lambda q^\beta(X_n(t+u), X_n(t))} | \mathcal{F}_t^n] \leq E[e^{\gamma_n(\delta,\lambda,T)} | \mathcal{F}_t^n]. \tag{4.6}$$

Puhalskii [**97**], Theorem 4.4, (see also Puhalskii [**100**], Theorem 3.2.3) gives a condition similar to (c) that would correspond to taking the supremum over $u$ inside the expectation in the definition of $C_n(\delta, \lambda, T)$ (a condition that is substantially more difficult to verify). Under Puhalskii's condition, it is, in fact, unnecessary to consider stopping times, that is, the supremum over $\tau \in S_0^n(T)$ can be replaced by the supremum over $0 \leq t \leq T$, and $C$-exponential tightness (Definition 4.12) follows without additional assumptions. Theorem 4.2 of Puhalskii [**97**] (Theorem 3.2.1 of [**100**]) gives the equivalence of (a) and (d).

Note also that the minimum in the exponent can be replaced by the product. In particular, if $0 \leq a, b \leq 1$ and $\beta > 0$, then $a^\beta b^\beta \leq a^\beta \wedge b^\beta \leq a^{\beta/2} b^{\beta/2}$.

Before proving Theorem 4.1, we need the following analogue of Ethier and Kurtz [**36**], Lemma 3.8.4. We suppress the $n$ to simplify the notation and let $a_\beta = 2^{(\beta-1)\vee 0}$ so that $q^\beta(x, y) \leq a_\beta(q^\beta(x, z) + q^\beta(z, y))$.

LEMMA 4.3. *Fix $\beta > 0$. For $\delta, \lambda, T > 0$, define*

$$C(\delta,\lambda,T) \equiv \sup_{\tau\in S_0(T)} \sup_{u\leq\delta} E[\sup_{v\leq\delta\wedge\tau} e^{\lambda q^\beta(X(\tau+u),X(\tau))q^\beta(X(\tau),X(\tau-v))}].$$

*Let $\tau \in S(T)$, and let $\widehat{\tau}$ be a stopping time with $\widehat{\tau} \geq \tau$ a.s. Then*

$$\begin{aligned} &E[\sup_{v\leq\delta\wedge\tau} e^{\lambda q^\beta(X(\widehat{\tau}\wedge(\tau+\delta)),X(\tau))q^\beta(X(\tau),X(\tau-v))}] \\ &\qquad \leq C(3\delta, 24a_\beta^4\lambda, T+2\delta). \end{aligned}$$

PROOF. Let $\tau \in S(T+\delta)$, and let $M_\tau(\delta)$ denote the collection of $\mathcal{F}_\tau$-measurable random variables $U$ satisfying $0 \leq U \leq \delta$. Note that $\tau + U \in S(T+2\delta)$. It follows from Ethier and Kurtz [**36**] (3.8.23) and the Schwartz and Jensen inequalities, that for $\tau \in S(T+\delta)$ and $U \in M_\tau(\delta)$

$$\begin{aligned} &E[\sup_{v\leq\delta\wedge\tau} e^{\lambda q^\beta(X(\tau+U)),X(\tau))q^\beta(X(\tau),X(\tau-v))}] \\ &\qquad \leq \sqrt{C(2\delta, 2a_\beta\lambda, T+\delta)C(3\delta, 8a_\beta^2\lambda, T+2\delta)} \\ &\qquad \leq C(3\delta, 8a_\beta^2\lambda, T+2\delta)\ . \end{aligned} \tag{4.7}$$

(Note that $C(\delta,\lambda,T)$ is nondecreasing in all three variables.) For $\tau \in S(T)$, observe that

$$\begin{aligned} &q^\beta(X(\widehat{\tau}\wedge(\tau+\delta)),X(\tau))q^\beta(X(\tau),X(\tau-v)) \\ &\quad \leq a_\beta q^\beta(X(\tau+\delta),X(\tau))q^\beta(X(\tau),X(\tau-v)) \\ &\qquad +a_\beta^2 q^\beta(X(\tau+\delta),X(\widehat{\tau}\wedge(\tau+\delta)))q^\beta(X(\widehat{\tau}\wedge(\tau+\delta)),X(\tau)) \\ &\qquad +a_\beta^2 q^\beta(X(\tau+\delta),X(\widehat{\tau}\wedge(\tau+\delta)))q^\beta(X(\widehat{\tau}\wedge(\tau+\delta)),X(\tau-v)). \end{aligned}$$

Observing that $U = \tau+\delta-\widehat{\tau}\wedge(\tau+\delta)$ is $\mathcal{F}_{\widehat{\tau}\wedge(\tau+\delta)}$-measurable so that (4.7) can be applied, Hölder's inequality and (4.7 give

$$\begin{aligned} &E[\sup_{v\leq\delta\wedge\tau} e^{\lambda q^\beta(X(\widehat{\tau}\wedge(\tau+\delta)),X(\tau))q^\beta(X(\tau),X(\tau-v))}] \\ &\qquad \leq \left(C(3\delta, 24a_\beta^3\lambda, T+\delta)C(3\delta, 24a_\beta^4\lambda, T+2\delta)^2\right)^{1/3} \\ &\qquad \leq C(3\delta, 24a_\beta^4\lambda, T+2\delta). \end{aligned}$$

□

PROOF. (of Theorem 4.1). ($a$ implies $b$) Fix $\beta > 0$. For compact $K \subset D_E[0,\infty)$, define

$$\gamma_K(\delta,T) = \sup_{x\in K} \sup_{0\leq t\leq T} \sup_{0\leq u\leq\delta, 0\leq v\leq\delta\wedge t} q^\beta(x(t+u),x(t)) \wedge q^\beta(x(t),x(t-v))$$

and note that $\lim_{\delta\to 0}\gamma_K(\delta,T) = 0$. By the assumption of exponential tightness, for each $\lambda > 0$, there is a compact $K_\lambda$ satisfying

$$\limsup_{n\to\infty} \frac{1}{n} \log P\{X_n \notin K_\lambda\} \leq -2\lambda.$$

Then, setting $\gamma_n(\delta,\lambda,T) = n\lambda(\gamma_{K_\lambda}(\delta,T) + I_{\{X_n\notin K_\lambda\}})$, (4.1) is trivially satisfied and noting that

$$E[e^{n\lambda I_{\{X_n\notin K_\lambda\}}}] \leq 1 + e^{n\lambda}P\{X_n \notin K_\lambda\} \to 1,$$

we have

$$\lim_{n\to\infty}\frac{1}{n}\log E[e^{\gamma_n(\delta,\lambda,T)}]=\lambda\gamma_{K_\lambda}(\delta,T)$$

and (4.2) follows. A similar argument gives (4.3).

($b$ implies $c$) Note that if (4.1) holds for $0\le t\le T$, then the inequality also holds with $t$ replaced by $\tau\in S_0^n(T)$, and hence, (4.2) implies (4.4).

($c$ implies $d$) We follow the development for weak convergence in Section 3.8 of Ethier and Kurtz [**36**]; however, the details are actually simpler in the present setting. Let $0<\epsilon\le 1$. Suppressing the index $n$ for the moment, let $\tau_0=\sigma_0=0$ and for $k=1,2,\ldots$, define

$$\tau_k=\inf\{t>\tau_{k-1}:r(X(t),X(\tau_{k-1}))\ge\epsilon\},$$

if $\tau_{k-1}<\infty$, and

$$\sigma_k=\sup\{t\le\tau_k:r(X(t),X(\tau_k))\vee r(X(t-),X(\tau_k))\ge\epsilon\},$$

if $\tau_k<\infty$. Set $\sigma_k=\infty$ if $\tau_k=\infty$. It follows (see Section 3.8 of Ethier and Kurtz [**36**]) that $\min\{\tau_{k+1}-\sigma_k:\tau_k<T+\delta/2\}>\delta$ implies $w'(X,\delta/2,T)\le 2\epsilon$, so

$$P\{w'(X,\delta/2,T)>2\epsilon\}\le\sum_{k=1}^{[T/\delta]+2}P\{\tau_{k+1}-\sigma_k\le\delta,\tau_k<T+\delta/2\}.$$

Consequently, it is sufficient to show that

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\max_k\frac{1}{n}\log P\{\tau_{k+1}-\sigma_k\le\delta,\tau_k<T+\delta/2\}=-\infty. \tag{4.8}$$

Let $\tau$ be any stopping time with $\tau\le T+\delta/2$, and define

$$\tau^+=\inf\{t>\tau:r(X(t),X(\tau))\ge\epsilon\}$$

and

$$\tau^-=\sup\{t\le\tau:r(X(t),X(\tau))\vee r(X(t-),X(\tau))\ge\epsilon\}.$$

Then for $0<\epsilon\le 1$ and any $\lambda>0$,

(4.9)

$$P\{\tau^+-\tau^-\le\delta\}\le e^{-n\lambda\epsilon^{2\beta}}E[\sup_{v\le\delta\wedge\tau}e^{n\lambda q^\beta(X_n(\tau^+\wedge(\tau+\delta)),X_n(\tau))q^\beta(X_n(\tau),X_n(\tau-v))}]\,.$$

By Lemma 4.3, the right side of (4.9) is bounded by

$$e^{-n\lambda\epsilon^{2\beta}}C_n(3\delta,24a_\beta^4\lambda,T+3\delta).$$

By (4.4), it follows that

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\sup_{\tau\in S(T+\delta/2)}\frac{1}{n}\log P\{\tau^+-\tau^-\le\delta\}\le-\lambda\epsilon^{2\beta}.$$

Since $\lambda$ is arbitrary and for $\tau=\tau_k\wedge(T+\delta/2)$,

$$P\{\tau_{k+1}-\sigma_k\le\delta,\tau_k<T+\delta/2\}\le P\{\tau^+-\tau^-\le\delta\},$$

(4.8) follows.

($d$ implies $a$) Let $t_1,t_2,\ldots$ be some ordering of $\mathcal{T}_0$. For each $\lambda>0$ and $k=1,2,\ldots$, there exists $\delta_k>0$ and compact $\Gamma_k\subset E$ such that

$$P\{w'(X_n,\delta_k,k)>k^{-1}\}\le e^{-n\lambda k}$$

and

$$P\{X_n(t_k)\notin\Gamma_k\}\le e^{-n\lambda k},$$

for $n = 1, 2, \ldots$. Let $H_k^1 = \{x : w'(x, \delta_k, k) \le k^{-1}\}$ and $H_k^2 = \{x : x(t_k) \in \Gamma_k\}$. Let $K_\lambda$ be the closure of $\cap_{k=1}^\infty H_k^1 \cap H_k^2$. Then $K_\lambda$ is compact and

$$P\{X_n \notin K_\lambda\} \le 2 \sum_{k=1}^\infty e^{-n\lambda k} \le \frac{2e^{-n\lambda}}{1 - e^{-n\lambda}},$$

so $\limsup_{n\to\infty} \frac{1}{n} \log P\{X_n \notin K_\lambda\} \le -\lambda$. □

The next theorem from Schied [**106**] (see Theorem A.1 in [**30**]) reduces the verification of the exponential tightness of $\{X_n\}$ to that of $\{f(X_n)\}$ for real-valued functions $f$. The weak convergence analogue is in Kurtz [**73**]. (See Ethier and Kurtz [**36**], Theorem 3.9.1. Jakubowski [**62**] gives simpler conditions on the collection of functions and a simpler proof.).

Theorem 4.4. *A sequence $\{X_n\}$ is exponentially tight in $D_E[0, \infty)$ if and only if*

a) *for each $T > 0$ and $a > 0$, there exists a compact $K_{a,T} \subset E$ such that*

$$\limsup_{n\to\infty} \frac{1}{n} \log P(\exists t \le T \ni X_n(t) \notin K_{a,T}) \le -a; \tag{4.10}$$

b) *there exists a family of functions $F \subset C(E)$ that is closed under addition and separates points in $E$ such that for each $f \in F$, $\{f(X_n)\}$ is exponentially tight in $D_R[0, \infty)$.*

Remark 4.5. We will refer to Condition (a) as the *exponential compact containment condition.*

Proof. Necessity of the two conditions follows from the definition of exponential tightness and continuity of the mapping $x \in D_E[0, \infty) \to f \circ x \in D_R[0, \infty)$ for $f \in C(E)$.

Exponential tightness of $\{f \circ X_n\}$, for all $f \in F$ implies, by Lemma 3.6, that for $f_1, f_2, \ldots \in F$ $\{Z_n\} \equiv \{(f_1 \circ X_n, f_2 \circ X_n, \ldots)\}$ is exponentially tight in $D_R[0, \infty) \times D_R[0, \infty) \times \cdots$. Since $F$ is closed under addition, exponential tightness in $D_{R^\infty}[0, \infty)$ then follows. (See Ethier and Kurtz [**36**], Problem 3.11.22.) For compact $K \subset E$, let $\{f_i^K\}$ separate points in $K$. Then

$$q_K(x, y) = \sum_{i=1}^\infty 2^{-i} 1 \wedge |f_i^K(x) - f_i^K(y)|$$

is a metric equivalent to $r$ on $K$. In particular, there exists a nondecreasing function $\rho_K : [0, \infty) \to [0, \infty)$ with $\lim_{u\to 0} \rho_K(u) = 0$ such that $r(x, y) \le \rho_K(q_K(x, y))$, $x, y \in K$. Let $Z_n^K = (f_1^K \circ X_n, f_2^K \circ X_n, \ldots)$, and let the metric on $R^\infty$ be given by $r_{R^\infty}(u, v) = \sum_{i=1}^\infty 2^{-i} 1 \wedge |u_i - v_i|$.

It follows that

$$\begin{aligned}&\{w'(X_n, \delta, T) > \epsilon\}\\ &\quad \subset \{X_n(t) \notin K, \text{ some } t \le T\} \cup \{X_n(t) \in K, t \le T, \rho_K(w'(Z_n^K, \delta, T)) > \epsilon\},\end{aligned}$$

and the exponential tightness of $\{Z_n^K\}$ then implies

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log P\{w'(X_n, \delta, T) > \epsilon\} \le \limsup_{n\to\infty} \frac{1}{n} \log P\{X_n(t) \notin K, \text{ some } t \le T\}.$$

Since $K$ is an arbitrary compact set, (4.10) then implies (4.5). □

The following localization result may simplify certain arguments, particularly for Markov processes, where if generators $A$ and $A_K$ satisfy $Af(x) = A_K f(x)$, $x \in K$, $f \in \mathcal{D}(A) = \mathcal{D}(A_K)$, then solutions of the corresponding martingale problems should agree in distribution up to the first time they leave $K$. (See Lemma 4.5.16 of Ethier and Kurtz [**36**].)

LEMMA 4.6. *Suppose $\{X_n\}$ satisfies the exponential compact containment condition and for $a > 0$, $T > 0$, let $K_{a,T}$ satisfy (4.10). For $x \in D_E[0,\infty)$, let $\tau_{a,T}(x) = \inf\{t : x(t) \notin K_{a,T}\}$. Suppose that for each $a > 0$, $T > 0$, and $n = 1, 2, \ldots$, $X_n^{a,T}(\cdot \wedge \tau_{a,T}(X_n^{a,T}))$ has the same distribution as $X_n(\cdot \wedge \tau_{a,T}(X_n))$ and that for each $a > 0$ and $T > 0$, $\{X_n^{a,T}\}$ satisfies the large deviation principle with good rate function $I^{a,T}$. Then $\{X_n\}$ satisfies the large deviation principle with good rate function given by*

$$I(x) = \lim_{\delta \to 0} \liminf_{a,T \to \infty} \inf_{y \in B_\delta(x)} I^{a,T}(y) = \lim_{\delta \to 0} \limsup_{a,T \to \infty} \sup_{y \in B_\delta(x)} I^{a,T}(y)$$

PROOF. Without loss of generality, we can assume that $X_n(\cdot \wedge \tau_{a,T}(X_n)) = X_n^{a,T}(\cdot \wedge \tau_{a,T}(X_n^{a,T}))$. (See Lemma 5.15 of Ethier and Kurtz [**36**].) Setting $X_n^{\eta,a} = X_n^{a,T}$ for $\eta = e^{-T}$, the lemma follows by Lemma 3.14. □

## 4.2. Large deviations under changes of time-scale

Continuous time large deviation problems are frequently approached by discrete time approximations. The following lemma demonstrates the generality of this approach (cf. Dupuis and Ellis [**35**] Lemma 10.3.4).

LEMMA 4.7. *For $X_n$ in $D_E[0,\infty)$ and $\epsilon_n > 0$, define $Y_n(t) = X_n([t/\epsilon_n]\epsilon_n)$. Then the following hold.*

a) *If $\{X_n\}$ is exponentially tight, then $\{Y_n\}$ is exponentially tight.*

b) *If $\epsilon_n \to 0$ and $\{X_n\}$ is exponentially tight, then for each $f \in C_b(D_E[0,\infty))$*

$$\limsup_{n \to \infty} \frac{1}{n} \log E[e^{nf(X_n)}] = \limsup_{n \to \infty} \frac{1}{n} \log E[e^{nf(Y_n)}] \tag{4.11}$$

*and*

$$\liminf_{n \to \infty} \frac{1}{n} \log E[e^{nf(X_n)}] = \liminf_{n \to \infty} \frac{1}{n} \log E[e^{nf(Y_n)}], \tag{4.12}$$

*and hence, the large deviation principle holds for $\{X_n\}$ if and only if it holds for $\{Y_n\}$ and if it holds, $\{X_n\}$ and $\{Y_n\}$ have the same rate function.*

PROOF. Clearly, if $\{X_n\}$ satisfies the exponential compact containment condition, then so does $\{Y_n\}$. By the necessity of part (b) of Theorem 4.1, there exist $\gamma_n(\delta, \lambda, T)$ satisfying (4.1). Consequently, $\{Y_n\}$ will satisfy (4.1) with $\gamma_n(\delta, \lambda, T)$ replaced by

$$\widehat{\gamma}_n(\delta, \lambda, T) = \begin{cases} 0, & \delta < \epsilon_n \\ \gamma_n(2\delta, \lambda, T), & \delta \geq \epsilon_n. \end{cases}$$

Since $\widehat{\gamma}_n(\delta, \lambda, T)$ clearly satisfies (4.2), exponential tightness for $\{Y_n\}$ follows.

Part (b) will follow from Lemma 3.13 provided we verify (3.13). But, defining $x^\epsilon$ by $x^\epsilon(t) = x([t/\epsilon]\epsilon)$, if $K \subset D_E[0,\infty)$ is compact, then for the metric given by (1.5),

$$\lim_{\epsilon \to 0} \sup_{x \in K} d(x, x^\epsilon) = 0,$$

and (3.13) follows. (See Lemma 4.8 below.) □

Lemma 4.7 can be generalized to other time changes. The following property of the Skorohod topology provides the basis for these extensions.

LEMMA 4.8. *Let $\zeta$ be a cadlag, nonnegative, and nondecreasing function on $[0,\infty)$, and let $\eta > 0$ and $T > 0$ satisfy $\eta \geq \sup_{t \leq T} |\zeta(t) - t|$. Then for $\delta > 2\eta$ and the metric $d$ given by (1.5),*

$$d(x, x \circ \zeta) \leq \log \frac{\delta}{\delta - 2\eta} \vee (w'(x, \delta, T + \eta) + \frac{\eta(T+\eta)}{\delta} + e^{-T}). \tag{4.13}$$

PROOF. Fix $\epsilon > 0$. Let $0 = t_0 < t_1 < \cdots < t_{m-1} < T + \eta \leq t_m$ satisfy

$$\max_i \sup_{s,t \in [t_{i-1}, t_i)} r(x(s), x(t)) \leq w'(x, \delta, T + \eta) + \epsilon.$$

For each $i$, define $\tau_i = \inf\{t : \zeta(t) \geq t_i\}$, and for $0 \leq t \leq t_m$, let $\lambda(t)$ be the linear interpolation of the points $(t_i, \tau_i)$. For $t > t_m$, let $\lambda'(t) = 1$. Since $|t_i - \tau_i| \leq \eta$, $\frac{\delta}{\delta+2\eta} \leq \lambda'(t) \leq \frac{\delta}{\delta - 2\eta}$, and $\gamma(\lambda) \leq \log \frac{\delta}{\delta-2\eta}$. For $t \vee \lambda(t) \leq T + \eta$,

$$r(x \circ \zeta \circ \lambda(t), x(t)) \leq w'(x, \delta, T + \eta) + \epsilon,$$

and it follows that

$$\int_0^T e^{-u} \sup_{t \geq 0} r(x \circ \zeta(\lambda(t) \wedge u), x(t \wedge u)) du \leq w'(x, \delta, T + \eta) + \epsilon + 2m\eta.$$

Note that $2m\eta$ bounds the Lebesgue measure of the set of $u$ for which there exists a $t$ and a $t_i$ satisfying

$$\lambda(t) \wedge u < t_i \leq t \wedge u \quad \text{or} \quad t \wedge u < t_i \leq \lambda(t) \wedge u \tag{4.14}$$

since either of the inequalities in (4.14) implies $|u - t_i| \leq \eta$. Since $m \leq (T + \eta)/\delta$ and $\epsilon > 0$ is arbitrary, (4.13) follows. □

LEMMA 4.9. *For each $n$, let $X_n$ be a process in $D_E[0,\infty)$, let $\Lambda_n$ be a nonnegative, nondecreasing process independent of $X_n$, and define*

$$Y_n(t) = X_n(\Lambda_n(t)).$$

*Suppose that for each $t > 0$ and $\eta > 0$,*

$$\lim_{n\to\infty} \frac{1}{n} \log P\{\sup_{s \leq t} |s - \Lambda_n(s)| > \eta\} = -\infty. \tag{4.15}$$

a) *If $\{X_n\}$ is exponentially tight, then $\{Y_n\}$ is exponentially tight.*
b) *If $\{X_n\}$ is exponentially tight, then for each $f \in C_b(D_E[0,\infty))$*

$$\limsup_{n\to\infty} \frac{1}{n} \log E[e^{nf(X_n)}] = \limsup_{n\to\infty} \frac{1}{n} \log E[e^{nf(Y_n)}] \tag{4.16}$$

*and*

$$\liminf_{n\to\infty} \frac{1}{n} \log E[e^{nf(X_n)}] = \liminf_{n\to\infty} \frac{1}{n} \log E[e^{nf(Y_n)}], \tag{4.17}$$

*and hence, the large deviation principle holds for $\{X_n\}$ if and only if it holds for $\{Y_n\}$ and if it holds, $\{X_n\}$ and $\{Y_n\}$ have the same rate function.*

REMARK 4.10. An obvious question is what happens to $\{Y_n\}$ if $\Lambda_n$ does not converge as in 4.15? Russell [**103**] gives a systematic discussion of this question.

PROOF. Lemma 4.8 implies (3.13), and the result follows by Lemma 3.13. □

## 4.3. Compactification

The requirement in Theorem 4.1 that $\{X_n(t)\}$ be exponentially tight for each $t \in \mathcal{T}_0$ and the stronger requirement in Condition (a) of Theorem 4.4 are, of course, trivially satisfied if $E$ is compact. As in the weak convergence setting, these conditions can sometimes be finessed in the noncompact case by working with some compactification $\widehat{E}$ of $E$. The following result gives conditions under which the large deviation principle in $D_{\widehat{E}}[0,\infty)$ can be used to obtain the large deviation principle in $D_E[0,\infty)$. The result also covers situations where the topology on $\widehat{E}$ is weaker than the topology on $E$. For example, $\widehat{E}$ could be the space of probability measures on a metric space with the weak topology while $E$ is the space of probability measures on the same metric space having a density with respect to some fixed reference measure $\nu$ with the metric on $E$ given by the $L_1(\nu)$-norm.

Theorem 4.11. *Suppose $(\widehat{E},\widehat{r})$ and $(E,r)$ are metric spaces, $E \subset \widehat{E}$, $\mathcal{B}(E) \subset \mathcal{B}(\widehat{E})$, and if $\{z_n\} \subset E$ converges to $z \in E$ under $r$, then it also converges under $\widehat{r}$ (that is, $r$ generates a stronger topology on $E$ than $\widehat{r}$). Let $\{P_n\}$ be a sequence of probability measures on $D_{(\widehat{E},\widehat{r})}[0,\infty)$ such that for each $n$, $P_n(D_{(E,r)}[0,\infty)) = 1$. (We include the metric in the notation for clarity. Note that $D_{(E,r)}[0,\infty) \subset D_{(E,\widehat{r})}[0,\infty)$, but the two spaces need not be equal.) Let*

$$\widehat{\Gamma} = D_{(\widehat{E},\widehat{r})}[0,\infty) - D_{(E,r)}[0,\infty).$$

*Suppose that the large deviation principle for $\{P_n\}$ holds in $D_{\widehat{E}}[0,\infty)$ with good rate function $\widehat{I}$.*

a) *If the exponential compact containment condition holds in $(E,r)$, then $\widehat{I}(x) = \infty$, for all $x \in \widehat{\Gamma}$, and the large deviation principle holds for $\{P_n\}$ on $D_{(E,r)}[0,\infty)$.*

b) *Suppose that the topology on $E$ generated by $\widehat{r}$ is the same as the topology generated by $r$ so*

$$\widehat{\Gamma} = \{x \in D_{\widehat{E}}[0,\infty) : \exists t \geq 0 \ni x(t) \text{ or } x(t-) \in \widehat{E} - E\}.$$

*If $\widehat{I}(x) = \infty$ for each $x \in \widehat{\Gamma}$, then the large deviation principle holds for $\{P_n\}$ on $D_E[0,\infty)$ with rate function $I$ given by $I(x) = \widehat{I}(x)$ for $x \in D_E[0,\infty)$. In particular, the exponential compact containment condition holds in $E$.*

Proof. Note that if $K$ is compact in $(E,r)$, then it is also compact in $(\widehat{E},\widehat{r})$ (but not conversely) and the topology on $K$ given by $r$ is the same as the topology on $K$ given by $\widehat{r}$. In particular, $D_{(K,r)}[0,\infty) = D_{(K,\widehat{r})}[0,\infty)$. If $\mathcal{K} \subset D_{(\widehat{E},\widehat{r})}[0,\infty)$ is compact, then $\mathcal{K}_K = \mathcal{K} \cap D_{(K,r)}[0,\infty)$ is a compact subset of $D_{(E,r)}[0,\infty)$. (Apply the characterization of Skorohod convergence given in [**36**], Proposition 3.6.5, and the equivalence of $r$ and $\widehat{r}$ on $K$.)

Under the conditions of Part (a), for $a, \eta > 0$ and $e^{-T} < \eta$, let $K = K_{a,T}$ be a compact subset of $E$ (under $r$) satisfying (4.10), and let $\mathcal{K}$ be a compact subset of $D_{(\widehat{E},\widehat{r})}[0,\infty)$ satisfying

$$\limsup_{n\to\infty} \frac{1}{n} \log P\{X_n \notin \mathcal{K}\} \leq -a.$$

Let $\mathcal{K}_K = \mathcal{K} \cap D_{(K,r)}[0,\infty)$. Since $\mathcal{K}_K^\epsilon \supset \mathcal{K} \cap (D_{(K,r)}[0,\infty))^\epsilon$,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n} \log P\{X_n \notin \mathcal{K}_K^\epsilon\} \\
&\qquad \le \max\{\limsup_{n\to\infty} \frac{1}{n} P\{X_n \notin \mathcal{K}\}, \limsup_{n\to\infty} \frac{1}{n} \log P\{X_n \notin (D_{(K,r)}[0,\infty))^\epsilon\} \\
&\qquad \le -a,
\end{aligned}$$

where the $\epsilon$-"fattening" is with respect to the metric (1.5) given by $r$. It follows that $\{X_n\}$ is exponentially tight, and Part (a) follows by Lemma 3.12.

For Part (b), suppose that $\widehat{I}(x) = \infty$ for each $x \in \widehat{\Gamma}$. If $A$ is an open subset of $D_E[0,\infty)$, then by the equivalence of the topologies, there exists an open subset $\widehat{A}$ of $D_{\widehat{E}}[0,\infty)$ such that $A = \widehat{A} \cap D_E[0,\infty)$. Consequently,

$$\begin{aligned}
\liminf_{n\to\infty} \frac{1}{n} \log P_n(A) &= \liminf_{n\to\infty} \frac{1}{n} \log P_n(\widehat{A} \cap D_E[0,\infty)) \\
&= \liminf_{n\to\infty} \frac{1}{n} \log P_n(\widehat{A}) \\
&\ge -\inf_{x\in\widehat{A}} \widehat{I}(x) \\
&= -\inf_{x\in A} \widehat{I}(x),
\end{aligned}$$

where the last equality follows from the fact that $x \in \widehat{A} - A$ implies $\widehat{I}(x) = \infty$. Similarly, if $B$ is a closed subset of $D_E[0,\infty)$, then there exists a closed subset $\widehat{B}$ of $D_{\widehat{E}}[0,\infty)$ such that $B = \widehat{B} \cap D_E[0,\infty)$. Consequently,

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n} \log P_n(B) &= \limsup_{n\to\infty} \frac{1}{n} \log P_n(\widehat{B} \cap D_E[0,\infty)) \\
&= \limsup_{n\to\infty} \frac{1}{n} \log P_n(\widehat{B}) \\
&\le -\inf_{x\in\widehat{B}} \widehat{I}(x) \\
&= -\inf_{x\in B} \widehat{I}(x),
\end{aligned}$$

and Part (b) follows. ☐

## 4.4. Large deviations in the compact uniform topology

DEFINITION 4.12. [$C$-exponential tightness] A sequence of stochastic processes $\{X_n\}$ that is exponentially tight in $D_E[0,\infty)$ is *$C$-exponentially tight* if for each $\eta > 0$ and $T > 0$,

$$\limsup_{n\to\infty} \frac{1}{n} \log P(\sup_{s\le T} r(X_n(s), X_n(s-)) \ge \eta) = -\infty. \tag{4.18}$$

The weak convergence analogue of this condition is simply that

$$\sup_{s\le T} r(X_n(s), X_n(s-)) \Rightarrow 0$$

for each $T > 0$ which implies that any weak limit point of $\{X_n\}$ must be continuous. The definition of $C$-exponential tightness given here is not the same as that given in Puhalskii [**98**] (Definition 3.2.2 of [**100**]); however, the following theorem shows that the two definitions are equivalent.

THEOREM 4.13. *An exponentially tight sequence* $\{X_n\}$ *in* $D_E[0,\infty)$ *is* $C$*-exponentially tight if and only if each rate function* $I$ *that gives the large deviation principle for a subsequence* $\{X_{n(k)}\}$*, satisfies* $I(x)=\infty$ *for each* $x\in D_E[0,\infty)$ *such that* $x\notin C_E[0,\infty)$*.*

PROOF. Let $A_{\eta,T}=\{x\in D_E[0,\infty):\sup_{s\leq T} r(x(s),x(s-))\geq\eta\}$ and $B_{\eta,T}=\{x\in D_E[0,\infty):\sup_{s<T} r(x(s),x(s-))>\eta\}$. Then $A_{\eta,T}$ is closed and $B_{\eta,T}$ is open. If $\{X_n\}$ is exponentially tight and (4.18) holds, then for any subsequence satisfying the large deviation principle with a rate function $I$ (see Theorem 3.7), we have

$$\begin{aligned}
-\inf_{x\in B_{\eta,T}} I(x) &\leq \liminf_{k\to\infty}\frac{1}{n(k)}\log P(X_{n(k)}\in B_{\eta,T})\\
&\leq \limsup_{k\to\infty}\frac{1}{n(k)}\log P(X_{n(k)}\in A_{\eta,T})=-\infty.
\end{aligned}$$

Therefore, $I(x)=\infty$ for each $x\in B_{\eta,T}$. By the arbitrariness of $\eta$ and $T$, $I(x)=\infty$ for every $x\in D_E[0,\infty)-C_E[0,\infty)$.

Now, assume that $\{X_n\}$ is exponentially tight but that (4.18) does not hold. Then there exists a subsequence $n(k)$ such that

$$\liminf_{k\to\infty}\frac{1}{n(k)}\log P(X_{n(k)}\in A_{\eta,T})>-\infty, \tag{4.19}$$

and by exponential tightness a subsubsequence along which the large deviation principle holds with a rate function $I$. It follows that

$$-\inf_{x\in A_{\eta,T}} I(x)\geq \liminf_{k\to\infty}\frac{1}{n(k)}\log P(X_{n(k)}\in A_{\eta,T})>-\infty,$$

that is, there exists $x\in A_{\eta,T}\subset D_E[0,\infty)-C_E[0,\infty)$ such that $I(x)<\infty$. ☐

With reference to (1.5), for $x,y\in D_E[0,\infty)$, let

$$d_u(x,y)=\int_0^\infty e^{-u}\sup_{t\leq u} q(x(t),y(t))du.$$

Then $d_u$ is a metric corresponding to the compact uniform topology. Let $\mathcal{S}_E$ be the Borel $\sigma$-algebra of $(D_E[0,\infty),d)$. Note that $f_x(y)=d_u(x,y)$ defines a $\mathcal{S}_E$-measurable function and hence $B^u_\epsilon(x)=\{y\in D_E[0,\infty):d_u(x,y)<\epsilon\}$ and $\overline{B}^u_\epsilon(x)=\{y\in D_E[0,\infty):d_u(x,y)\leq\epsilon\}$ are in $\mathcal{S}_E$. On $C_E[0,\infty)$, $d_u$ and $d$ are equivalent metrics, and in the theory of weak convergence, if $X_n\Rightarrow X$ in the Skorohod topology and $X$ is continuous, then $X_n\Rightarrow X$ in the compact uniform topology in the sense that

$$\lim_{n\to\infty}E[f(X_n)]=E[f(X)]$$

for every bounded function that is $d_u$-continuous and $\mathcal{S}_E$-measurable. $C$-exponential tightness is the large deviation analogue of the limit being continuous in weak convergence.

THEOREM 4.14. *Suppose that* $\{P_n\}$ *is a sequence of probability measures on* $D_E[0,\infty)$ *that is* $C$*-exponentially tight and that the large deviation principle holds with rate function* $I$ *(in the Skorohod topology). Then the large deviation principle holds in the compact uniform topology with the same rate function.*

PROOF. The result is an immediate consequence of the contraction principle, Lemma 3.11. Take $S = (D_E[0,\infty), d)$, $S' = (D_E[0,\infty), d_u)$, and $F(x) = x$. Then $F$ is continuous at each point in $C_E[0,\infty)$ and by Theorem 4.13, $I(x) = \infty$ if $x \notin C_E[0,\infty)$. $\square$

## 4.5. Exponential tightness for solutions of martingale problems

DEFINITION 4.15. Let $A$ be the generator of a Markov process, and let $\mathcal{H}_\dagger$ be the collection of $(f,g) \in M(E) \times M(E)$ such that for every solution $X$ of the martingale problem for $A$,

$$Z_{f,g}(t) = e^{f(X(t))-f(Y(0))-\int_0^t g(X(s))ds} \tag{4.20}$$

is a cadlag supermartingale, or, in the discrete time case with $X(t) = Y_{[t/\epsilon]}$,

$$Z_{f,g}(t) = e^{f(X(t))-f(X(0))-\int_0^{[t/\epsilon]\epsilon} g(X(s))ds} \tag{4.21}$$

is a supermartingale.

Of course, $\mathcal{H}$ defined in (1.7) is a subset of $\mathcal{H}_\dagger$, and, for example, if $A$ gives a diffusion and $L$ is the second order differential operator such that $Af = Lf$ for $f \in \mathcal{D}(A)$, then for $f \in C^2$ and $h = e^{-f}Le^f$, (4.20) is a local martingale by Itô's formula. Since every nonnegative local martingale is a supermartingale, $(f, e^{-f}Le^f) \in \mathcal{H}_\dagger$. More generally, if $f_k \in \mathcal{D}(\mathcal{H})$, $\inf_k \inf_x f_k(x) > -\infty$, $\sup_k \sup_x \mathcal{H}f_k(x) < \infty$, and for each $x \in E$, $(f_k(x), \mathcal{H}f_k(x)) \to (f(x), h(x))$, then (4.20) is a supermartingale.

For $n = 1, 2, \ldots$, let $E_n$ be a metric space and let $\eta_n : E_n \to E$ be a Borel measurable mapping into a complete, separable metric space $E$. Let $A_n \subset B(E_n) \times B(E_n)$ and $H_n = \{(f, \frac{1}{n}e^{-nf}g) : f \in B(E_n), (e^{nf}, g) \in A_n\}$. Let $H_{\dagger,n}$ be the collection of $(f,g) \in M(E_n) \times M(E_n)$ such that

$$e^{nf(Y_n(t))-nf(Y_n(0))-\int_0^t ng(Y_n(s))ds}$$

is a supermartingale for every solution $Y_n$ of the martingale problem for $A_n$. Of course, $H_n \subset H_{\dagger,n}$.

Similarly, in discrete time, let $T_n$ be a transition operator on $B(E_n)$, $\epsilon_n > 0$, and

$$H_n = \frac{1}{n\epsilon_n} \log e^{-nf} T_n e^{nf}, \quad f \in B(E_n).$$

Then $H_{\dagger,n}$ is the collection of $(f,g) \in B(E_n) \times B(E_n)$ such that

$$e^{nf(Y_n(t))-nf(Y_n(0))-\int_0^{[t/\epsilon_n]\epsilon_n} ng(Y_n(s))ds}$$

is a supermartingale for every Markov chain with time step $\epsilon_n$ satisfying

$$E[f(Y_n(t))|Y_n(0) = y] = T_n^{[t/\epsilon_n]} f(y).$$

In other words,

$$e^{-g(x)} T_n e^{nf}(x) \le e^{nf(x)}, \quad x \in E_n.$$

As above, $q = r \wedge 1$

DEFINITION 4.16. $D \subset C_b(E)$ *approximates the metric* $q$ if for each compact $K \subset E$ and $z \in K$, there exists $f_n \in D$ such that

$$\lim_{n\to\infty} \sup_{x\in K} |f_n(x) - q(x,z)| = 0.$$

We want to give conditions implying exponential tightness for the sequence $X_n = \eta_n(Y_n)$. For $f \in B(E)$, define

$$\eta_n f = f \circ \eta_n.$$

We assume that $X_n$ is cadlag.

Corollary 4.17. *Let $Y_n$, $X_n$, $H_n$, and $H_{\dagger,n}$ be as above in either the continuous or discrete time case, and in the discrete time case, assume that $\epsilon_n \to 0$. Let $F \subset C_b(E)$ and $S \subset R$. Suppose that*

a) *$\{X_n\}$ satisfies the exponential compact containment condition.*

b) *Either $F$ is closed under addition and separates points in $E$ and $S = R$, or $F$ approximates the metric $q$ and $S = (0, \infty)$.*

c) *For each $\lambda \in S$ and $f \in F$, there exists $\{(f_n, g_n)\}$ such that $(\lambda f_n, g_n) \in H_{\dagger,n}$, $\sup_n \|f_n\| < \infty$, for each compact $K \subset E$,*

$$\lim_{n\to\infty} \sup_{x\in\eta_n^{-1}(K)} |f_n(x) - \eta_n f(x)| = 0 \tag{4.22}$$

*and*

$$\sup_n \sup_{x\in\eta_n^{-1}(K)} g_n(x) = C_\lambda(f, K) < \infty. \tag{4.23}$$

*Then $\{X_n\}$ is exponentially tight.*

Remark 4.18. In many of the examples mentioned in the Introduction, $E_n = E$, $\eta_n$ is the identity map, and it suffices to let $f_n = f$. In others including Examples 1.9 and 1.12, the more general condition is needed.

Proof. First, assume that $F$ is closed under addition and separates points in $E$ and $S = R$. We consider the continuous-time case. The proof for the discrete-time case is essentially the same.

Since for $\lambda > 0$,

$$\begin{aligned} &E[e^{n\lambda|f(X_n(t+s))-f(X_n(t))|}|\mathcal{F}^n_t] \\ &\qquad \le E[e^{n\lambda(f(X_n(t+s))-f(X_n(t)))}|\mathcal{F}^n_t] + E[e^{-n\lambda(f(X_n(t+s))-f(X_n(t)))}|\mathcal{F}^n_t], \end{aligned}$$

by Theorems 4.1 and 4.4, it is enough to show that for each $\lambda \in R$, $T > 0$ and $n$, there exists $\gamma_n(s, \lambda, T)$, increasing in $s$ and satisfying

$$E[\exp\{n\lambda f(X_n(t+s)) - n\lambda f(X_n(t))\}|\mathcal{F}^n_t] \le E[\exp\{\gamma_n(s, \lambda, T)\}|\mathcal{F}^n_t]$$

for $0 \le t \le T$, such that

$$\lim_{s\to 0} \limsup_{n\to\infty} \frac{1}{n} \log E[e^{\gamma_n(s,\lambda,T)}] = 0.$$

For compact $K \subset E$, let $\Gamma_n(K, T) = \{X_n(t) \in K, t \le T\} = \{Y_n(t) \in \eta_n^{-1}(K), t \le T\}$. If $f \in F$ and $(\lambda f_n, g_n) \in H_n$ satisfies Condition (c), then

$$\begin{aligned} &E[I_{\Gamma_n(K,T)} e^{n\lambda f_n(Y_n(t+s))-n\lambda f_n(Y_n(t))}|\mathcal{F}^n_t] \\ &\qquad \le e^{nC_\lambda(f,K)s} E[e^{n\lambda f_n(Y_n(t+s))-n\lambda f_n(Y_n(t))-\int_0^s ng_n(Y_n(t+u))du}|\mathcal{F}^n_t] \\ &\qquad \le e^{nC_\lambda(f,K)s}. \end{aligned} \tag{4.24}$$

By assumption, there exists a compact set $K = K(\lambda, f, T) \subset E$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \log P(\exists t \le T, X_n(t) \notin K) \le -(1 + 2|\lambda|\|f\|).$$

Therefore, for $t, s \geq 0$, $t+s \leq T$,

$$\begin{aligned}
&E[e^{n\lambda f(X_n(t+s))-n\lambda f(X_n(t))}|\mathcal{F}_t^n] \\
&\quad\leq E[e^{n\lambda f(X_n(t+s))-n\lambda f(X_n(t))} I_{\Gamma_n(K,T)}|\mathcal{F}_t^n] \\
&\qquad\qquad +E[e^{n\lambda f(X_n(t+s))-n\lambda f(X_n(t))} I_{\Gamma_n(K,T)^c}|\mathcal{F}_t^n] \\
&\quad\leq e^{2n|\lambda| \sup_{y\in\eta_n^{-1}(K)}|\eta_n f(y)-f_n(y)|} E[I_{\Gamma_n(K,T)} e^{n\lambda f_n(Y_n(t+s))-n\lambda f_n(Y_n(t))}|\mathcal{F}_t^n] \\
&\qquad\qquad +e^{2n|\lambda|\|f\|} P\{\Gamma_n(K,T)^c|\mathcal{F}_t^n\} \\
&\quad\leq e^{2n|\lambda| \sup_{y\in\eta_n^{-1}(K)}|f_n(y)-\eta_n f(y)|+nC_\lambda(f,K)s} + e^{2n\lambda\|f\|} P\{\Gamma_n(K,T)^c|\mathcal{F}_t^n\}.
\end{aligned}$$

Taking

$$\gamma_n(s,\lambda,T) = \log\{e^{2n|\lambda| \sup_{y\in\eta_n^{-1}(K)}|f_n(y)-\eta_n f(y)|+nC_\lambda(f,K)s} + e^{2n|\lambda|\|f\|} I_{\Gamma_n(K,T)^c}\},$$

then

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n}\log E[e^{\gamma_n(s,\lambda,T)}] \\
&\qquad = \max\{C_\lambda(f,K)s, 2|\lambda|\|f\| + \limsup_{n\to\infty}\frac{1}{n}\log P\{\Gamma_n(K,T)^c\}\} \\
&\qquad = C_\lambda(f,K)s.
\end{aligned}$$

Hence, by Theorems 4.1 and 4.4, the conclusion follows.

Now assume that $F$ approximates the metric $q$ and $S=(0,\infty)$. For $\lambda \in S$, select compact $K \subset E$ so that

$$\limsup_{n\to\infty}\frac{1}{n}\log P(\exists t \leq T, X_n(t) \notin K) \leq -(1+\lambda).$$

For each $\epsilon > 0$, there exist $N_\epsilon > 0$, $\{\xi_1, \ldots, \xi_{N_\epsilon}\} \subset E$, and $\{f_1^\epsilon, \ldots, f_{N_\epsilon}^\epsilon) \in F$ such that $\min_{k=1,\ldots,N_\epsilon} q(x,\xi_k) < \epsilon$, for all $x \in K$, and $\max_{1\leq k\leq N_\epsilon}\sup_{z\in K}|q(z,\xi_k) - f_k(z)| \leq \epsilon$.

For $y \in K$, there exists $\xi_k \in \{\xi_1, \ldots, \xi_N\}$ such that $q(y,\xi_k) < \epsilon$. Then, for $x \in K$,

$$q(x,y) \leq q(x,\xi_k) + q(y,\xi_k) \leq q(x,\xi_k) + \epsilon \leq q(x,\xi_k) - q(y,\xi_k) + 2\epsilon,$$

and

$$q(x,y) \leq f_k^\epsilon(x) - f_k^\epsilon(y) + 4\epsilon.$$

It follows that

$$e^{n\lambda q(x,y)} \leq e^{4n\lambda\epsilon}\sum_{k=1}^{N(\epsilon)} e^{n\lambda f_k(x) - n\lambda f_k(y)}.$$

By (4.22) and (4.23), there exist $(\lambda f_{n,k}^\epsilon, g_{n,k}^\epsilon) \in H_{\dagger,n}$ such that for $n$ sufficiently large

$$\max_k \sup_{y\in\eta_n^{-1}(K)} |f_{n,k}^\epsilon(y) - \eta_n f_k^\epsilon(y)| \leq \epsilon$$

and $C_\lambda(f_k^\epsilon, K) \equiv \max_k \sup_{y \in \eta_n^{-1}(K)} |g_{n,k}^\epsilon(y)| < \infty$, and hence,

$$\begin{aligned} & I_{\Gamma_n(K,T)} e^{n\lambda q(X_n(t+s),X_n(t))} \\ & \quad \leq I_{\Gamma_n(K,T)} e^{6\epsilon n\lambda} \sum_{k=1}^{N_\epsilon} e^{n\lambda f_{n,k}^\epsilon(Y_n(t+s)) - n\lambda f_{n,k}^\epsilon(Y_n(t))} \\ & \quad \leq I_{\Gamma_n(K,T)} e^{6\epsilon n\lambda + sn \vee_k C_\lambda(f_k^\epsilon, K)} \\ & \qquad\qquad \sum_{k=1}^{N_\epsilon} e^{n\lambda f_{n,k}^\epsilon(Y_n(t+s)) - n\lambda f_{n,k}^\epsilon(Y_n(t)) - \int_t^{t+s} n g_{n,k}^\epsilon(Y_n(r))dr} \\ & \quad \leq e^{6\epsilon n\lambda + sn \vee_k C_\lambda(f_k^\epsilon, K)} \sum_{k=1}^{N_\epsilon} e^{n\lambda f_{n,k}^\epsilon(Y_n(t+s)) - n\lambda f_{n,k}^\epsilon(Y_n(t)) - \int_t^{t+s} n g_{n,k}^\epsilon(Y_n(r))dr}. \end{aligned}$$

Therefore

$$\begin{aligned} E[e^{n\lambda q(X_n(t+s),X_n(t))} | \mathcal{F}_t^n] & \leq E[I_{\Gamma_n(K,T)} e^{n\lambda q(X_n(t+s),X_n(t))} | \mathcal{F}_t^n] \\ & \qquad + E[I_{\Gamma_n(K,T)^c} e^{n\lambda q(X_n(t+s),X_n(t))} | \mathcal{F}_t^n] \\ & \leq e^{6\epsilon n\lambda + sn \vee_{k=1}^{N_\epsilon} C_\lambda(f_k^\epsilon, K)} + e^{n\lambda} P\{\Gamma_n(K,T)^c | \mathcal{F}_t^n\}. \end{aligned}$$

For $\delta > 0$, let $\epsilon_0(\delta) = \inf\{\epsilon > 0 : \vee_{k=1}^{N_\epsilon} C_\lambda(f_k^\epsilon, K) \leq \delta^{-1/2}\}$. Note that $\epsilon_0(\delta) \to 0$ as $\delta \to 0$, and select $\epsilon(\delta)$ such that $\epsilon(\delta) < \epsilon_0(\delta) + \delta$ and $\vee_{k=1}^{N_{\epsilon(\delta)}} C_\lambda(f_k^{\epsilon(\delta)}, K) \leq \delta^{-1/2}$. Finally, define

$$\gamma_n(\delta, \lambda, T) = \log\{e^{6\epsilon(\delta)n\lambda + \delta n \vee_{k=1}^{N_{\epsilon(\delta)}} C_\lambda(f_k^{\epsilon(\delta)}, K)} + e^{n\lambda} I_{\Gamma_n(K,T)^c}\},$$

and observe that

$$\begin{aligned} & \limsup_{n\to\infty} \frac{1}{n} \log E[e^{\gamma_n(s,\lambda,T)}] \\ & \quad = \max\{6\epsilon(\delta)\lambda + \delta \vee_{k=1}^{N_{\epsilon(\delta)}} C_\lambda(f_k^{\epsilon(\delta)}, K), \lambda + \limsup_{n\to\infty} \frac{1}{n} \log P(\Gamma_n(K,T)^c)\} \\ & \quad = 6\epsilon(\delta)\lambda + \delta \vee_{k=1}^{N_{\epsilon(\delta)}} C_\lambda(f_k^{\epsilon(\delta)}, K). \end{aligned}$$

Since the right side goes to zero as $\delta \to 0$, the conclusion follows by Theorem 4.1. □

The following technical modification of Corollary 4.17 is occasionally useful. The proof is essentially the same.

COROLLARY 4.19. *Let $Y_n$, $X_n$, $H_n$, and $H_{\dagger,n}$ be as above in either the continuous or discrete time case, and in the discrete time case, assume that $\epsilon_n \to 0$. Let $F \subset C_b(E)$ and $S \subset R$. Let $\mathcal{Q}$ be an index set and for $n = 1, 2, \ldots$ and $q \in \mathcal{Q}$, let $K_n^q$ be a measurable subset of $E_n$. Suppose that*

a) *For each $q \in \mathcal{Q}$, $\cup_n \eta_n(K_n^q)$ is relatively compact in $E$.*
b) *For each $a > 0$ and $T > 0$, there exists $q(a,T) \in \mathcal{Q}$ such that*

$$\limsup_{n\to\infty} \frac{1}{n} \log P\{\exists t \leq T \ni Y_n(t) \notin K_n^{q(a,T)}\} \leq -a. \tag{4.25}$$

*(In particular, $\{X_n\}$ satisfies the exponential compact containment condition.)*

c) *Either $F$ is closed under addition and separates points in $E$ and $S = R$, or $F$ approximates the metric $q$ and $S = (0, \infty)$.*

d) *For each $\lambda \in S$ and $f \in F$, there exists $\{(f_n, g_n)\}$ such that $(\lambda f_n, g_n) \in H_{\dagger,n}$, $\sup_n \|f_n\| < \infty$, for each $q \in \mathcal{Q}$,*

$$\lim_{n\to\infty} \sup_{x \in K_n^q} |f_n(x) - \eta_n f(x)| = 0 \tag{4.26}$$

*and*

$$\sup_n \sup_{x\in K_n^q} g_n(x) = C_\lambda(f, K) < \infty. \tag{4.27}$$

*Then $\{X_n\}$ is exponentially tight.*

## 4.6. Verifying compact containment

The usual approach to verifying a property like the exponential compact containment condition (Condition (a) of Theorem 4.4 and Corollary 4.17) is with a Lyapunov function technique.

Lemma 4.20. *Let $K$ be compact and $G \supset K$ be open. Let $A$ be the generator for a Markov process, and let $\mathcal{H}_\dagger$ be given by Definition 4.15. Let $(f, g) \in \mathcal{H}_\dagger$, and define*

$$\beta \equiv \inf_{x\in G^c} f(x) - \sup_{x\in K} f(x), \quad \gamma \equiv \sup_{x\in G} g(x) \vee 0.$$

*Let $X$ be a solution of the martingale problem for $A$, and suppose that $Z_{f,g}$ defined in (4.20) is right continuous. Then for $T > 0$,*

$$P\{X(0) \in K, X(t) \notin G \text{ some } t \le T\} \le P\{X(0) \in K\} e^{-\beta + T\gamma}. \tag{4.28}$$

Remark 4.21. Suppose, as in Corollary 4.17, $Y$ takes valued in $E_0$, $\eta : E_0 \to E$, $X = \eta(Y)$, and $(f, g) \in \mathcal{H}_\dagger \subset M(E_0) \times M(E_0)$. If $K \subset E$ is compact, $G \supset K$ is open and

$$\beta \equiv \inf_{y\in \eta^{-1}(G)^c} f(y) - \sup_{y\in\eta^{-1}(K)} f(y), \quad \gamma \equiv \sup_{y \in \eta^{-1}(G)} g(y),$$

then for $T > 0$,

$$P\{X(0) \in K, X(t) \notin G \text{ some } t \le T\} \le P\{X(0) \in K\} e^{-\beta + T\gamma}. \tag{4.29}$$

Proof. Let

$$\tau = I_{\{X(0)\in K\}} \inf\{t : X(t) \notin G\}.$$

Then, since $Z_{f,g}$, given in (4.20), is a right continuous supermartingale, by the optional sampling theorem

$$\begin{aligned} P\{X(0) \in K, X(t) \notin G \text{ some } t \le T\} e^{\beta - T\gamma} + P\{X(0) \notin K\} &\le E[Z_{f,g}(\tau \wedge T)] \\ &\le 1, \end{aligned}$$

and (4.28) follows. □

The following lemma is an immediate consequence of the previous lemma.

Lemma 4.22. *Let $X_n$ be a solution of the martingale problem for $A_n$. Let $K$ be compact, and let $G \supset K$ be open. Let $(f_n, g_n) \in H_{\dagger,n} = \{(f, g) : (nf, ng) \in \mathcal{H}_{\dagger,n}\}$, and assume that the corresponding supermartingale is right continuous. Define*

$$\beta(K, G) \equiv \liminf_{n\to\infty} (\inf_{x\in G^c} f_n(x) - \sup_{x\in K} f_n(x)), \quad \gamma(G) \equiv \limsup_{n\to\infty} \sup_{x\in G} g_n(x).$$

*Then*

$$\limsup_{n\to\infty}\frac{1}{n}\log P\{X_n(t)\notin G \text{ some } t\le T\}$$
$$\le \max\{-\beta(K,G)+T\gamma(G),\limsup_{n\to\infty}\frac{1}{n}\log P\{X_n(0)\notin K\}\}.$$

If $E$ is locally compact, Lemma 4.22 can be applied to open $G$ with compact closure to verify the exponential compact containment condition.

Example 4.23. Suppose in Example 1.5 there exists $C>0$ such that

$$\sum_i x_ib_i(x)\le C(1+|x|^2),\quad |a_{ij}(x)|\le C(1+|x|^2), \tag{4.30}$$

and there exists $\alpha>0$ such that

$$C(\eta,\alpha)\equiv\sup_{x\in R^d}\int_{R^d}\left(\frac{|z|}{1+|x|}\right)^2 e^{\alpha\frac{|z|}{1+|x|}}\eta(x,dz)<\infty. \tag{4.31}$$

Then for $\delta>0$ satisfying $6\delta<\alpha$, let $f_n(x)=f(x)=\delta\log(1+|x|^2)$ and

$$g_n(x)=\frac{1}{2n}\sum_{ij}a_{ij}(x)\partial_i\partial_jf(x)+\frac{1}{2}\sum_{ij}a_{ij}(x)\partial_if(x)\partial_jf(x)+\sum_i b_i(x)\partial_if(x)$$
$$+\int_{R^d}(e^{n\delta\log\left(1+\frac{n^{-1}2x\cdot z+n^{-2}|z|^2}{1+|x|^2}\right)}-1-\frac{2\delta x\cdot z}{1+|x|^2})\eta(x,dz).$$

The derivative terms are uniformly bounded in $x$ and $n$ by a constant depending on $C$ in (4.30) and $\delta$. The integral term satisfies

$$\int_{R^d}(e^{n\delta\log\left(1+\frac{n^{-1}2x\cdot z+n^{-2}|z|^2}{1+|x|^2}\right)}-1-\frac{2\delta x\cdot z}{1+|x|^2})\eta(x,dz)$$
$$\le\int_{R^d}(e^{n\delta\log\left(1+\frac{n^{-1}2x\cdot z+n^{-2}|z|^2}{1+|x|^2}\right)}-e^{\frac{2\delta x\cdot z}{1+|x|^2}})\eta(x,dz)$$
$$+\int_{R^d}(e^{\frac{2\delta x\cdot z}{1+|x|^2}}-1-\frac{2\delta x\cdot z}{1+|x|^2})\eta(x,dz).$$

The second term on the right is bounded by $2\delta^2C(\eta,4\delta)<\infty$. Setting $\Gamma_{n,x}=\{z:|z|>n(1+|x|)\}$, the first term on the right is bounded by

$$\int_{\{z:|z|\le n(1+|x|)\}}e^{\delta\frac{|2x\cdot z|+n^{-1}|z|^2}{1+|x|^2}}\frac{\delta|z|^2}{n(1+|x|^2)}\eta(x,dz)$$
$$+\int_{\{z:|z|>n(1+|x|)\}}\left(1+\frac{n^{-1}2x\cdot z+n^{-2}|z|^2}{1+|x|^2}\right)^{n\delta}\eta(x,dz)$$
$$\le\frac{2\delta}{n}C(\eta,6\delta)+\int_{\Gamma_{n,x}}\left(1+\frac{4}{n}\frac{|z|}{1+|x|}+\frac{2}{n^2}\left(\frac{|z|}{1+|x|}\right)^2\right)^{n\delta}\eta(x,dz)$$
$$\le\frac{2\delta}{n}C(\eta,6\delta)+\int_{\Gamma_{n,x}}e^{4\delta\frac{|z|}{1+|x|}}\eta(x,dz)$$
$$\le\frac{2\delta}{n}C(\eta,6\delta)+\frac{1}{n^2}C(\eta,4\delta),$$

where $C(\eta,4\delta)\le C(\eta,6\delta)<\infty$.

It follows that $(f_n, g_n) \in H_{\dagger,n}$ and for any bounded $G$,

$$\gamma(G) \leq \limsup_{n\to\infty} \sup_{x\in R^d} g_n(x) < \infty.$$

Suppose $K = \{x : |x| \leq c\}$. For $\beta > 0$, let $d$ satisfy $\delta \log(1+d^2) = \beta + \delta \log(1+c^2)$. Then for

$$G = \{x : |x| < d\},$$

the conditions of Lemma 4.22 are satisfied. Consequently, if $\{X_n(0)\}$ is exponentially tight, the exponential compact containment condition holds for $\{X_n\}$.

Furthermore, if (4.31) holds for all $\alpha > 0$, Conditions (b) and (c) of Corollary 4.17 are easy to check, and hence $\{X_n\}$ is exponentially tight.

If $E$ is not locally compact, then, with Lemma 3.3 in mind, it may be useful to work with a sequence of open sets $\{G_k\}$.

LEMMA 4.24. *Let $X_n$ be a solution of the martingale problem for $A_n$. Let $K \subset E$ be compact, and for $k = 1, 2, \ldots$, let $G_k$ be open and contained in a finite union of open balls of radius $k^{-1}$. Let $\widehat{K}$ be the closure of $\cap_k G_k$. (Note that $\widehat{K}$ will be complete and totally bounded, hence compact.) Assume that $K \subset \widehat{K}$ Let $(f_{k,n}, g_{k,n}) \in H_{\dagger,n}$, and assume that the corresponding supermartingale is right continuous. Define*

$$\beta_{k,n} \equiv \inf_{x\in G_k^c} f_{k,n}(x) - \sup_{x\in K} f_{k,n}(x), \quad \gamma_{k,n} \equiv \sup_{x\in G_k} g_{k,n}(x).$$

*If $\gamma = \sup_{k,n} \gamma_{k,n} < \infty$ and $\beta_{k,n} \geq \beta + \frac{2}{n}\log k$, then*

$$\limsup_{n\to\infty} \frac{1}{n} \log P\{X_n(t) \notin \widehat{K} \text{ some } t \leq T\}$$
$$\leq \max\{-\beta + \gamma T, \limsup_{n\to\infty} \frac{1}{n} \log P\{X_n(0) \notin K\}\}.$$

PROOF. By Lemma 4.20,

$$P\{X_n(0) \in K, X_n(t) \notin G_k \text{ some } t \leq T\} \leq P\{X_n(0) \in K\} e^{-n\beta_{k,n} + nT\gamma_{k,n}},$$

and hence

$$\limsup_{n\to\infty} \frac{1}{n} \log P\{X_n(t) \notin \widehat{K} \text{ some } t \leq T\} \tag{4.32}$$
$$\leq \max\{\limsup_{n\to\infty} \frac{1}{n} \log \sum_k e^{-n\beta_{k,n} + n\gamma_{k,n}T}, \limsup_{n\to\infty} \frac{1}{n} \log P\{X_n(0) \notin K\}\}.$$

Under the assumptions on $\gamma$ and $\beta$, the right side of (4.32) is bounded by

$$\max\{-\beta + \gamma T, \limsup_{n\to\infty} \frac{1}{n} \log P\{X_n(0) \notin K\}\}.$$

□

## 4.7. Finite dimensional determination of the process rate function

The following lemma shows that finite dimensional rate functions give lower bounds for process rate functions.

LEMMA 4.25. *Let* $\{X_n\}$ *satisfy the large deviation principle with rate function* $I$. *Let* $x \in D_E[0,\infty)$, *and let* $\Delta_x$ *denote the set of discontinuities of* $x$. *If* $t_1,\ldots,t_m \in \Delta_x^c$ *and the large deviation principle holds for* $\{(X_n(t_1),\ldots,X_n(t_m))\}$ *with rate function* $I_{t_1,\ldots,t_m}$, *then*

$$I(x) \geq I_{t_1,\ldots,t_m}(x(t_1),\ldots,x(t_m)). \tag{4.33}$$

*Consequently, if for each finite subset* $\{t_1,\ldots,t_m\} \subset \mathcal{T}_0$, *the large deviation principle holds for* $\{(X_n(t_1),\ldots,X_n(t_m))\}$ *with rate function* $I_{t_1,\ldots,t_m}$, *then*

$$I(x) \geq \sup_{\{t_i\}\subset\mathcal{T}_0\cap\Delta_x^c} I_{t_1,\ldots,t_m}(x(t_1),\ldots,x(t_m)). \tag{4.34}$$

PROOF. For $\epsilon > 0$ and $t_1,\ldots,t_m \notin \Delta_x$, there exists $\epsilon' > 0$ such that

$$B_{\epsilon'}(x) \subset \{y : (y(t_1),\ldots,y(t_m)) \in B_\epsilon^m((x(t_1),\ldots,x(t_m)))\}$$

where $B_\epsilon^m$ denotes a ball in $E^m$. Consequently, (4.33) follows from (3.3). □

The assumption that $t_1,\ldots,t_m \in \Delta_x^c$ can be dropped in (4.33) if the right side is replaced by the minimum of $I_{t_1,\ldots,t_m}(y_1,\ldots,y_m)$ over the $2^m$ choices where $y_i$ is $x(t_i)$ or $x(t_i-)$. This minimum is in fact needed unless other conditions are introduced. For example, let $P\{X_n = I_{[1+1/n,\infty)}\} = 1$. Then for $x = I_{[1,\infty)}$, $I(x) = 0$, but $I_1(1) = \infty$.

Our next goal is to estimate $I$ from above in terms of finite dimensional rate functions. Let $y \in D_E[0,\infty)$. Suppose that $0 \leq t_1 < t_2 < \cdots$ satisfies $\lim_{i\to\infty} t_i = \infty$, and define $Py$ by

$$Py(t) = y(t_1), \quad 0 \leq t < t_2,$$

and for $t \geq t_2$,

$$Py(t) = y(t_i), \quad t_i \leq t < t_{i+1}.$$

Note that if we set $\zeta(t) = t_1$ for $0 \leq t < t_2$ and $\zeta(t) = t_i$ for $t_i \leq t < t_{i+1}$, $i = 2,3,\ldots$, then $Py(t) = y \circ \zeta(t)$.

LEMMA 4.26. *Let* $\mathcal{T}_0$ *be dense in* $[0,\infty)$. *Let* $x \in D_E[0,\infty)$. *For* $\epsilon > 0$ *and compact* $K \subset D_E[0,\infty)$, *there exist* $t_1,\ldots,t_m \in \mathcal{T}_0$ *and* $\epsilon' > 0$ *such that*

$$B_\epsilon(x) \supset \{y : (y(t_1),\ldots,y(t_m)) \in B_{\epsilon'}^m((x(t_1),\ldots,x(t_m)))\} \cap K. \tag{4.35}$$

PROOF. Without loss of generality, we can assume that there exist compact $\Gamma_T \subset E$ and a function $h(\delta,T) > 0$ defined for $T,\delta > 0$ satisfying $\lim_{\delta\to0} h(\delta,T) = 0$ such that

$$K = \{y : y(t) \in \Gamma_T, t \leq T, w'(y,\delta,T+\delta) \leq h(\delta,T)\}.$$

Let $2\eta < \delta$, and let $0 \leq t_1 < t_2 < \cdots$ satisfy $t_i \in \mathcal{T}_0$, $t_i \to \infty$, $t_2 \leq \eta$, and $t_{i+1} - t_i \leq \eta$. Select $m$ so that $t_m \geq T + \delta$. If $y$ is in the set on the right of (4.35), then by Lemma 4.8,

$$\begin{aligned} d(x,y) &\leq d(x,Px) + d(Px,Py) + d(Py,y) \\ &\leq d(x,Px) + \epsilon' + (w'(y,\delta,T+\eta) + \frac{\eta(T+\eta)}{\delta} + e^{-T}) \vee \log\frac{\delta}{\delta-2\eta} \\ &\leq d(x,Px) + \epsilon' + (h(\delta,T) + \frac{\eta(T+\eta)}{\delta} + e^{-T}) \vee \log\frac{\delta}{\delta-2\eta}. \end{aligned}$$

Select $\epsilon' < \epsilon/5$, $T$ so that $e^{-T} \leq \epsilon'$, and $\delta$ so that $h(\delta,T) \leq \epsilon'$. Then select $\eta$ so that $d(x,Px) \leq \epsilon'$, $\eta(T+\eta)/\delta \leq \epsilon'$, and $\log(\delta/(\delta-2\eta)) \leq 3\epsilon'$. Then $d(x,y) < \epsilon$ for all $y$ in the set on the right of (4.35). □

LEMMA 4.27. *Let $\{X_n\}$ be exponentially tight in $D_E[0,\infty)$, and assume that the large deviation principle holds with rate function $I$ (otherwise select a subsequence). Let $\mathcal{T}_0$ be dense in $[0,\infty)$, and suppose that for each finite subset $\{t_1,\ldots,t_m\}\subset\mathcal{T}_0$, the large deviation principle holds for $\{(X_n(t_1),\ldots,X_n(t_m))\}$ with rate function $I_{t_1,\ldots,t_m}$. Then either $\liminf_{n\to\infty}\frac{1}{n}\log P\{X_n\in B_\epsilon(x)\}=-\infty$ and (by exponential tightness)*

$$\sup_{\{t_i\}\subset\mathcal{T}_0} I_{t_1,\ldots,t_m}(x(t_1),\ldots,x(t_m))=\infty=I(x) \tag{4.36}$$

*or*

$$\liminf_{n\to\infty}\frac{1}{n}\log P\{X_n\in B_\epsilon(x)\}\geq -I_{t_1,\ldots,t_m}(x(t_1),\ldots,x(t_m)) \tag{4.37}$$

*and*

$$I(x)\leq \sup_{\{t_i\}\subset\mathcal{T}_0} I_{t_1,\ldots,t_m}(x(t_1),\ldots,x(t_m)) \tag{4.38}$$

PROOF. Lemma 4.26 implies that for each $\epsilon>0$ and compact set $K\subset D_E[0,\infty)$, there exists $t_1,\ldots,t_m\in\mathcal{T}_0$ and $\epsilon'>0$ such that

$$\begin{aligned}
&\liminf_{n\to\infty}\frac{1}{n}\log P\{X_n\in B_\epsilon(x)\}\vee\liminf_{n\to\infty}\frac{1}{n}\log P\{X_n\in K^c\}\\
&\quad=\liminf_{n\to\infty}\frac{1}{n}\log P(\{X_n\in B_\epsilon(x)\cup K^c\})\}\\
&\quad\geq\liminf_{n\to\infty}\frac{1}{n}\log P\{(X(t_1),\ldots,X(t_m))\in B^m_{\epsilon'}(x(t_1),\ldots,x(t_m))\}\\
&\quad\geq -I_{t_1,\ldots,t_m}(x(t_1),\ldots,x(t_m)).
\end{aligned}$$

Since for each $a>0$ there exists a compact $K_a$ such that

$$\limsup_{n\to\infty}\frac{1}{n}\log P\{X_n\in K_a^c\}\leq -a,$$

the lemma follows. □

Lemmas 4.25 and 4.27 give the following theorem.

THEOREM 4.28. *Assume that $\{X_n\}$ is exponentially tight in $D_E[0,\infty)$ and that for each $0\leq t_1<t_2<\cdots<t_m$, $\{(X_n(t_1),\ldots,X_n(t_m))\}$ satisfies the large deviation principle in $E^m$ with rate function $I_{t_1,\ldots,t_m}$. Then $\{X_n\}$ satisfies the large deviation principle in $D_E[0,\infty)$ with good rate function*

$$I(x)=\sup_{\{t_i\}\subset\Delta_x^c} I_{t_1,\ldots,t_m}(x(t_1),\ldots,x(t_m)). \tag{4.39}$$

PROOF. Let $I$ be the rate function corresponding to a subsequence of $\{X_n\}$ for which the large deviation principle holds. Exponential tightness implies $I$ will be good. Taking $\mathcal{T}_0=\Delta_x^c$, Lemmas 4.25 and 4.27 imply that $I$ satisfies (4.39), which in turn implies that there is only one possible rate function and hence the large deviation principle holds for the full sequence $\{X_n\}$ with rate function given by (4.39). □

The following corollary is a consequence of Lemma 3.23.

COROLLARY 4.29. *Suppose that $D \subset C_b(E)$ is bounded above and isolates points. (See Definition 3.18.) Assume that $\{X_n\}$ is exponentially tight in $D_E[0,\infty)$ and that for each $0 \leq t_1 \leq \ldots \leq t_m$ and $f_1,\ldots,f_m \in D$*

$$\Lambda(t_1,\ldots,t_m,f_1,\ldots,f_m) = \lim_{n\to\infty} \frac{1}{n} \log E[e^{n(f_1(X_n(t_1))+\cdots+f_m(X_n(t_m)))}]$$

*exists. Then $\{X_n\}$ satisfies the large deviation principle in $D_E[0,\infty)$ with good rate function*

$$I(x) = \sup_m \sup_{\{t_1,\ldots,t_m\}\subset\Delta_x^c} \sup_{f_1,\ldots,f_m\in D} \{f_1(x(t_1)) + \cdots + f_m(x(t_m)) \\ -\Lambda(t_1,\ldots,t_m,f_1,\ldots,f_m)\}.$$

If $C$-exponential tightness holds, then Theorem 4.28 can be simplified. The next theorem is essentially Puhalskii [**97**], Theorem 4.5 (Theorem 3.2.8 of [**100**]). He only considers $R^d$-valued processes, but his proof extends easily to the metric-valued case. See also de Acosta [**24**] for generalizations to projective systems.

THEOREM 4.30. *Assume that $\{X_n\}$ is $C$-exponentially tight in $D_E[0,\infty)$. Let $\mathcal{T}_0$ be a dense subset of $[0,\infty)$, and suppose that for each $0 \leq t_1 \leq \ldots \leq t_m \in \mathcal{T}_0$, $\{(X_n(t_1),\ldots,X_n(t_m))\}$ satisfies the large deviation principle in $E^m$ with rate function $I_{t_1,\ldots,t_m}$. Then $\{X_n\}$ satisfies the large deviation principle in $D_E[0,\infty)$ with good rate function*

$$I(x) = \sup_{\{t_i\}\subset\mathcal{T}_0} I_{t_1,\ldots,t_m}(x(t_1),\ldots,x(t_m)). \tag{4.40}$$

PROOF. By $C$-exponential tightness, if $x \notin C_E[0,\infty)$, then $I(x) = \infty$ and (4.40) follows by (4.36). Otherwise, (4.40) follows by (4.34) and (4.38). $\square$

# Part 2

# Large deviations for Markov processes and semigroup convergence