1. Stochastic equations for Markov processes

- Filtrations and the Markov property
- Ito equations for diffusion processes
- Poisson random measures
- Ito equations for Markov processes with jumps

Filtrations and the Markov property

$(\Omega, \mathcal{F}, P)$ a probability space

Available information is modeled by a sub-σ-algebra of $\mathcal{F}$

$\mathcal{F}_t$ information available at time t

$\{\mathcal{F}_t\}$ is a *filtration.* $\quad t < s$ implies $\mathcal{F}_t \subset \mathcal{F}_s$

A stochastic process X is *adapted* to $\{\mathcal{F}_t\}$ if $X(t)$ is $\mathcal{F}_t$-measurable for each $t \geq 0$.

An E-valued stochastic process X adapted to $\{\mathcal{F}_t\}$ is $\{\mathcal{F}_t\}$-*Markov* if

$$E[f(X(t+r))|\mathcal{F}_t] = E[f(X(t+r))|X(t)], \quad t, r \geq 0, \ f \in B(E)$$

An $\mathbb{R}$-valued stochastic process M adapted to $\{\mathcal{F}_t\}$ is an $\{\mathcal{F}_t\}$-*martingale* if

$$E[M(t+r)|\mathcal{F}_t] = M(t), \quad t, r \geq 0$$

τ is an $\{\mathcal{F}_t\}$-*stopping time* if for each $t \geq 0$, $\{\tau \leq t\} \in \mathcal{F}_t$. For a stopping time τ,

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : \{\tau \leq t\} \cap A \in \mathcal{F}_t, t \geq 0\}$$

Ito integrals

W *standard Browian motion* (W has independent increments with $W(t+s)-W(t)$ normally distributed with mean zero and variance s.)

W is *compatible* with a filtration $\{\mathcal{F}_t\}$ if W is $\{\mathcal{F}_t\}$-adapted and $W(t+s)-W(t)$ is independent of $\mathcal{F}_t$ for all $s,t \geq 0$.

If X is $\mathbb{R}$-valued, cadlag and $\{\mathcal{F}_t\}$-adapted, then

$$\int_0^t X(s-)dW(s) = \lim_{\max(t_{i+1}-t_i)\to 0} \sum X(t_i)(W(t\wedge t_{i+1})-W(t\wedge t_i))$$

exists, and the integral satisfies

$$E[(\int_0^t X(s-)dW(s))^2] = E[\int_0^t X(s)^2 ds]$$

if the right side is finite.

The integral extends to all measurable and adapted processes satisfying

$$\int_0^t X(s)^2 ds < \infty$$

W is an $\mathbb{R}^m$-valued *standard Brownian motion* if $W=(W_1,\ldots,W_m)^T$ for $W_1,\ldots,W_m$ independent one-dimensional standard Brownian motions.

Ito equations for diffusion processes

$\sigma : \mathbb{R}^d \to \mathbb{M}^{d\times m}$ and $b : \mathbb{R}^d \to \mathbb{R}^d$

Consider

$$X(t) = X(0) + \int_0^t \sigma(X(s))dW(s) + \int_0^t b(X(s))ds$$

where $X(0)$ is independent of the standard Brownian motion W.

Why is X Markov? Suppose W is compatible with $\{\mathcal{F}_t\}$, X is $\{\mathcal{F}_t\}$-adapted, and X is unique (among $\{\mathcal{F}_t\}$-adapted solutions). Then

$$X(t+r) = X(t) + \int_t^{t+r} \sigma(X(s))dW(s) + \int_t^{t+r} b(X(s))ds$$

and $X(t+r) = H(t,r,X(t),W_t)$ where $W_t(s) = W(t+s) - W(t)$. Since W_t is independent of $\mathcal{F}_t$,

$$E[f(X(t+r))|\mathcal{F}_t] = \int f(H(t,r,X(t),w))\mu_W(dw).$$

Gaussian white noise.

(U, d_U) a complete, separable metric space; $\mathcal{B}(U)$, the Borel sets

μ a (Borel) measure on U

$\mathcal{A}(U) = \{A \in \mathcal{B}(U) : \mu(A) < \infty\}$

$W(A,t)$ Mean zero, Gaussian process indexed by $\mathcal{A}(U) \times [0,\infty)$

$E[W(A,t)W(B,s)] = t \wedge s\mu(A \cap B)$,

$W(\varphi,t) = \int \varphi(u)W(du,t)$

$\varphi(u) = \sum a_i I_{A_i}(u)$

$W(\varphi,t) = \sum_i a_i W(A_i,t)$

$E[W(\varphi_1,t)W(\varphi_2,s)] = t \wedge s \int_U \varphi_1(u)\varphi_2(u)\mu(du)$

Define $W(\varphi,t)$ for all $\varphi \in L_2(\mu)$.

Definition of integral

$X(t) = \sum_i \xi_i(t)\varphi_i$ process in $L_2(\mu)$

$I_W(X,t) = \int_{U\times[0,t]} X(s,u)W(du\times ds) = \sum_i \int_0^t \xi_i(s)dW(\varphi_i, s)$

$$\begin{aligned} E[I_W(X,t)^2] &= E\left[\sum_{i,j}\int_0^t \xi_i(s)\xi_j(s)ds\int_U \varphi_i\varphi_j d\mu\right] \\ &= E\left[\int_0^t\int_U X(s,u)^2\mu(du)ds\right] \end{aligned}$$

The integral extends to adapted processes satisfying

$$\int_0^t\int_U X(s,u)^2\mu(du)ds < \infty \quad a.s.$$

so that

$$(I_W(X,t))^2 - \int_0^t\int_U X(s,u)^2\mu(du)ds$$

is a local martingale.

Poisson random measures

ν a σ-finite measure on U, (U, d_U) a complete, separable metric space.

ξ a Poisson random measure on $U \times [0, \infty)$ with mean measure $\nu \times m$.

For $A \in \mathcal{B}(U) \times \mathcal{B}([0, \infty))$, $\xi(A)$ has a Poisson distribution with expectation $\nu \times m(A)$ and $\xi(A)$ and $\xi(B)$ are independent if $A \cap B = \emptyset$.

$\xi(A, t) \equiv \xi(A \times [0, t])$ is a Poisson process with parameter $\nu(A)$.

$\widetilde{\xi}(A, t) \equiv \xi(A \times [0, t] - \nu(A)t$ is a martingale.

ξ is $\{\mathcal{F}_t\}$ compatible, if for each $A \in \mathcal{B}(U)$, $\xi(A, \cdot)$ is $\{\mathcal{F}_t\}$ adapted and for all $t, s \geq 0$, $\xi(A \times (t, t + s])$ is independent of $\mathcal{F}_t$.

Stochastic integrals for Poisson random measures

X cadlag, $L_1(\nu)$-valued, $\{\mathcal{F}_t\}$-adapted

We define

$$I_\xi(X,t) = \int_{U\times[0,t]} X(u,s-)\xi(du\times ds)$$

in such a way that

$$\begin{aligned} E\left[|I_\xi(X,t)|\right] &\leq E\left[\int_{U\times[0,t]} |X(u,s-)|\xi(du\times ds)\right] \\ &= \int_{U\times[0,t]} E[|X(u,s)|]\nu(du)ds \end{aligned}$$

If the right side is finite, then

$$E\left[\int_{U\times[0,t]} X(u,s-)\xi(du\times ds)\right] = \int_{U\times[0,t]} E[X(u,s)]\nu(du)ds$$

Predictable integrands

The integral extends to *predictable* integrands satisfying

$$\int_{U\times[0,t]} |X(u,s)| \wedge 1\nu(du)ds < \infty \quad a.s.$$

so that

$$E[I_\xi(X,t\wedge\tau)] = E\left[\int_{U\times[0,t\wedge\tau]} X(u,s)\xi(du\times ds)\right]$$

for any stopping time satisfying

$$E\left[\int_{U\times[0,t\wedge\tau]} |X(u,s)|\nu(du)ds\right] < \infty \tag{1}$$

If (1) holds for all t, then

$$\int_{U\times[0,t\wedge\tau]} X(u,s)\xi(du\times ds) - \int_{U\times[0,t\wedge\tau]} X(u,s)\nu(du)ds$$

is a martingale.

X is predictable if it is the pointwise limit of adapted, left-continuous processes.

Stochastic integrals for centered Poisson random measures

X cadlag, $L_2(\nu)$-valued, $\{\mathcal{F}_t\}$-adapted

We define

$$I_{\widetilde{\xi}}(X,t) = \int_{U\times[0,t]} X(u,s-)\widetilde{\xi}(du\times ds)$$

in such a way that $E\left[I_{\widetilde{\xi}}(X,t)^2\right] = \int_{U\times[0,t]} E[X(u,s)^2]\nu(du)ds$ if the right side is finite. Then $I_{\widetilde{\xi}}(X,\cdot)$ is a square-integrable martingale.

The integral extends to *predictable* integrands satisfying

$$\int_{U\times[0,t]} |X(u,s)|^2 \wedge |X(u,s)|\nu(du)ds < \infty \quad a.s.$$

so that $I_{\widetilde{\xi}}(X, t\wedge\tau)$ is a martingale for any stopping time satisfying

$$E\left[\int_{U\times[0,t\wedge\tau]} |X(u,s)|^2 \wedge |X(u,s)|\nu(du)ds\right] < \infty$$

Itô equations for Markov processes with jumps

W Gaussian white noise on $U_0 \times [0,\infty)$ with $E[W(A,t)W(B,t)] = \mu(A \cap B)t$

ξ_i Poisson random measure on $U_i \times [0,\infty)$ with mean measure ν_i.

Let

$$\widetilde{\xi}_1(A,t) = \xi_1(A,t) - \nu_1(A)t$$

$$\widehat{\sigma}^2(x) = \int \sigma^2(x,u)\mu(du) < \infty$$

$$\int \alpha_1^2(x,u) \wedge |\alpha_1(x,u)|\nu_1(du) < \infty, \quad \int |\alpha_2(x,u)| \wedge 1\nu_2(du) < \infty$$

and

$$\begin{aligned} X(t) = X(0) \quad &+ \int_{U_0\times[0,t]} \sigma(X(s-),u)W(du \times ds) + \int_0^t \beta(X(s-))ds \\ &+ \int_{U_1\times[0,t]} \alpha_1(X(s-),u)\widetilde{\xi}_1(du \times ds) \\ &+ \int_{U_2\times[0,t]} \alpha_2(X(s-),u)\xi_2(du \times ds) \ . \end{aligned}$$

A martingale inequality

Discrete time version: Burkholder (1973).

Continuous time version: Lenglart, Lepingle, and Pratelli (1980).

Easy proof: Ichikawa (1986).

Lemma 1 *For $0 < p \leq 2$ there exists a constant C_p such that for any locally square integrable martingale M with Meyer process $\langle M \rangle$ and any stopping time τ*

$$E[\sup_{s \leq \tau} |M(s)|^p] \leq C_p E[\langle M \rangle_\tau^{p/2}]$$

$\langle M \rangle$ is the (essentially unique) predictable process such that $M^2 - \langle M \rangle$ is a local martingale. (A left-continuous, adapted process is predictable.)

Graham's uniqueness theorem

Lipschitz condition

$$\sqrt{\int |\sigma(x,u)-\sigma(y,u)|^2\mu(du)} + |\beta(x)-\beta(y)|$$
$$+\sqrt{\int |\alpha_1(x,u)-\alpha_1(y,u)|^2\nu_1(du)} + \int |\alpha_2(x,u)-\alpha_2(y,u)|\nu_2(du)$$
$$\leq M|x-y|$$

Estimate

X and $\widetilde{X}$ solution of SDE.

$$
\begin{aligned}
&E[\sup_{s\le t}|X(s)-\widetilde{X}(s)|] \\
&\le E[|X(0)-\widetilde{X}(0)|]+C_1E[(\int_0^t\int_{U_0}|\sigma(X(s-),u)-\sigma(\widetilde{X}(s-),u)|^2\mu(du)ds)^{\frac{1}{2}}] \\
&\quad +C_1E[(\int_0^t\int_{U_1}|\alpha_1(X(s-),u)-\alpha_1(\widetilde{X}(s-),u)|^2\nu_1(du)ds)^{\frac{1}{2}}] \\
&\quad +E[\int_0^t\int_{U_2}|\alpha_2(X(s-),u)-\alpha_2(\widetilde{X}(s-),u)|\nu_2(du)ds] \\
&\quad +E[\int_0^t|\beta(X(s-))-\beta(\widetilde{X}(s-))|ds] \\
&\le E[|X(0)-\widetilde{X}(0)|]+D(\sqrt{t}+t)E[\sup_{s\le t}|X(s)-\widetilde{X}(s)|]
\end{aligned}
$$

Boundary conditions

X has values in D and satisfies

$$X(t) = X(0) + \int_0^t \sigma(X(s))dW(s) + \int_0^t b(X(s))ds + \int_0^t \eta(X(s))d\lambda(s)$$

where λ is nondecreasing and increases only when $X(t) \in \partial D$.

X has values in D and satisfies

$$X(t) = X(0) + \int_0^t \sigma(X(s))dW(s) + \int_0^t b(X(s))ds + \int_0^t \alpha(X(s-), \zeta_{N(s-)+1})dN(s)$$

where $\zeta_1, \zeta_2, \ldots$ are iid and independent of $X(0)$ and W and $N(t)$ is the number of times X has hit the boundary by time t.

2. Martingale problems for Markov processes

- Levy and Watanabe characterizations
- Ito's formula and martingales associated with solutions of stochastic equations
- Generators and martingale problems for Markov processes
- Equivalence between stochastic equations and martingale problems

Martingale characterizations

Brownian motion (Levy)

W a continuous $\{\mathcal{F}_t\}$-martingale

$W(t)^2 - t$ an $\{\mathcal{F}_t\}$-martingale

Then W is a standard Brownian motion compatible with $\{\mathcal{F}_t\}$.

Poisson process (Watanabe)

N a counting process adapted to $\{\mathcal{F}_t\}$

$N(t) - \lambda t$ an $\{\mathcal{F}_t\}$-martingale

Then N is a Poisson process with parameter λ compatible with $\{\mathcal{F}_t\}$

Diffusion processes

$\sigma : \mathbb{R}^d \to \mathbb{M}^{d\times m}$ and $b : \mathbb{R}^d \to \mathbb{R}^d$

Consider

$$X(t) = X(0) + \int_0^t \sigma(X(s))dW(s) + \int_0^t b(X(s))ds$$

By Itô's formula

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds = \int_0^t \nabla f(X(s))^T \sigma(X(s))dW(s)$$

where

$$Af(x) = \frac{1}{2}\sum a_{ij}(x)\partial_i\partial_j f(x) + b(x)\cdot\nabla f(x)$$

with $((a_{ij}(x)\ )) = \sigma(x)\sigma(x)^T$.

Martingale problem for A

Assume a_{ij} and b_i are locally bounded, and let $\mathcal{D}(A) = C_c^2(\mathbb{R}^d)$. X is a solution of the *martingale problem* for A if there exists a filtration $\{\mathcal{F}_t\}$ such that X is $\{\mathcal{F}_t\}$-adapted and

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds$$

is an $\{\mathcal{F}_t\}$-martingale for each $f \in \mathcal{D}(A)$.

Any solution of the SDE is a solution of the martingale problem.

General SDE

Let

$$\begin{aligned} X(t) = X(0) \quad &+ \quad \int_{U_0\times[0,t]} \sigma(X(s-),u)W(du\times ds) + \int_0^t \beta(X(s-))ds \\ &+ \quad \int_{U_1\times[0,t]} \alpha_1(X(s-),u)\widetilde{\xi}_1(du\times ds) \\ &+ \quad \int_{U_2\times[0,t]} \alpha_2(X(s-),u)\xi_2(du\times ds)\ . \end{aligned}$$

Then

$$\begin{aligned} & f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds \\ &= \int_{U_0\times[0,t]} \nabla f(X(s))\cdot\sigma(X(s),u)W(du\times ds) \\ &\quad + \int_{U_1} (f(X(s-)+\alpha_1(X(s-),u)) - f(X(s-))\widetilde{\xi}_1(du\times ds) \\ &\quad + \int_{U_2} (f(X(s-)+\alpha_2(X(s-),u)) - f(X(s-))\widetilde{\xi}_2(du\times ds) \end{aligned}$$

Form of the generator

$$
\begin{aligned}
Af(x) &= \frac{1}{2}\sum a_{ij}(x)\partial_i\partial_j f(x) + b(x)\cdot\nabla f(x) \\
&\quad + \int_{U_1} (f(x+\alpha_1(x,u)) - f(x) - \alpha_1(x,u)\cdot\nabla f(x))\nu_1(du) \\
&\quad + \int_{U_2} (f(x+\alpha_2(x,u)) - f(x))\nu_2(du)
\end{aligned}
$$

Let $\mathcal{D}(A)$ be a collection of functions for which Af is bounded. Then a solution of the SDE is a solution of the martingale problem for A.

The martingale problem for A

X is a solution for the martingale problem for (A, ν_0), $\nu_0 \in \mathcal{P}(E)$, if $PX(0)^{-1} = \nu_0$ and there exists a filtration $\{\mathcal{F}_t\}$ such that

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds$$

is an $\{\mathcal{F}_t\}$-martingale for all $f \in \mathcal{D}(A)$.

Theorem 2 *If any two solutions of the martingale problem for A satisfying $PX_1(0)^{-1} = PX_2(0)^{-1}$ also satisfy $PX_1(t)^{-1} = PX_2(t)^{-1}$ for all $t \geq 0$, then the f.d.d. of a solution X are uniquely determined by $PX(0)^{-1}$.*

If X is a solution of the MGP for A and $Y_a(t) = X(a+t)$, then Y_a is a solution of the MGP for A.

Theorem 3 *If the conclusion of the above theorem holds, then any solution of the martingale problem for A is a Markov process.*

A is called the *generator* for the Markov process.

Weak solutions of SDEs

$\widetilde{X}$ is a *weak solution* of the SDE if there exists a probability space on which are defined X, W, ξ_1, and ξ_2 satisfying the SDE and $\widetilde{X}$ has the same distribution as X.

Theorem 4 *Suppose that the a_{ij} and b_i are locally bounded and that for each $f \in C_c^2(\mathbb{R}^d)$*

$$\sup_x \int_{U_1} |f(x+\alpha_1(x,u)) - f(x) - \alpha_1(x,u)\cdot\nabla f(x)|\nu_1(du) < \infty$$

and

$$\sup_x \int_{U_2} |f(x+\alpha_2(x,u)) - f(x)|\nu_2(du) < \infty.$$

Let $\mathcal{D}(A) = C_c^2(\mathbb{R}^d)$. Then any solution of the martingale problem for A is a weak solution of the SDE.

Nonsingular diffusions

Consider the SDE

$$X(t) = X(0) + \int_0^t \sigma(X(s))dW(s) + b(X(s))ds$$

and assume that $d = m$ (that is, σ is square). Let

$$Af(x) = \frac{1}{2}\sum a_{ij}(x)\partial_i\partial_j f(x) + b(x)\cdot\nabla f(x).$$

Assume that $\sigma(x)$ is invertible for each x and that $|\sigma(x)^{-1}|$ is locally bounded. If $\widetilde{X}$ is a solution of the martingale problem for A, then

$$M(t) = \widetilde{X}(t) - \int_0^t b(\widetilde{X}(s))ds$$

is a local martingale and

$$\widetilde{W}(t) = \int_0^t \sigma(\widetilde{X}(s))^{-1}dM(s)$$

is a standard Brownian motion compatible with $\{\mathcal{F}_t^{\widetilde{X}}\}$. It follows that

$$\widetilde{X}(t) = \widetilde{X}(0) + \int_0^t \sigma(\widetilde{X}(s))d\widetilde{W}(s) + \int_0^t b(\widetilde{X}(s))ds.$$

Natural form for the jump terms

Consider the generator for a simple pure jump process

$$Af(x) = \lambda(x) \int_{\mathbb{R}^d} (f(z) - f(x))\mu(x, dz),$$

where $\lambda \geq 0$ and $\mu(x, \cdot)$ is a probability measure. There exists $\gamma : \mathbb{R}^d \times [0,1] \to \mathbb{R}^d$ such that

$$\int_0^1 f(x + \gamma(x,u))du = \int_{\mathbb{R}^d} f(z)\mu(x, dz).$$

Let ξ be a Poisson random measure on $[0,\infty) \times [0,1] \times [0,\infty)$ with Lebesgue mean measure. Then

$$X(t) = X(0) + \int_{[0,\infty)\times[0,1]\times[0,t]} I_{[0,\lambda(X(s-))]}(v)\gamma(X(s-),u)\xi(dv \times du \times ds)$$

is a stochastic differential equation corresponding to A. Note that this SDE is of the form above with $U_2 = [0,\infty) \times [0,1]$ and $\alpha_2(x,u,v) = I_{[0,\lambda(x)]}(v)\gamma(x,u)$ and that

$$\int_{[0,\infty)\times[0,1]} |\alpha_2(x,u,v) - \alpha_2(y,u,v)|dvdu \leq |\lambda(x) - \lambda(y)| \int_{[0,1]} \gamma(x,u)du$$
$$+\lambda(y) \int_{[0,1]} |\gamma(x,u) - \gamma(y,u)|du$$

More general jump terms

$$X(t) = X(0) + \int_{[0,\infty)\times U\times[0,t]} I_{[0,\lambda(X(s-),u)]}(v)\gamma(X(s-),u)\xi(dv\times du\times ds)$$

where ξ is a Poisson random measure with mean measure $m\times\nu\times m$. The generator is of the form

$$Af(x) = \int_U \lambda(x,u)(f(x+\gamma(x,u)) - f(x))\nu(du).$$

If ξ is replaced by $\widetilde{\xi}$, then

$$Af(x) = \int_U \lambda(x,u)(f(x+\gamma(x,u)) - f(x) - \gamma(x,u)\cdot\nabla f(x))\nu(du).$$

Note that many different choices of λ, γ, and ν will produce the same generator.

Reflecting diffusions

X has values in D and satisfies

$$X(t) = X(0) + \int_0^t \sigma(X(s))dW(s) + \int_0^t b(X(s))ds + \int_0^t \eta(X(s))d\lambda(s)$$

where λ is nondecreasing and increases only when $X(t) \in \partial D$. By Itô's formula

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds - \int_0^t Bf(X(s))d\lambda(s) = \int_0^t \nabla f(X(s))^T \sigma(X(s))dW(s)$$

where

$$Af(x) = \frac{1}{2}\sum a_{ij}(x)\partial_i\partial_j f(x) + b(x)\cdot\nabla f(x) \qquad Bf(x) = \eta(x)\cdot\nabla f(x)$$

Either take $\mathcal{D}(A) = \{f \in C_c^2(D) : Bf(x) = 0, x \in \partial D\}$ or formulate a *constrained* martingale problem with solution (X, λ) by requiring X to take values in D, λ to be nondecreasing and increase only when $X \in \partial D$, and

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds - \int_0^t Bf(X(s))d\lambda(s)$$

to be an $\{\mathcal{F}_t^{X,\lambda}\}$-martingale.

Instantaneous jump conditions

X has values in D and satisfies

$$X(t) = X(0) + \int_0^t \sigma(X(s))dW(s) + \int_0^t b(X(s))ds + \int_0^t \alpha(X(s-), \zeta_{N(s-)+1})dN(s)$$

where $\zeta_1, \zeta_2, \ldots$ are iid and independent of $X(0)$ and W and $N(t)$ is the number of times X has hit the boundary by time t. Then

$$\begin{aligned} f(X(t)) - f(X(0)) &- \int_0^t Af(X(s))ds - \int_0^t Bf(X(s-))dN(s) \\ = &\int_0^t \nabla f(X(s))^T \sigma(X(s))dW(s) \\ &+ \int_0^t (f(X(s-) + \alpha(X(s-), \zeta_{N(s-)+1})) \\ &\qquad - \int_U f(X(s-) + \alpha(X(s-), u))\nu(du))dN(s) \end{aligned}$$

where

$$Af(x) = \frac{1}{2} \sum a_{ij}(x) \partial_i \partial_j f(x) + b(x) \cdot \nabla f(x)$$

and

$$Bf(x) = \int_U (f(x + \alpha(x, u)) - f(x))\nu(du).$$

3. Weak convergence for stochastic processes

- General definition of weak convergence
- Prohorov's theorem
- Skorohod representation theorem
- Skorohod topology
- Conditions for tightness in the Skorohod topology

Topological proof of convergence

(S, d) metric space

$F_n : S \to \mathbb{R}$

$F_n \to F$ in some sense (e.g., $x_n \to x$ implies $F_n(x_n) \to F(x)$)

$F_n(x_n) = 0$

1. Show that $\{x_n\}$ is compact
2. Show that any limit point of $\{x_n\}$ satisfies $F(x) = 0$
3. Show that the equation $F(x) = 0$ has a unique solution x_0
4. Conclude that $x_n \to x_0$

Convergence in distribution

(S,d) complete, separable metric space

X_n S-valued random variable

$\{X_n\}$ *converges in distribution* to X ($\{P_{X_n}\}$ *converges weakly* to P_X) if for each $f \in \overline{C}(S)$

$$\lim_{n\to\infty} E[f(X_n)] = E[f(X)].$$

Denote convergence in distribution by $X_n \Rightarrow X$.

Equivalent statements

$\{X_n\}$ converges in distribution to X if and only if

$$\liminf_{n\to\infty} P\{X_n \in A\} \geq P\{X \in A\}, \text{ each open } A,$$

or equivalently

$$\limsup_{n\to\infty} P\{X_n \in B\} \leq P\{X \in B\}, \text{ each closed } B,$$

Tightness and Prohorov's theorem

A sequence $\{X_n\}$ is *tight* if for each $\epsilon > 0$, there exists a compact set $K_\epsilon \subset S$ such that

$$\sup_n P\{X_n \notin K_\epsilon\} \leq \epsilon.$$

Theorem 5 *Suppose that* $\{X_n\}$ *is tight. Then there exists a subsequence* $\{n(k)\}$ *along which the sequence converges in distribution.*

Skorohod topology on $D_E[0,\infty)$

(E,r) complete, separable metric space

$D_E[0,\infty)$ space of cadlag, E-valued functions

$x_n \to x \in D_E[0,\infty)$ in the Skorohod (J_1) topology if and only if there exist strictly increasing λ_n mapping $[0,\infty)$ onto $[0,\infty)$ such that for each $T>0$,

$$\lim_{n\to\infty} \sup_{t\le T}(|\lambda_n(t)-t| + r(x_n\circ\lambda_n(t), x(t))) = 0.$$

The Skorohod topology is metrizable so that $D_E[0,\infty)$ is a complete, separable metric space.

Note that $I_{[1+\frac{1}{n},\infty)} \to I_{[1,\infty)}$ in $D_{\mathbb{R}}[0,\infty)$, but $(I_{[1+\frac{1}{n},\infty)}, I_{[1,\infty)})$ does *not* converge in $D_{\mathbb{R}^2}[0,\infty)$.

Conditions for tightness

$S_0^n(T)$ collection of discrete $\{\mathcal{F}_t^n\}$-stopping times $\qquad q(x,y) = 1 \wedge r(x,y)$

Theorem 6 *Suppose that for $t \in \mathcal{T}_0$, a dense subset of $[0,\infty)$, $\{X_n(t)\}$ is tight. Then the following are equivalent.*

a) $\{X_n\}$ *is tight in* $D_E[0,\infty)$.

b) (Kurtz) For $T > 0$, *there exist* $\beta > 0$ *and random variables* $\gamma_n(\delta, T)$ *such that for* $0 \le t \le T$, $0 \le u \le \delta$, *and* $0 \le v \le t \wedge \delta$

$$E[q^\beta(X_n(t+u), X_n(t)) \wedge q^\beta(X_n(t), X_n(t-v)) | \mathcal{F}_t^n] \le E[\gamma_n(\delta, T) | \mathcal{F}_t^n]$$

$$\lim_{\delta \to 0} \limsup_{n \to \infty} E[\gamma_n(\delta, T)] = 0,$$

and

$$\lim_{\delta \to 0} \limsup_{n \to \infty} E[q^\beta(X_n(\delta), X_n(0))] = 0. \tag{2}$$

c) (Aldous) Condition (2) holds, and for each $T > 0$, *there exists* $\beta > 0$ *such that*

$$C_n(\delta, T) \equiv \sup_{\tau \in S_0^n(T)} \sup_{u \le \delta} E[\sup_{v \le \delta \wedge \tau} q^\beta(X_n(\tau+u), X_n(\tau)) \wedge q^\beta(X_n(\tau), X_n(\tau - v))]$$

satisfies $\lim_{\delta \to 0} \limsup_{n \to \infty} C_n(\delta, T) = 0$.

Example

$\eta_1, \eta_2, \ldots$ iid, $E[\eta_i] = 0$, $\sigma^2 = E[\eta_i^2] < \infty$

$$X_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \eta_i$$

Then

$$E[(X_n(t+u) - X_n(t))^2 | \mathcal{F}_t^{X_n}] = \frac{[n(t+u)] - [nt]}{n} \sigma^2 \leq (\delta + \frac{1}{n}) \sigma^2$$

for $u \leq \delta$.

Uniqueness of limit

Theorem 7 *If* $\{X_n\}$ *is tight in* $D_E[0,\infty)$ *and*

$$(X_n(t_1),\ldots,X_n(t_k)) \Rightarrow (X(t_1),\ldots,X(t_k))$$

for $t_1,\ldots,t_k \in \mathcal{T}_0$, $\mathcal{T}_0$ *dense in* $[0,\infty)$, *then* $X_n \Rightarrow X$.

For the example, this condition follows from the central limit theorem.

Skorohod representation theorem

Theorem 8 *Suppose that* $X_n \Rightarrow X$*. Then there exists a probability space* $(\Omega, \mathcal{F}, P)$ *and random variables,* $\widetilde{X}_n$ *and* $\widetilde{X}$*, such that* $\widetilde{X}_n$ *has the same distribution as* X_n*,* $\widetilde{X}$ *has the same distribution as* X*, and* $\widetilde{X}_n \to \widetilde{X}$ *a.s.*

Continuous mapping theorem

Corollary 9 *Let* $G(X) : S \to E$ *and define* $C_G = \{x \in S : G \text{ is continuous at } x\}$*. Suppose* $X_n \Rightarrow X$ *and that* $P\{X \in C_G\} = 1$*. Then* $G(X_n) \Rightarrow G(X)$*.*

Some mappings on $D_E[0,\infty)$

$$\pi_t : D_E[0,\infty) \to E \qquad \pi_t(x) = x(t)$$

$$C_{\pi_t} = \{x \in D_E[0,\infty) : x(t) = x(t-)\}$$

$$G_t : D_{\mathbb{R}}[0,\infty) \to \mathbb{R} \qquad G_t(x) = \sup\nolimits_{s\le t} x(s)$$

$$C_{G_t} = \{x \in D_{\mathbb{R}}[0,\infty) : \lim_{s\to t-} G_s(x) = G_t(x)\} \supset \{x \in D_{\mathbb{R}}[0,\infty) : x(t) = x(t-)\}$$

$$G : D_{\mathbb{R}}[0,\infty) \to D_{\mathbb{R}}[0,\infty), \qquad G(x)(t) = G_t(x), \quad \text{is continous}$$

$$H_t : D_E[0,\infty) \to \mathbb{R} \qquad H_t(x) = \sup\nolimits_{s\le t} r(x(s), x(s-))$$

$$C_{H_t} = \{x \in D_E[0,\infty) : \lim_{s\to t-} H_s(x) = H_t(x)\} \supset \{x \in D_E[0,\infty) : x(t) = x(t-)\}$$

$$H : D_E[0,\infty) \to D_{\mathbb{R}}[0,\infty), \quad H(x)(t) = H_t(x), \quad \text{is continuous}$$

Level crossing times

$$\tau_c : D_{\mathbb{R}}[0,\infty) \to [0,\infty) \qquad \tau_c(x) = \inf\{t : x(t) > c\}$$

$$\tau_c^- : D_{\mathbb{R}}[0,\infty) \to [0,\infty) \qquad \tau_c^-(x) = \inf\{t : x(t) \geq c \text{ or } x(t-) \geq c\}$$

$$G_{\tau_c} = G_{\tau_c^-} = \{x : \tau_c(x) = \tau_c^-(x)\}$$

Note that $\tau_c^-(x) \leq \tau_c(x)$ and that $x_n \to x$ implies

$$\tau_c^-(x) \leq \liminf_{n\to\infty} \tau_c^-(x_n) \leq \limsup_{n\to\infty} \tau_c(x_n) \leq \tau_c(x)$$

Localization

Theorem 10 *Suppose that for each $\alpha > 0$, τ_n^α satisfies*

$$P\{\tau_n^\alpha > \alpha\} \leq \alpha^{-1}$$

and $\{X_n(\cdot \wedge \tau_n^\alpha)\}$ is relatively compact. Then $\{X_n\}$ is relatively compact.

Compactification of the state space

Theorem 11 *Let $E \subset E_0$ where E_0 is compact and the topology on E is the restriction of the topology on E_0. Suppose that for each n, X_n is a process with sample paths in $D_E[0,\infty)$ and that $X_n \Rightarrow X$ in $D_{E_0}[0,\infty)$. If X has sample paths in $D_E[0,\infty)$, then $X_n \Rightarrow X$ in $D_E[0,\infty)$.*

4. Convergence for Markov processes characterized by stochastic equations

- Martingale central limit theorem
- Convergence for stochastic integrals
- Convergence for SDEs driven by semimartingales
- Diffusion approximations for Markov chains
- Limit theorems involving Poisson random measures and Gaussian white noise

Martingale central limit theorem

Let M_n be a martingale such that

$$\lim_{n\to\infty} E[\sup_{s\le t} |M_n(s) - M_n(s-)|] = 0$$

and

$$[M_n]_t \to ct$$

in probability.

Then $M_n \Rightarrow \sqrt{c}W$.

Vector-valued version: If for each $1 \le i \le d$

$$\lim_{n\to\infty} E[\sup_{s\le t} |M_n^i(s) - M_n^i(s-)|] = 0$$

and for each $1 \le i, j \le d$,

$$[M_n^i, M_n^j]_t \to c_{ij}t,$$

then $M_n \Rightarrow \sigma W$, where W is d-dimensional standard Brownian motion and σ is a symmetric $d \times d$-matrix satisfying $\sigma^2 = c = ((c_{ij}))$.

Example: Products of random matrices

Let $A^{(1)}, A^{(2)}, \ldots$ be iid random matrices with $E[A^{(k)}] = 0$ and $E[|A^{(k)}|^2] < \infty$. Set $X_0 = I$

$$X(k+1) = (I + \frac{1}{\sqrt{n}} A^{(k+1)}) X(k) = (I + \frac{1}{\sqrt{n}} A^{(k+1)}) \cdots (I + \frac{1}{\sqrt{n}} A^{(1)})$$

$$X_n(t) = X([nt])$$

and

$$M_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} A^{(k)}.$$

Then

$$X_n(t) = X_n(0) + \int_0^t dM_n(s) X_n(s-).$$

$M_n \Rightarrow M$ where M is a Brownian motion with

$$E[M_{ij}(t) M_{kl}(t)] = E[A_{ij} A_{kl}] t,$$

Can we conclude $X_n \Rightarrow X$ satisfying

$$X(t) = X(0) + \int_0^t dM(s) X(s)?$$

Example: Markov chains

$X_{k+1} = H(X_k, \xi_{k+1})$ where $\xi_1, \xi_2 \ldots$ are iid

$$P\{X_{k+1} \in B | X_0, \xi_1, \ldots, \xi_k\} = P\{X_{k+1} \in B | X_k\}$$

Example:

$$X^n_{k+1} = X^n_k + \sigma(X^n_k)\frac{1}{\sqrt{n}}\xi_{k+1} + b(X^n_{k+1})\frac{1}{n}$$

Assume $E[\xi_k] = 0$ and $Var(\xi_k) = 1$.

Define $\quad X_n(t) = X^n_{[nt]} \qquad A_n(t) = \frac{[nt]}{n} \qquad W_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \xi_k.$

$$X_n(t) = X_n(0) + \int_0^t \sigma(X_n(s-)dW_n(s) + \int_0^t b(X_n(s-))dA_n(s)$$

Can we conclude $X_n \Rightarrow X$ satisfying

$$X(t) = X(0) + \int_0^t \sigma(X(s))dW(s) + \int_0^t b(X(s))ds \ ?$$

Stochastic integration

Definition: For cadlag processes X, Y,

$$\int_0^t X(s-)dY(s) = \lim_{\max|t_{i+1}-t_i|\to 0} \sum X(t_i)(Y(t_{i+1}\wedge t) - Y(t_i \wedge t))$$

whenever the limit exists in probability.

Sample paths of bounded variation: If Y is a finite variation process, the stochastic integral exists (apply dominated convergence theorem) and

$$\int_0^t X(s-)dY(s) = \int_{(0,t]} X(s-)\alpha_Y(ds)$$

α_Y is the signed measure with

$$\alpha_Y(0,t] = Y(t) - Y(0)$$

Existence for square integrable martingales

If M is a square integrable martingale, then

$$E[(M(t+s)-M(t))^2|\mathcal{F}_t]=E[[M]_{t+s}-[M]_t|\mathcal{F}_t]$$

Let X be bounded, cadlag and adapted. Then For partitions $\{t_i\}$ and $\{r_i\}$

$$\begin{aligned}&E[(\sum X(t_i)(M(t_{i+1}\wedge t)-M(t_i\wedge t))-\sum X(r_i)(M(r_{i+1}\wedge t)-M(r_i\wedge t)))^2]\\&\quad=E\left[\int_0^t(X(t(s-))-X(r(s-)))^2d[M]_s\right]\\&\quad=E\left[\int_{(0,t]}(X(t(s-))-X(r(s-)))^2\alpha_{[M]}(ds)\right]\end{aligned}$$

$t(s)=t_i$ for $s\in[t_i,t_{i+1})$ $\quad r(s)=r_i$ for $s\in[r_i,r_{i+1})$

Let $\sigma_c=\inf\{t:|X(t)|\geq c\}$ and

$$X^c(t)=\begin{cases}X(t) & t<\sigma_c\\ X(c-) & t\geq\sigma_c\end{cases}$$

Then for

$$\int_0^{t\wedge\tau_c}X(s-)dY(s)=\int_0^{t\wedge\tau_c}X^c(s-)dY(s)$$

Semimartingales

Definition: A cadlag, $\{\mathcal{F}_t\}$-adapted process Y is an $\{\mathcal{F}_t\}$-*semimartingale* if:

$$Y = M + V$$

M a local, square integrable $\{\mathcal{F}_t\}$-martingale

V an $\{\mathcal{F}_t\}$-adapted, finite variation process

Total variation: For a *finite variation process*

$$T_t(Z) \equiv \sup_{\{t_i\}} \sum |Z(t_{i+1} \wedge t) - Z(t_i \wedge t)| < \infty$$

Quadratic variation: For cadlag semimartingale

$$[Y]_t = \lim_{\max|t_{i+1}-t_i|\to 0} \sum (Y(t_{i+1} \wedge t) - Y(t_i \wedge t))^2$$

Covariation: For cadlag semimartingales

$$[Y, Z]_t = \lim \sum (Y(t_{i+1} \wedge t) - Y(t_i \wedge t))(Z(t_{i+1} \wedge t) - Z(t_i \wedge t))$$

Probability estimates for SIs

$Y = M + V$

M local square-integrable martingale

V finite variation process

Assume $|X| \leq 1$.

$$
\begin{aligned}
P\{\sup_{s\leq t}|\int_0^s X(r-)dY(r)| > K\} \\
&\leq P\{\sigma \leq t\} + P\{\sup_{s\leq t\wedge\sigma}|\int_0^s X(r-)dM(r)| > K/2\} \\
&\qquad + P\{\sup_{s\leq t}|\int_0^s X(r-)dV(r)| > K/2\} \\
&\leq P\{\sigma \leq t\} + \frac{16E[\int_0^{t\wedge\sigma} X^2(s-)d[M]_s]}{K^2} + P\{\int_0^t |X(s-)|dT_s(V) \geq K/2\} \\
&\leq P\{\sigma \leq t\} + \frac{16E[[M]_{t\wedge\sigma}]}{K^2} + P\{T_t(V) \geq K/2\}
\end{aligned}
$$

Good integrator condition

Let $\mathcal{S}^0$ be the collection of $X = \sum \xi_i I_{[\tau_i,\tau_{i+1})}$, where $\tau_1 < \cdots < \tau_m$ are stopping times and ξ_i is $\mathcal{F}_{\tau_i}$-measurable. Then

$$\int_0^t X(s-)dY(s) = \sum \xi_i (Y(t \wedge \tau_{i+1}) - Y(t \wedge \tau_i)).$$

If Y is a semimartingale, then

$$\{\int_0^t X(s-)dY(s) : X \in \mathcal{S}^0, |X| \leq 1\}$$

is stochastically bounded.

Y satisfying this stochastic boundedness condition is a *good integrator.*

Bichteler-Dellacherie: Y is a good integrator if and only if Y is a semimartingale. (See, for example, Protter (1990), Theorem III.22.)

Uniformity conditions

$Y_n = M_n + A_n$, a semimartingale adapted to $\{\mathcal{F}_t^n\}$

$T_t(A_n)$, the total variation of A_n on $[0,t]$

$[M_n]_t$, the quadratic variation of M_n on $[0,t]$

Uniform tightness (UT) [JMP]:

$$\mathcal{H}_t^0 = \cup_{n=1}^{\infty}\{|\int_0^t Z(s-)dY_n(s)| : Z \in \mathcal{S}_0^n, \sup_{s\le t}|Z(s)| \le 1\}$$

is stochastically bounded.

Uniformly controlled variations (UCV) [KP]: $\{T_t(A_n), n = 1,2,\ldots\}$ is stochastically bounded, and there exist stopping times $\{\tau_n^\alpha\}$ such that $P\{\tau_n^\alpha \le \alpha\} \le \alpha^{-1}$ and $\sup_n E[[M_n]_{t\wedge\tau_n^\alpha}] < \infty$

A sequence of semimartingales $\{Y_n\}$ that converges in distribution and satisfies either UT or UCV will be called *good*.

Basic convergence theorem

Theorem. (Jakubowski, Mémin and Pagès; Kurtz & Protter)

(X_n, Y_n) $\{\mathcal{F}_t^n\}$-adapted in $D_{\mathbb{M}^{km}\times\mathbb{R}^m}[0,\infty)$.

$Y_n = M_n + A_n$ an $\{\mathcal{F}_t^n\}$-semimartingale

Assume that $\{Y_n\}$ satisfies either UT or UCV.

If $(X_n, Y_n) \Rightarrow (X, Y)$ in $D_{\mathbb{M}^{km}\times\mathbb{R}^m}[0,\infty)$ with the Skorohod topology.

THEN

$$(X_n, Y_n, \int X_n(s-)dY_n(s)) \Rightarrow (X, Y, \int X(s-)dY(s))$$

in $D_{\mathbb{M}^{km}\times\mathbb{R}^m\times\mathbb{R}^k}[0,\infty)$

If $(X_n, Y_n) \to (X, Y)$ in probability in the Skorohod topology on $D_{\mathbb{M}^{km}\times\mathbb{R}^m}[0,\infty)$

THEN

$$(X_n, Y_n, \int X_n(s-)dY_n(s)) \to (X, Y, \int X(s-)dY(s))$$

in probability in $D_{\mathbb{M}^{km}\times\mathbb{R}^m\times\mathbb{R}^k}[0,\infty)$

"IN PROBABILITY" CANNOT BE REPLACED BY "A.S."

Stochastic differential equations

Suppose

$$F : D_{\mathbb{R}^d}[0,\infty) \to D_{\mathbb{M}^{d\times m}}[0,\infty)$$

is *nonanticipating* in the sense that

$$F(x,t) = F(x^t,t)$$

where $x^t(s) = x(s \wedge t)$.

U cadlag, adapted, with values in $\mathbb{R}^d$

Y an $\mathbb{R}^m$-valued semimartingale.

Consider

$$X(t) = U(t) + \int_0^t F(X,s-)dY(s) \tag{3}$$

(X,τ) is a local solution of (3) if X is adapted to a filtration $\{\mathcal{F}_t\}$ such that Y is an $\{\mathcal{F}_t\}$-semimartingale, τ is an $\{\mathcal{F}_t\}$-stopping time, and

$$X(t\wedge\tau) = U(t\wedge\tau) + \int_0^{t\wedge\tau} F(X,s-)dY(s), \quad t \geq 0.$$

Strong local uniqueness holds if for any two local solutions (X_1,τ_1), (X_2,τ_2) with respect to a common filtration, we have $X_1(\cdot\wedge\tau_1\wedge\tau_2) = X_2(\cdot\wedge\tau_1\wedge\tau_2)$.

Sequences of SDE's

$$X_n(t) = U_n(t) + \int_0^t F_n(X_n, s-)dY_n(s).$$

Structure conditions

$T_1[0,\infty) = \{\gamma : \gamma$ nondecr. and maps $[0,\infty)$ onto $[0,\infty), \gamma(t+h) - \gamma(t) \leq h\}$

C.a F_n **behaves well under time changes:** If $\{x_n\} \subset D_{\mathbb{R}^d}[0,\infty)$, $\{\gamma_n\} \subset T_1[0,\infty)$, and $\{x_n \circ \gamma_n\}$ is relatively compact in $D_{\mathbb{R}^d}[0,\infty)$, then $\{F_n(x_n) \circ \gamma_n\}$ is relatively compact in $D_{\mathbb{M}^{d\times m}}[0,\infty)$.

C.b F_n **converges to** F**:** If $(x_n, y_n) \to (x, y)$ in $D_{\mathbb{R}^d \times \mathbb{R}^m}[0,\infty)$, then $(x_n, y_n, F_n(x_n)) \to (x, y, F(x))$ in $D_{\mathbb{R}^d \times \mathbb{R}^m \times \mathbb{M}^{d\times m}}[0,\infty)$.

Note that C.b implies continuity of F, that is, $(x_n, y_n) \to (x, y)$ implies $(x_n, y_n, F(x_n)) \to (x, y, F(x))$.

Examples

$F(x,t) = f(x(t)), \quad f$ continuous

$F(x,t) = f(\int_0^t h(x(s))ds) \qquad f,\ h$ continuous

$F(x,t) = \int_0^t h(t-s, s, x(s))ds, \qquad h$ continuous

Convergence theorem

Theorem 12 *Suppose*

- (U_n, X_n, Y_n) *satisfies*

$$X_n(t) = U_n(t) + \int_0^t F_n(X_n, s-)dY_n(s).$$

- $(U_n, Y_n) \Rightarrow (U, Y)$
- $\{Y_n\}$ *is good*
- $\{F_n\}$, F *satisfy structure conditions*
- $\sup_n \sup_x \|F_n(x, \cdot)\| < \infty$.

Then $\{(U_n, X_n, Y_n)\}$ *is relatively compact and any limit point satisfies*

$$X(t) = U(t) + \int_0^t F(X, s-)dY(s)$$

If, in addition, $(U_n, Y_n) \to (U, Y)$ *in probability and strong uniqueness holds, then* $(U_n, X_n, Y_n) \to (U, X, Y)$ *in probability.*

Euler scheme

Let $0 = t_0 < t_1 < \cdots$ The Euler approximation for

$$X(t) = U(t) + \int_0^t F(X, s-)dY(s)$$

is given by

$$\widehat{X}(t_{k+1}) = \widehat{X}(t_k) + U(t_{k+1}) - U(t_k) + F(\widehat{X}, t_k)(Y(t_{k+1}) - Y(t_k)).$$

Consistency

Let $\{t_k^n\}$ satisfy $\max_k |t_{k+1}^n - t_k^n| \to 0$, and define

$$\begin{pmatrix} Y_n(t) \\ U_n(t) \end{pmatrix} = \begin{pmatrix} Y(t_k^n) \\ U(t_k^n) \end{pmatrix}, \qquad t_k^n \le t < t_{k+1}^n.$$

Then $(U_n, Y_n) \Rightarrow (U, Y)$ and $\{Y_n\}$ is good.

The Euler scheme corresponding to $\{t_k^n\}$ satisfies

$$\widehat{X}_n(t) = U_n(t) + \int_0^t F(\widehat{X}_n, s-)dY_n(s)$$

If F satisfies the structure conditions and strong uniqueness holds, then $\widehat{X}_n \to X$ in probability. (In the diffusion case, Maruyama (1955))

Sequences of Poisson random measures

ξ_n Poisson random measures with mean measures $n\nu \times m$.

h measurable

For $A \in \mathcal{B}(U)$ satisfying $\int_A h^2(u)\nu(du) < \infty$, define

$$M_n(A,t) = \frac{1}{\sqrt{n}} \int_A h(u)(\xi_n(du \times [0,t]) - nt\nu(du)).$$

M_n is an orthogonal martingale random measure with

$$[M_n(A), M_n(B)]_t = \tfrac{1}{n} \int_{A\cap B} h(u)^2 \ \xi_n(du \times [0,t])$$

$$\langle M_n(A), M_n(B)\rangle_t = t \int_{A\cap B} h(u)^2 \nu(du).$$

M_n converges to Gaussian white noise W with

$$E[W(A,t)W(B,s)] = t \wedge s \int_{A\cap B} h(u)^2 \nu(du)$$

$$E[W(\varphi_1,t)W(\varphi_2,s)] = t \wedge s \int \varphi_1(u)\varphi_2(u)h(u)^2\nu(du)$$

$$X_n(t) = X_n(0) + \frac{1}{\sqrt{n}} \int_{U\times[0,t]} \alpha_1(X_n(s-),u)\xi_n(du \times ds) + \frac{1}{n} \int_{U\times[0,t]} \alpha_2(X_n(s-),u)\xi_n(du \times ds)$$

Assume $\int_U \alpha_1(x,u)\nu(du) = 0$. Then

$$X_n(t) = X_n(0) + \frac{1}{\sqrt{n}} \int_{U\times[0,t]} \alpha_1(X_n(s-),u)\widetilde{\xi}_n(du \times ds) + \frac{1}{n} \int_{U\times[0,t]} \alpha_2(X_n(s-),u)\xi_n(du \times ds)$$

Can we conclude $X_n \Rightarrow X$ satisfying

$$X(t) = X(0) + \int_{U\times[0,t]} \alpha_1(X(s),u)W(du \times ds) + \int_0^t \int_U \alpha_2(X(s-),u)\nu(du)ds \ ?$$

Discrete-time Markov chains

Consider

$$X_{k+1}^n = X_k^n + \sigma(X_k^n, \xi_{k+1})\frac{1}{\sqrt{n}} + b(X_k^n, \zeta_{k+1})\frac{1}{n}$$

where $\{(\xi_k, \zeta_{\mathrm{k}})\}$ is iid in $U_1 \times U_2$.

μ the distribution of ξ_k

ν the distribution of ζ_k

Assume $\int_{U_1} \sigma(x, u_1)\mu(du_1) = 0$

Define

$$\begin{aligned} M_n(A,t) &= \frac{1}{\sqrt{n}}\sum_{k=1}^{[nt]}(I_A(\xi_k) - \mu(A)) \\ V_n(B,t) &= \frac{1}{n}\sum_{k=1}^{[nt]} I_B(\zeta_k) \end{aligned}$$

Stochastic equation driven by random measure

Then $X_n(t) \equiv X^n_{[nt]}$ satisfies

$$X_n(t) = X_n(0) + \int_0^t \int_{U_1} \sigma_n(X_n(s), u) M_n(du \times ds) + \int_0^t \int_{U_2} b_n(X_n(s), u) V_n(du \times ds)$$

$$V_n(A, t) \to t\nu(A) \qquad M_n(A, t) \Rightarrow M(A, t)$$

M is Gaussian with covariance

$$E[M(A, t)M(B, s)] = t \wedge s(\mu(A \cap B) - \mu(A)\mu(B))$$

Can we conclude that $X_n \Rightarrow X$ satisfying

$$X(t) = X(0) + \int_0^t \int_{U_1} \sigma(X(s), u) M(du \times ds) + \int_0^t \int_{U_2} b(X(s), u)\nu(du)ds \ ?$$

Good integrator condition

H a separable Banach space ($H = L_2(\nu)$, $L_1(\nu)$, $L_2(\mu)$, etc.)

$Y(\varphi, t)$ a semimartingale for each $\varphi \in H$

$Y(\sum a_k\varphi_k, t) = \sum_k a_k Y(\varphi_k, t)$

Let $\mathcal{S}$ be the collection of cadlag, adapted processes of the form $Z(t) = \sum_{k=1}^m \xi_k(t)\varphi_k$, $\varphi_k \in H$.

Define

$$I_Y(Z,t) = \int_{U\times[0,t]} Z(u,s-)Y(du\times ds) = \sum_k \int_0^t \xi_k(s-)dY(\varphi_k, s).$$

Basic assumption:

$$\mathcal{H}_t = \{\sup_{s\le t} |I_Y(Z,s)| : Z \in \mathcal{S}, \sup_{s\le t} \|Z(s)\|_H \le 1\}$$

is stochastically bounded. (Call Y a *good* $H^{\#}$-semimartingale.)

The integral extends to all cadlag, adapted H-valued processes.

Convergence for $\mathbf{H}^{\#}$-semimartingales

H	a separable Banach space of functions on U
Y_n	an $\{\mathcal{F}_t^n\}$-$H^{\#}$-semimartingale (for each $\varphi \in H$, $Y(\varphi, \cdot)$ is an $\{\mathcal{F}_t^n\}$-semimartingale)
$\{X_n\}$	cadlag, H-valued processes

$(X_n, Y_n) \Rightarrow (X, Y)$, if

$$(X_n, Y_n(\varphi_1, \cdot), \ldots, Y_n(\varphi_m, \cdot)) \Rightarrow (X, Y(\varphi_1, \cdot), \ldots, Y(\varphi_m, \cdot))$$

in $D_{H\times\mathbb{R}^m}[0, \infty)$ for each choice of $\varphi_1, \ldots, \varphi_m \in H$.

Convergence for Stochastic Integrals

Let

$$\mathcal{H}_{n,t} = \{\sup_{s\le t} |I_{Y_n}(Z,s)| \ : \ Z \in \mathcal{S}_n, \sup_{s\le t} \|Z(s)\|_H \le 1\}.$$

Definition: $\{Y_n\}$ is *uniformly tight* if $\cup_n \mathcal{H}_{n,t}$ is stochastically bounded for each t.

Theorem 13 *Protter and Kurtz (1996). Assume that $\{Y_n\}$ is uniformly tight. If $(X_n, Y_n) \Rightarrow (X, Y)$, then there is a filtration $\{\mathcal{F}_t\}$, such that Y is an $\{\mathcal{F}_t\}$-adapted, good, $H^\#$-semimartingale, X is $\{\mathcal{F}_t\}$-adapted and*

$$(X_n, Y_n, I_{Y_n}(X_n)) \Rightarrow (X, Y, I_Y(X)) \ .$$

If $(X_n, Y_n) \to (X, Y)$ in probability, then $(X_n, Y_n, I_{Y_n}(X_n)) \to (X, Y, I_Y(X))$ in probability.

Cho (1994) for distribution-valued martingales

Jakubowski (1995) for Hilbert-space-valued semimatingales.

Sequences of SDE's

$$X_n(t) = U_n(t) + \int_{U\times[0,t]} F_n(X_n, s-, u) Y_n(du \times ds).$$

Structure conditions

$T_1[0,\infty) = \{\gamma : \gamma$ nondecreasing and maps $[0,\infty)$ onto $[0,\infty), \gamma(t+h) - \gamma(t) \le h\}$

C.a F_n **behaves well under time changes:** If $\{x_n\} \subset D_{\mathbb{R}^d}[0,\infty)$, $\{\gamma_n\} \subset T_1[0,\infty)$, and $\{x_n \circ \gamma_n\}$ is relatively compact in $D_{\mathbb{R}^d}[0,\infty)$, then $\{F_n(x_n) \circ \gamma_n\}$ is relatively compact in $D_{H^d}[0,\infty)$.

C.b F_n **converges to** F**:** If $(x_n, y_n) \to (x, y)$ in $D_{\mathbb{R}^d\times\mathbb{R}^m}[0,\infty)$, then $(x_n, y_n, F_n(x_n)) \to (x, y, F(x))$ in $D_{\mathbb{R}^d\times\mathbb{R}^m\times H^d}[0,\infty)$.

Note that C.b implies continuity of F, that is, $(x_n, y_n) \to (x, y)$ implies $(x_n, y_n, F(x_n)) \to (x, y, F(x))$.

SDE convergence theorem

Theorem 14 *Suppose that* (U_n, X_n, Y_n) *satisfies*

$$X_n(t) = U_n(t) + \int_{U\times[0,t]} F_n(X_n, s-, u)Y_n(du \times ds),$$

that $(U_n, Y_n) \Rightarrow (U, Y)$, *and that* $\{Y_n\}$ *is uniformly tight. Assume that* $\{F_n\}$ *and* F *satisfy the structure condition and that* $\sup_n \sup_x \|F_n(x, \cdot)\|_{H^d} < \infty$. *Then* $\{(U_n, X_n, Y_n)\}$ *is relatively compact and any limit point satisfies*

$$X(t) = U(t) + \int_{U\times[0,t]} F(X, s-, u)Y(du \times ds)$$

5. Convergence for Markov processes characterized by martingale problems

- Tightness estimates based on generators
- Convergence of processes based on convergence of generators
- Averaging
- A measure-valued limit

Compactness conditions based on compactness of real functions

Compact containment condition: For each $T > 0$ and $\epsilon > 0$, there exists a compact $K \subset E$ such that

$$\liminf_{n\to\infty} P\{X_n(s) \in K, s \leq T\} \geq 1 - \epsilon.$$

Theorem 15 *Let D be dense in $\overline{C}(E)$ in the compact uniform topology. $\{X_n\}$ is relatively compact (in distribution in $D_E[0,\infty)$) iff $\{X_n\}$ satisfies the compact containment conditions and $\{f \circ X_n\}$ is relatively compact for each $f \in D$.*

Note that

$$\begin{aligned} E[(f(X_n(t+u)) - f(X_n(t)))^2|\mathcal{F}_t^n] \\ = E[f^2(X_n(t+u)) - f^2(X_n(t))|\mathcal{F}_t^n] \\ -2f(X_n(t))E[f(X_n(t+u)) - f(X_n(t))|\mathcal{F}_t^n] \end{aligned}$$

Limits of generators

$$X_n(t) = \frac{1}{\sqrt{n}}(N_b(nt) - N_d(nt)),$$

where N_b, N_d are independent, unit Poisson processes.

For $f \in C_c^3(\mathbb{R})$

$$A_n f(x) \quad = \quad n\left(\frac{f(x+\frac{1}{\sqrt{n}})+f(x-\frac{1}{\sqrt{n}})}{2} - f(x)\right) = \frac{1}{2}f''(x) + O(\frac{1}{\sqrt{n}}) \ .$$

Set $Af = \frac{1}{2}f''$.

$$\begin{aligned} & E[(f(X_n(t+u)) - f(X_n(t)))^2|\mathcal{F}_t^n] \\ & \quad = E[f^2(X_n(t+u)) - f^2(X_n(t))|\mathcal{F}_t^n] \\ & \qquad -2f(X_n(t))E[f(X_n(t+u)) - f(X_n(t))|\mathcal{F}_t^n] \\ & \quad \le u(\|A_n f^2\| + 2\|f\|\|A_n f\|) \ . \end{aligned}$$

It follows that $\{X_n\}$ is relatively compact in $D_{\mathbb{R}^\Delta}[0,\infty)$, or using the fact that

$$P\{\sup_{s\le t}|X_n(s)| \ge k\} \le \frac{4E[X_n^2(t)]}{k^2} = \frac{4t}{k^2} \ ,$$

the compact containment condition holds and $\{X_n\}$ is relatively compact in $D_{\mathbb{R}}[0,\infty)$.

Limits of martingales

Lemma 16 *For each $n = 1, 2, \ldots$, let M_n and Z_n be cadlag stochastic processes and let M_n be a $\{\mathcal{F}_t^{(M_n,Z_n)}\}$-martingale. Suppose that $(M_n, Z_n) \Rightarrow (M, Z)$. If for each $t \geq 0$, $\{M_n(t)\}$ is uniformly integrable, then M is a $\{\mathcal{F}_t^{(M,Z)}\}$-martingale.*

Recall that if a sequence of real-valued random variables $\{\psi_n\}$ is uniformly integrable and $\psi_n \Rightarrow \psi$, then $E[\psi_n] \to E[\psi]$.

It follows that if X is the limit of a subsequence of $\{X_n\}$, then

$$f(X_n(t)) - \int_0^t A_n f(X_n(s))ds \Rightarrow f(X(t)) - \int_0^t Af(X(s))ds$$

(along the subsequence) and X is a solution of the martingale problem for A.

Elementary convergence theorem

E compact, E_n, $n = 1, 2, \ldots$ complete, separable metric space.

$\eta_n : E_n \to E$

Y_n Markov in E_n with generator A_n, $X_n = \eta_n(Y_n)$ cadlag

$A \subset C(E) \times C(E)$ (for simplicity, we write $Af = g$ if $(f, g) \in A$).

$\mathcal{D}(A) = \{f : (f, g) \in A\}$

For each $(f, g) \in A$, there exist $f_n \in \mathcal{D}(A_n)$ such that

$$\sup_{x \in E_n} \left(|f_n(x) - f \circ \eta_n(x)| + |A_n f_n(x) - g \circ \eta_n(x)| \right) \to 0.$$

THEN $\{X_n\}$ is relatively compact and any limit point is a solution of the martingale problem for A.

Reflecting random walk

$E_n = \{0, \frac{1}{\sqrt{n}}, \frac{2}{\sqrt{n}}, \dots\}$

$$A_n f(x) = n\lambda(f(x + \frac{1}{\sqrt{n}}) - f(x)) + n\lambda I_{\{x>0\}}(f(x - \frac{1}{\sqrt{n}}) - f(x))$$

Let $f \in C_c^3[0,\infty)$. Then

$$\begin{aligned} A_n f(x) &= \lambda f''(x) + O(\frac{1}{\sqrt{n}}) \qquad x > 0 \\ A_n f(0) &= \sqrt{n}\lambda f'(0) + \frac{\lambda}{2} f''(0) + O(\frac{1}{\sqrt{n}}) \end{aligned}$$

Assume $f'(0) = 0$, but still have discontinuity at 0.

Let $f_n = f + \frac{1}{\sqrt{n}} h, \quad f \in \{f \in C_c^3[0,\infty) : f'(0) = 0\}, h \in C_c^3[0,\infty)$. Then

$$\begin{aligned} A_n f_n(x) &= \lambda f''(x) + \frac{1}{\sqrt{n}} \lambda h''(x) + O(\frac{1}{\sqrt{n}}) \qquad x > 0 \\ A_n f_n(0) &= \frac{\lambda}{2} f''(0) + \lambda h'(0) + O(\frac{1}{\sqrt{n}}) \end{aligned}$$

Assume that $h'(0) = \frac{1}{2} f''(0)$. Then, noting that $\eta_n(x) = x$, $x \in E_n$,

$$\sup_{x \in E_n} (|f_n(x) - f(x)| + |A_n f_n(x) - Af(x)|) \to 0$$

Averaging

Y stationary with marginal distribution ν, ergodic, and independent of W

$Y_n(t) = Y(\beta_n t)$, $\beta_n \to \infty$

$$X_n(t) = X(0) + \int_0^t \sigma(X_n(s), Y_n(s))dW(s) + \int_0^t b(X_n(s), Y_n(s))ds$$

Lemma 17 *Let $\{\mu_n\}$ be measures on $U \times [0,\infty)$ satisfying $\mu_n(U \times [0,t]) = t$. Suppose*

$$\int_U \varphi(u)\mu_n(du \times [0,t]) \to \int_U \varphi(u)\mu(du \times [0,t]),$$

for each $\varphi \in \overline{C}(U)$ and $t \geq 0$, and that $x_n \to x$ in $D_E[0,\infty)$. Then

$$\int_{U\times[0,t]} h(u, x_n(s))\mu_n(du \times ds) \to \int_{U\times[0,t]} h(u, x(s))\mu(du \times ds)$$

for each $h \in \overline{C}(U \times E)$ and $t \geq 0$.

Convergence of averaged generator

Assume that σ and b are bounded and continuous. Then for $f \in C_c^2(\mathbb{R})$,

$$f(X_n(t)) - f(X_n(0)) - \int_0^t Af(X_n(s), Y_n(s))ds = \int_0^t \sigma(X_n(s), Y_n(s))f'(X_n(s))dW(s)$$

$$Af(x,y) = \frac{1}{2}\sigma^2(x,y)f''(x) + b(x,y)f'(x)$$

is a martingale, $\{X_n\}$ is relatively compact,

$$\int_0^t \varphi(Y_n(s))ds = \frac{1}{\beta_n}\int_0^{\beta_n t} \varphi(Y(s))ds \to t\int_U \varphi(u)\nu(du),$$

so any limit point of $\{X_n\}$ is a solution of the martingale problem for

$$\overline{A}f(x) = \int_U Af(x,u)\nu(du).$$

Khas'minskii (1966), Kurtz (1992), Pinsky (1991)

Coupled system

$$X_n(t) = X(0) + \int_0^t \sigma(X_n(s), Y_n(s))dW_1(s) + \int_0^t b(X_n(s), Y_n(s))ds$$

$$Y_n(t) = Y(0) + \int_0^t \sqrt{n}\alpha(X_n(s), Y_n(s))dW_2(s) + \int_0^t n\beta(X_n(s), Y_n(s))ds$$

Consequently,

$$f(X_n(t)) - f(X_n(0)) - \int_0^t Af(X_n(s), Y_n(s))ds$$

and

$$g(Y_n(t)) - g(Y(0)) - \int_0^t nBg(X_n(s), Y_n(s))ds$$

$$Bg(x,y) = \frac{1}{2}\alpha^2(x,y)g''(y) + \beta(x,y)g'(y)$$

are martingales.

Estimates

Suppose that for $g \in C_c^2(\mathbb{R})$, $Bg(x,y)$ is bounded, and that, taking $g(y)=y^2$,

$$Bg(x,y) \leq K_1 - K_2 y^2.$$

Then, assuming $E[Y(0)^2] < \infty$,

$$E[Y_n(t)^2] \leq E[Y(0)^2] + \int_0^t (K_1 - K_2 E[Y_n(s)^2])ds$$

which implies

$$E[Y_n(t)^2] \leq E[Y(0)^2]e^{-K_2 t} + \frac{K_1}{K_2}(1 - e^{-K_2 t}).$$

The sequence of measures defined by

$$\int_{\mathbb{R}\times[0,t]} \varphi(\mathrm{y})\Gamma_n(d\mathrm{y} \times ds) = \int_0^t \varphi(Y_n(s))ds$$

is relatively compact.

Convergence of the averaged process

If σ and b are bounded, then $\{X_n\}$ is relatively compact and any limit point of $\{(X_n, \Gamma_n)\}$ must satisfy:

$$f(X(t)) - f(X(0)) - \int_{\mathbb{R}\times[0,t]} Af(X(s), \mathrm{y})\Gamma(d\mathrm{y} \times ds)$$

is a martingale for each $f \in C_c^2(\mathbb{R})$ and

$$\int_{\mathbb{R}\times[0,t]} Bg(X(s), \mathrm{y})\Gamma(d\mathrm{y} \times ds) \tag{4}$$

is a martingale for each $g \in C_c^2(\mathbb{R})$.

But (4) is continuous and of finite variation. Therefore

$$\int_{\mathbb{R}\times[0,t]} Bg(X(s), \mathrm{y})\Gamma(d\mathrm{y} \times ds) = \int_0^t \int_{\mathbb{R}} Bg(X(s), \mathrm{y})\gamma_s(d\mathrm{y})ds = 0.$$

Characterizing γ_s

For almost every s

$$\int_{\mathbb{R}} Bg(X(s),y)\gamma_s(dy) = 0, \quad g \in C_c^2(\mathbb{R})$$

But, fixing x, and setting $B_x g(y) = Bg(x,y)$

$$\int_{\mathbb{R}} B_x g(y)\pi(dy) = 0, \quad g \in C_2(\mathbb{R}),$$

implies π is a stationary distribution for the diffusion with generator

$$B_x g(y) = \frac{1}{2}\alpha_x^2(y)g''(y) + \beta_x(y)g'(y), \quad \alpha_x(y) = \alpha(x,y),\ \beta_x(y) = \beta(x,y)$$

If $\alpha_x(y) > 0$ for all y, then the stationary distribution π_x is uniquely determined. If uniqueness hold for all x, $\Gamma(dy \times ds) = \pi_{X(s)}(dy)ds$ and X is a solution of the martingale problem for

$$\overline{A}f(x) = \int_{\mathbb{R}} Af(x,y)\pi_x(dy).$$

Moran models in population genetics

E *type space*

B generator of *mutation process*

σ *selection coefficient*

Generator with state space E^n

$$A^n f(x) = \sum_{i=1}^n B_i f(x) + \frac{1}{2(n-2)} \sum_{1 \le i \ne j \ne k \le n} (1 + \frac{2}{n}\sigma(x_i, x_j))(f(\eta_k(x|x_i)) - f(x))$$

$\eta_k(x|z) = (x_1, \ldots, x_{k-1}, z, x_{k+1}, \ldots x_n)$

Note that if $(X_1, \ldots, X_n)$ is a solution of the martingale problem for A, then for any permutation σ, $(X_{\sigma_1}, \ldots, X_{\sigma_n})$ is a solution of the martingale problem for A.

Conditioned martingale lemma

Lemma 18 *Suppose U and V are $\{\mathcal{F}_t\}$-adapted,*

$$U(t) - \int_0^t V(s)ds$$

is an $\{\mathcal{F}_t\}$-martingale, and $\mathcal{G}_t \subset \mathcal{F}_t$. Then

$$E[U(t)|\mathcal{G}_t] - \int_0^t E[V(s)|\mathcal{G}_s]ds$$

is a $\{\mathcal{G}_t\}$-martingale.

Generator for measure-valued process

$\mathcal{P}^n(E) = \{\mu \in \mathcal{P}(E) : \mu = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}\}$

$$Z(t) = \frac{1}{n}\sum_{i=1}^n \delta_{X_i(t)}$$

For $f \in B(E^m)$ and $\mu \in \mathcal{P}^n(E)$

$$\langle f, \mu^{(m)}\rangle = \frac{1}{n\cdots(n-m+1)} \sum_{i_1\neq\cdots\neq i_m} f(x_{i_1},\ldots,x_{i_m}).$$

Symmetry and the conditioned martingale lemma imply

$$\langle f, Z^{(n)}(t)\rangle - \int_0^t \langle A^n f, Z^{(n)}(s)\rangle ds$$

is a $\{\mathcal{F}_t^Z\}$-martingale.

Define $F(\mu) = \langle f, \mu^{(n)}\rangle$

$$\mathbb{A}^n F(\mu) = \langle A^n f, \mu^{(n)}\rangle$$

Convergence of the generator

If f depends on m variables ($m < n$)

$$
\begin{aligned}
&A^n f(x) \\
&\quad = \sum_{i=1}^{m} B_i f(x) \\
&\quad + \frac{1}{2(n-2)} \sum_{k=1}^{m} \sum_{1\le i\ne k\le m} \sum_{1\le j\ne k,i\le n} (1+\frac{2}{n}\sigma(x_i,x_j))(f(\eta_k(x|x_i)) - f(x)) \\
&\quad + \frac{1}{2(n-2)} \sum_{k=1}^{m} \sum_{i=m+1}^{n} \sum_{1\le j\ne k,i\le n} (1+\frac{2}{n}\sigma(x_i,x_j))(f(\eta_k(x|x_i)) - f(x))
\end{aligned}
$$

$$
\begin{aligned}
\langle A^n f, \mu^{(n)}\rangle \;&=\; \sum_{i=1}^{m} \langle B_i f, \mu^{(m)}\rangle + \frac{1}{2} \sum_{1\le i\ne k\le m} (\langle \Phi_{ik} f, \mu^{(m-1)}\rangle - \langle f, \mu^{(m)}\rangle) + O(\frac{1}{n}) \\
&\quad + \sum_{k=1}^{m} (\langle \sigma_k f, \mu^{(m+1)}\rangle - \langle \sigma f, \mu^{(m+2)}\rangle) + O(\frac{1}{n})
\end{aligned}
$$

Conclusions

- If E is compact, compact containment condition is immediate ($\mathcal{P}(E)$ is compact).
- $\mathbb{A}^n F$ is bounded as long as $B_i f$ is bounded.
- Limit of uniformly integrable martingales is a martingale
- $Z^n \Rightarrow Z$ if uniqueness holds for limiting martingale problem.

6. The lecturer's whims

- Wong-Zakai corrections
- Processes with boundary conditions
- Averaging for stochastic equations

Not good (evil?) sequences

Markov chains: Let $\{\xi_k\}$ be an irreducible finite Markov chain with stationary distribution $\pi(x)$, $\sum g(x)\pi(x) = 0$, and let h satisfy $Ph-h = g$ $(Ph(x) = \sum p(x,y)h(y))$. Define

$$V_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} g(\xi_k).$$

Then $V_n \Rightarrow \sigma W$, where

$$\sigma^2 = \sum_{x,y} \pi(x)p(x,y) \left(Ph(x) - h(y)\right)^2.$$

$\{V_n\}$ is not "good" but

$$\begin{aligned} V_n(t) &= \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} (Ph(\xi_{k-1}) - h(\xi_k)) + \frac{1}{\sqrt{n}}(Ph(\xi_{[nt]}) - Ph(\xi_0)) \\ &= M_n(t) + Z_n(t) \end{aligned}$$

where $\{M_n\}$ is good sequence of martingales and $Z_n \Rightarrow 0$.

Piecewise linear interpolation of W:

$$W_n(t) = W(\frac{[nt]}{n}) + (t - \frac{[nt]}{n})n\left(W(\frac{[nt]+1}{n}) - W(\frac{[nt]}{n})\right)$$

(Classical Wong-Zakai example.)

$$\begin{aligned} W_n(t) &= W(\frac{[nt]+1}{n}) - (\frac{[nt]+1}{n} - t)n\left(W(\frac{[nt]+1}{n}) - W(\frac{[nt]}{n})\right) \\ &= M_n(t) + Z_n(t) \end{aligned}$$

where $\{M_n\}$ is good (take the filtration to be $\mathcal{F}_t^n = \mathcal{F}_{\frac{[nt]+1}{n}}$) and $Z_n \Rightarrow 0$.

Renewal processes: $N(t) = \max\{k : \sum_{i=1}^k \xi_i \leq t\}$, $\{\xi_i\}$ iid, positive, $E[\xi_k] = \mu$, $Var(\xi_k) = \sigma^2$.

$$V_n(t) = \frac{N(nt) - nt/\mu}{\sqrt{n}}.$$

Then $V_n \Rightarrow \alpha W$, $\alpha = \sigma/\mu^{3/2}$.

$$\begin{aligned} V_n(t) &= \frac{(N(nt)+1)\mu - S_{N(nt)+1}}{\mu\sqrt{n}} + \frac{S_{N(nt)+1} - nt}{\mu\sqrt{n}} \\ &= M_n(t) + Z_n(t) \end{aligned}$$

Not so evil after all

Assume $V_n(t) = Y_n(t) + Z_n(t)$ where $\{Y_n\}$ is good and $Z_n \Rightarrow 0$. In addition, assume $\{\int Z_n dZ_n\}$, $\{[Y_n, Z_n]\}$, and $\{[Z_n]\}$ are good.

$$\begin{aligned} X_n(t) &= X_n(0) + \int_0^t F(X_n(s-))dV_n(s) \\ &= X_n(0) + \int_0^t F(X_n(s-))dY_n(s) + \int_0^t F(X_n(s-))dZ_n(s) \end{aligned}$$

Integrate by parts using

$$F(X_n(t)) = F(X_n(0)) + \int_0^t F'(X_n(s-))F(X_n(s-))dV_n(s) + R_n(t)$$

where R_n can be estimated in terms of $[V_n] = [Y_n] + 2[Y_n, Z_n] + [Z_n]$.

Integration by parts

$$\int_0^t F(X_n(s-))dZ_n(s)$$
$$= F(X_n(t))Z_n(t) - F(X_n(0))Z_n(0) - \int_0^t Z_n(s-)dF(X_n(s)) - [F \circ X_n, Z_n]_t$$
$$\approx -\int_0^t Z_n(s-)F'(X_n(s-))F(X_n(s-))dY_n(s) - \int_0^t F'(X_n(s-))F(X_n(s-))Z_n(s-)dZ_n(s$$
$$-\int_0^t Z_n(s-)dR_n(s) - \int_0^t F'(X_n(s-))F(X_n(s-))d([Y_n, Z_n]_s + [Z_n]_s) - [R_n, Z_n]_t$$

Theorem 19 *Assume that* $V_n = Y_n + Z_n$ *where* $\{Y_n\}$ *is good,* $Z_n \Rightarrow 0$*, and* $\{\int Z_n dZ_n\}$ *is good. If* $(X_n(0), Y_n, Z_n, \int Z_n dZ_n, [Y_n, Z_n]) \Rightarrow (X(0), Y, 0, H, K)$*, then* $\{X_n\}$ *is relatively compact and any limit point satisfies*

$$X(t) = X(0) + \int_0^t F(X(s-))dY(s) + \int_0^t F'(X(s-))F(X(s-))d(H(s) - K(s))$$

Note: For all the examples, $H(t) - K(t) = ct$ *for some* c*.*

Reflecting diffusions

X has values in D and satisfies

$$X(t) = X(0) + \int_0^t \sigma(X(s))dW(s) + \int_0^t b(X(s))ds + \int_0^t \eta(X(s))d\lambda(s)$$

where λ is nondecreasing and increases only when $X(t) \in \partial D$. By Itô's formula

$$\begin{aligned} f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds - \int_0^t Bf(X(s))d\lambda(s) \\ = \int_0^t \nabla f(X(s))^T \sigma(X(s))dW(s) \end{aligned}$$

where

$$Af(x) = \frac{1}{2}\sum a_{ij}(x)\partial_i\partial_j f(x) + b(x)\cdot\nabla f(x) \qquad Bf(x) = \eta(x)\cdot\nabla f(x)$$

Either take $\mathcal{D}(A) = \{f \in C_c^2(D) : Bf(x) = 0, x \in \partial D\}$ or formulate a *constrained* martingale problem with solution (X, λ) by requiring X to take values in D, λ to be nondecreasing and increase only when $X \in \partial D$, and

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds - \int_0^t Bf(X(s))d\lambda(s)$$

to be an $\{\mathcal{F}_t^{X,\lambda}\}$-martingale.

Instantaneous jump conditions

X has values in D and satisfies

$$X(t) = X(0) + \int_0^t \sigma(X(s))dW(s) + \int_0^t b(X(s))ds + \int_0^t \alpha(X(s-), \zeta_{N(s-)+1})dN(s)$$

where $\zeta_1, \zeta_2, \ldots$ are iid and independent of $X(0)$ and W and $N(t)$ is the number of times X has hit the boundary by time t. Then

$$\begin{aligned} f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds - \int_0^t Bf(X(s-))dN(s) \\ = \int_0^t \nabla f(X(s))^T \sigma(X(s))dW(s) \\ + \int_0^t (f(X(s-) + \alpha(X(s-), \zeta_{N(s-)+1})) \\ - \int_U f(X(s-) + \alpha(X(s-), u))\nu(du))dN(s) \end{aligned}$$

where

$$Af(x) = \frac{1}{2} \sum a_{ij}(x)\partial_i \partial_j f(x) + b(x) \cdot \nabla f(x)$$

and

$$Bf(x) = \int_U (f(x + \alpha(x, u)) - f(x))\nu(du).$$

Approximation of reflecting diffusion

X has values in D and satisfies

$$X_n(t) = X(0) + \int_0^t \sigma(X_n(s))dW(s) + \int_0^t b(X_n(s))ds + \int_0^t \frac{1}{n}\eta(X_n(s-))dN_n(s)$$

$\lambda_n(t) \equiv \frac{N_n(t)}{n}$

Assume there exists $\varphi \in C^2$ such that φ, $A\varphi$ and $\nabla\varphi \cdot \sigma$ are bounded on D and $\inf_{x\in\partial D} \eta(x) \cdot \nabla\varphi(x) \geq \epsilon > 0$. Then

$$\begin{aligned}\inf_{x\in\partial D} n(\varphi(x + \frac{1}{n}\eta(x)) - \varphi(x))\lambda_n(t) \\ \leq 2\|\varphi\| + 2\|A\varphi\|t - \int_0^t \nabla\varphi(X_n(s))^T\sigma(X_n(s))dW(s)\end{aligned}$$

and $\{\lambda_n(t)\}$ is stochastically bouned

Convergence of time changed sequence

$\gamma_n(t) = \inf\{s : s + \lambda_n(s) > t\} \qquad t \leq \gamma_n(t) + \lambda_n \circ \gamma_n(t) \leq t + \frac{1}{n}$

$\widehat{X}_n(t) \equiv X_n \circ \gamma_n(t)$

$$\widehat{X}_n(t) = X(0) + \int_0^t \sigma(\widehat{X}_n(s))dW \circ \gamma_n(s) + \int_0^t b(\widehat{X}_n(s))d\gamma_n(s) + \int_0^t \eta(\widehat{X}_n(s-))d\lambda_n \circ \gamma_n(s)$$

Check relative compactness and goodness of $\{(W \circ \gamma_n, \gamma_n, \lambda_n \circ \gamma_n)\}$.

γ_n is Lipschitz with Lipschitz constant 1, $\lambda_n \circ \gamma_n$ is finite variation, nondecreasing and bounded by $t + \frac{1}{n}$, $M_n = W \circ \gamma_n$ is a martingale with $[M_n]_t = \gamma_n(t)$.

If σ and b are bounded and continuous, then (by the general theorem) $\{(\widehat{X}_n, W \circ \gamma_n, \gamma_n, \lambda_n \circ \gamma_n)\}$ is relatively compact and any limit point will satisfy

$$\widehat{X}(t) = X(0) + \int_0^t \sigma(\widehat{X}(s))dW \circ \gamma(s) + \int_0^t b(\widehat{X}(s))d\gamma(s) + \int_0^t \eta(\widehat{X}(s))d\widehat{\lambda}(s)$$

where $\gamma(t) + \widehat{\lambda}(t) = t$. (Note that γ and $\widehat{\lambda}$ are continuous.)

Convergence of original sequence

$$\widehat{X}(t) = X(0) + \int_0^t \sigma(\widehat{X}(s))dW \circ \gamma(s) + \int_0^t b(\widehat{X}(s))d\gamma(s) + \int_0^t \eta(\widehat{X}(s))d\widehat{\lambda}(s)$$

If γ is constant on $[a,b]$, then for $a \leq t \leq b$, $\widehat{X}(t) \in \partial D$ and

$$\widehat{X}(t) = \widehat{X}(a) + \int_a^t \eta(\widehat{X}(s))ds.$$

Under appropriate conditions on η, no such interval can exist with $a < b$ and γ must be strictly increasing.

Then (at least along a subsequence) $X_n \Rightarrow X \equiv \widehat{X} \circ \gamma^{-1}$ and setting $\lambda = \widehat{\lambda} \circ \gamma^{-1}$,

$$X(t) = X(0) + \int_0^t \sigma(X(s))dW(s) + \int_0^t b(X(s))ds + \int_0^t \eta(X(s))d\lambda(s)$$

Rapidly varying components

- W a standard Brownian motion in $\mathbb{R}$
- $X_n(0)$ independent of W
- ζ a stochastic process with state space U, independent of W and $X_n(0)$

Define $\zeta_n(t) = \zeta(nt)$.

$$X_n(t) = X_n(0) + \int_0^t \sigma(X_n(s), \zeta_n(s))dW(s) + \int_0^t b(X_n(s), \zeta_n(s))ds$$

Define

$$M_n(A, t) = \int_0^t I_A(\zeta_n(s))dW(s)$$

and

$$V_n(A, t) = \int_0^t I_A(\zeta_n(s))ds,$$

so that

$$X_n(t) = X_n(0) + \int_{U\times[0,t]} \sigma(X_n(s), u)M_n(du \times ds) + \int_{U\times[0,t]} b(X_n(s), u)V_n(du \times ds)$$

Convergence of driving processes

Define

$$\begin{aligned} M_n(\varphi, t) &= \int_0^t \varphi(\zeta_n(s))dW(s) \\ V_n(\varphi, t) &= \int_0^t \varphi(\zeta_n(s))ds \end{aligned}$$

Assume that

$$\frac{1}{t}\int_0^t \varphi(\zeta(s))ds \to \int_U \varphi(u)\nu(du)$$

in probability for each $\varphi \in \overline{C}(U)$.

Observe that

$$\begin{aligned} [M_n(\varphi_1, \cdot), M_n(\varphi_2, \cdot)]_t = \int_0^t \varphi_1(\zeta_n(s))\varphi_2(\zeta_n(s))ds \\ \to t \int_U \varphi_1(u)\varphi_2(u)\nu(du) \end{aligned}$$

Limiting equation

The functional central limit theorem for martingales implies $M_n(\varphi,t) \Rightarrow M(\varphi,t)$ where M is Gaussian with

$$E[M(\varphi_1,t)M(\varphi_2,s)] = t\wedge s\int_U \varphi_1(u)\varphi_2(u)\nu(du)$$

and

$$V_n(\varphi,t)\to t\int_U \varphi(u)\nu(du)$$

If $\{(M_n,V_n)\}$ is good, then $X_n \Rightarrow X$ satisfying

$$X(t)=X(0)+\int_0^t\int_U \sigma(X(s),u)M(du\times ds)+\int_0^t\int_U b(X(s),u)\nu(du)ds$$

Estimates for averaging problem

Recall

$$M_n(\varphi,t)=\int_0^t \varphi(\zeta_n(s))dW(s)$$

If

$$Z(t,u)=\sum_{k=1}^m \xi_k(t)\varphi_k(u)$$

then

$$\int_{U\times[0,t]} Z(s,u)M_n(du\times ds)=\sum_{k=1}^m\int_0^t \xi_k(s-)dM_n(\varphi_k,s)$$

and

$$\begin{aligned} E[(\int_{U\times[0,t]} Z(s,u)M_n(du\times ds))^2] &= E\left[\int_0^t \sum_{k,l}\xi_k(s)\xi_l(s)\varphi_k(\zeta_n(s))\varphi_l(\zeta_n(s))ds\right] \\ &= E\left[\int_0^t Z(s,\zeta_n(s))^2 ds\right] \end{aligned}$$

Assume U is locally compact, ψ is strictly positive and vanishes at ∞, $\|\varphi\|_H = \sup|\psi(u)\varphi(u)|$, and $\sup_T E[\frac{1}{T}\int_0^T \frac{1}{\psi(\zeta(s))^2}ds]<\infty$.