1. Stochastic equations for Markov processes

- Filtrations and the Markov property
- Ito equations for diffusion processes
- Poisson random measures
- Ito equations for Markov processes with jumps
Filtrations and the Markov property

$(\Omega, \mathcal{F}, P)$ a probability space

Available information is modeled by a sub-$\sigma$-algebra of $\mathcal{F}$

$\mathcal{F}_t$ information available at time $t$

$\{\mathcal{F}_t\}$ is a filtration. $t < s$ implies $\mathcal{F}_t \subset \mathcal{F}_s$

A stochastic process $X$ is adapted to $\{\mathcal{F}_t\}$ if $X(t)$ is $\mathcal{F}_t$-measurable for each $t \geq 0$.

An $E$-valued stochastic process $X$ adapted to $\{\mathcal{F}_t\}$ is $\{\mathcal{F}_t\}$-Markov if

$$E[f(X(t + r))|\mathcal{F}_t] = E[f(X(t + r))|X(t)], \quad t, r \geq 0, \quad f \in B(E)$$

An $\mathbb{R}$-valued stochastic process $M$ adapted to $\{\mathcal{F}_t\}$ is an $\{\mathcal{F}_t\}$-martingale if

$$E[M(t + r)|\mathcal{F}_t] = M(t), \quad t, r \geq 0$$

$\tau$ is an $\{\mathcal{F}_t\}$-stopping time if for each $t \geq 0$, $\{\tau \leq t\} \in \mathcal{F}_t$. For a stopping time $\tau$,

$$\mathcal{F}_{\tau} = \{ A \in \mathcal{F} : \{ \tau \leq t\} \cap A \in \mathcal{F}_t, t \geq 0 \}$$
Ito integrals

$W$ standard Brownian motion ($W$ has independent increments with $W(t+s) - W(t)$ normally distributed with mean zero and variance $s$.)

$W$ is compatible with a filtration $\{\mathcal{F}_t\}$ if $W$ is $\{\mathcal{F}_t\}$-adapted and $W(t+s) - W(t)$ is independent of $\mathcal{F}_t$ for all $s, t \geq 0$.

If $X$ is $\mathbb{R}$-valued, cadlag and $\{\mathcal{F}_t\}$-adapted, then

$$\int_0^t X(s-)dW(s) = \lim_{\max(t_{i+1}-t_i) \to 0} \sum X(t_i)(W(t \wedge t_{i+1}) - W(t \wedge t_i))$$

exists, and the integral satisfies

$$E[(\int_0^t X(s-)dW(s))^2] = E[\int_0^t X(s)^2ds]$$

if the right side is finite.

The integral extends to all measurable and adapted processes satisfying

$$\int_0^t X(s)^2ds < \infty$$

$W$ is an $\mathbb{R}^m$-valued standard Brownian motion if $W = (W_1, \ldots, W_m)^T$ for $W_1, \ldots, W_m$ independent one-dimensional standard Brownian motions.
Ito equations for diffusion processes

\[ \sigma : \mathbb{R}^d \to \mathbb{M}^{d \times m} \text{ and } b : \mathbb{R}^d \to \mathbb{R}^d \]

Consider

\[ X(t) = X(0) + \int_0^t \sigma(X(s))dW(s) + \int_0^t b(X(s))ds \]

where \( X(0) \) is independent of the standard Brownian motion \( W \).

Why is \( X \) Markov? Suppose \( W \) is compatible with \( \{ \mathcal{F}_t \} \), \( X \) is \( \{ \mathcal{F}_t \} \)-adapted, and \( X \) is unique (among \( \{ \mathcal{F}_t \} \)-adapted solutions). Then

\[ X(t + r) = X(t) + \int_t^{t+r} \sigma(X(s))dW(s) + \int_t^{t+r} b(X(s))ds \]

and \( X(t + r) = H(t, r, X(t), W_t) \) where \( W_t(s) = W(t + s) - W(t) \). Since \( W_t \) is independent of \( \mathcal{F}_t \),

\[ E[f(X(t + r))|\mathcal{F}_t] = \int f(H(t, r, X(t), w))\mu_W(dw). \]
Gaussian white noise.

$(U, d_U)$ a complete, separable metric space; $\mathcal{B}(U)$, the Borel sets

$\mu$ a (Borel) measure on $U$

$\mathcal{A}(U) = \{A \in \mathcal{B}(U) : \mu(A) < \infty\}$

$W(A, t)$ Mean zero, Gaussian process indexed by $\mathcal{A}(U) \times [0, \infty)$

$E[W(A, t)W(B, s)] = t \wedge s\mu(A \cap B)$,

$W(\varphi, t) = \int \varphi(u)W(du, t)$

$\varphi(u) = \sum a_i I_{A_i}(u)$

$W(\varphi, t) = \sum_i a_i W(A_i, t)$

$E[W(\varphi_1, t)W(\varphi_2, s)] = t \wedge s \int_U \varphi_1(u) \varphi_2(u)\mu(du)$

Define $W(\varphi, t)$ for all $\varphi \in L_2(\mu)$. 
Definition of integral

\[ X(t) = \sum_i \xi_i(t)\varphi_i \] process in \( L_2(\mu) \)

\[ I_W(X, t) = \int_{U \times [0,t]} X(s, u)W(du \times ds) = \sum_i \int_0^t \xi_i(s)dW(\varphi_i, s) \]

\[ E[I_W(X, t)^2] = E \left[ \sum_{i,j} \int_0^t \xi_i(s)\xi_j(s)ds \int_U \varphi_i\varphi_j d\mu \right] \]

\[ = E \left[ \int_0^t \int_U X(s, u)^2\mu(du)ds \right] \]

The integral extends to adapted processes satisfying

\[ \int_0^t \int_U X(s, u)^2\mu(du)ds < \infty \quad a.s. \]

so that

\[ (I_W(X, t))^2 - \int_0^t \int_U X(s, u)^2\mu(du)ds \]

is a local martingale.
Poisson random measures

$\nu$ a $\sigma$-finite measure on $U$, $(U, d_U)$ a complete, separable metric space.

$\xi$ a Poisson random measure on $U \times [0, \infty)$ with mean measure $\nu \times m$.

For $A \in \mathcal{B}(U) \times \mathcal{B}([0, \infty))$, $\xi(A)$ has a Poisson distribution with expectation $\nu \times m(A)$ and $\xi(A)$ and $\xi(B)$ are independent if $A \cap B = \emptyset$.

$$\xi(A, t) \equiv \xi(A \times [0, t])$$ is a Poisson process with parameter $\nu(A)$.

$$\tilde{\xi}(A, t) \equiv \xi(A \times [0, t]) - \nu(A)t$$ is a martingale.

$\xi$ is $\{\mathcal{F}_t\}$ compatible, if for each $A \in \mathcal{B}(U)$, $\xi(A, \cdot)$ is $\{\mathcal{F}_t\}$ adapted and for all $t, s \geq 0$, $\xi(A \times (t, t + s])$ is independent of $\mathcal{F}_t$. 
Stochastic integrals for Poisson random measures

\( X \) cadlag, \( L_1(\nu) \)-valued, \( \{ \mathcal{F}_t \} \)-adapted

We define

\[
I_\xi(X, t) = \int_{U \times [0, t]} X(u, s-) \xi(du \times ds)
\]

in such a way that

\[
E \left[ |I_\xi(X, t)| \right] \leq E \left[ \int_{U \times [0, t]} |X(u, s-)\xi(du \times ds) | \right]
\]

\[
= \int_{U \times [0, t]} E[|X(u, s)|] \nu(du)ds
\]

If the right side is finite, then

\[
E \left[ \int_{U \times [0, t]} X(u, s-) \xi(du \times ds) \right] = \int_{U \times [0, t]} E[X(u, s)] \nu(du)ds
\]
Predictable integrands

The integral extends to *predictable* integrands satisfying

\[ \int_{U\times[0,t]} |X(u, s)| \wedge 1\nu(du)ds < \infty \quad a.s. \]

so that

\[ E[I_\xi(X, t \wedge \tau)] = E \left[ \int_{U\times[0,t\wedge\tau]} X(u, s)\xi(du \times ds) \right] \]

for any stopping time satisfying

\[ E \left[ \int_{U\times[0,t\wedge\tau]} |X(u, s)|\nu(du)ds \right] < \infty \tag{1} \]

If (1) holds for all \( t \), then

\[ \int_{U\times[0,t\wedge\tau]} X(u, s)\xi(du \times ds) - \int_{U\times[0,t\wedge\tau]} X(u, s)\nu(du)ds \]

is a martingale.

\( X \) is predictable if it is the pointwise limit of adapted, left-continuous processes.
Stochastic integrals for centered Poisson random measures

\( X \) cadlag, \( L_2(\nu) \)-valued, \( \{F_t\} \)-adapted

We define

\[
I_{\tilde{\xi}}(X, t) = \int_{U \times [0,t]} X(u, s-) \tilde{\xi}(du \times ds)
\]

in such a way that \( E\left[ I_{\tilde{\xi}}(X, t)^2 \right] = \int_{U \times [0,t]} E[X(u, s)^2] \nu(du) ds \) if the right side is finite. Then \( I_{\tilde{\xi}}(X, \cdot) \) is a square-integrable martingale.

The integral extends to \textit{predictable} integrands satisfying

\[
\int_{U \times [0,t]} |X(u, s)|^2 \wedge |X(u, s)| \nu(du) ds < \infty \quad a.s.
\]

so that \( I_{\tilde{\xi}}(X, t \wedge \tau) \) is a martingale for any stopping time satisfying

\[
E\left[ \int_{U \times [0,t \wedge \tau]} |X(u, s)|^2 \wedge |X(u, s)| \nu(du) ds \right] < \infty
\]
Itô equations for Markov processes with jumps

$W$ Gaussian white noise on $U_0 \times [0, \infty)$ with $E[W(A, t)W(B, t)] = \mu(A \cap B)t$

$\xi$ Poisson random measure on $U_i \times [0, \infty)$ with mean measure $\nu_i$.

Let

$$\tilde{\xi}_i(A, t) = \xi_i(A, t) - \nu_i(A)t$$

$$\tilde{\sigma}^2(x) = \int \sigma^2(x, u)\mu(du) < \infty$$

$$\int \alpha^2_1(x, u) \wedge |\alpha_1(x, u)|\nu_1(du) < \infty, \quad \int |\alpha_2(x, u)| \wedge 1\nu_2(du) < \infty$$

and

$$X(t) = X(0) + \int_{U_0 \times [0, t]} \sigma(X(s-), u)W(du \times ds) + \int_0^t \beta(X(s-))ds$$

$$+ \int_{U_1 \times [0, t]} \alpha_1(X(s-), u)\tilde{\xi}_1(du \times ds)$$

$$+ \int_{U_2 \times [0, t]} \alpha_2(X(s-), u)\xi_2(du \times ds).$$
A martingale inequality

Continuous time version: Lenglart, Lepingle, and Pratelli (1980).
Easy proof: Ichikawa (1986).

Lemma 1 For $0 < p \leq 2$ there exists a constant $C_p$ such that for any locally square integrable martingale $M$ with Meyer process $\langle M \rangle$ and any stopping time $\tau$

$$E[\sup_{s \leq \tau} |M(s)|^p] \leq C_p E[\langle M \rangle_{\tau}^{p/2}]$$

$\langle M \rangle$ is the (essentially unique) predictable process such that $M^2 - \langle M \rangle$ is a local martingale. (A left-continuous, adapted process is predictable.)
Graham’s uniqueness theorem

Lipschitz condition

\[
\sqrt{\int |\sigma(x, u) - \sigma(y, u)|^2 \mu(du) + |\beta(x) - \beta(y)|} \\
+ \sqrt{\int |\alpha_1(x, u) - \alpha_1(y, u)|^2 \nu_1(du) + \int |\alpha_2(x, u) - \alpha_2(y, u)| \nu_2(du)} \\
\leq M |x - y|
\]
Estimate

$X$ and $\tilde{X}$ solution of SDE.

\[
E[\sup_{s \leq t} |X(s) - \tilde{X}(s)|] \\
\leq E[|X(0) - \tilde{X}(0)|] + C_1 E[\left( \int_0^t \int_{U_0} |\sigma(X(s), u) - \sigma(\tilde{X}(s), u)|^2 \mu(du) ds \right)^{\frac{1}{2}}] \\
+ C_1 E[\left( \int_0^t \int_{U_1} |\alpha_1(X(s), u) - \alpha_1(\tilde{X}(s), u)|^2 \nu_1(du) ds \right)^{\frac{1}{2}}] \\
+ E[\int_0^t \int_{U_2} |\alpha_2(X(s), u) - \alpha_2(\tilde{X}(s), u)| \nu_2(du) ds] \\
+ E[\int_0^t |\beta(X(s)) - \beta(\tilde{X}(s))| ds] \\
\leq E[|X(0) - \tilde{X}(0)|] + D(\sqrt{t} + t) E[\sup_{s \leq t} |X(s) - \tilde{X}(s)|]
Boundary conditions

$X$ has values in $D$ and satisfies

$$X(t) = X(0) + \int_0^t \sigma(X(s))dW(s) + \int_0^t b(X(s))ds + \int_0^t \eta(X(s))d\lambda(s)$$

where $\lambda$ is nondecreasing and increases only when $X(t) \in \partial D$.

$X$ has values in $D$ and satisfies

$$X(t) = X(0) + \int_0^t \sigma(X(s))dW(s) + \int_0^t b(X(s))ds + \int_0^t \alpha(X(s-), \zeta_{N(s-)+1})dN(s)$$

where $\zeta_1, \zeta_2, \ldots$ are iid and independent of $X(0)$ and $W$ and $N(t)$ is the number of times $X$ has hit the boundary by time $t$. 
2. Martingale problems for Markov processes

- Levy and Watanabe characterizations
- Ito’s formula and martingales associated with solutions of stochastic equations
- Generators and martingale problems for Markov processes
- Equivalence between stochastic equations and martingale problems
Martingale characterizations

**Brownian motion** (Levy)

$W$ a continuous $\{\mathcal{F}_t\}$-martingale

$W(t)^2 - t$ an $\{\mathcal{F}_t\}$-martingale

Then $W$ is a standard Brownian motion compatible with $\{\mathcal{F}_t\}$.

**Poisson process** (Watanabe)

$N$ a counting process adapted to $\{\mathcal{F}_t\}$

$N(t) - \lambda t$ an $\{\mathcal{F}_t\}$-martingale

Then $N$ is a Poisson process with parameter $\lambda$ compatible with $\{\mathcal{F}_t\}$.
Diffusion processes

\[ \sigma : \mathbb{R}^d \to M^{d \times m} \text{ and } b : \mathbb{R}^d \to \mathbb{R}^d \]

Consider

\[ X(t) = X(0) + \int_0^t \sigma(X(s))dW(s) + \int_0^t b(X(s))ds \]

By Itô’s formula

\[ f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds = \int_0^t \nabla f(X(s))^T \sigma(X(s))dW(s) \]

where

\[ Af(x) = \frac{1}{2} \sum a_{ij}(x) \partial_i \partial_j f(x) + b(x) \cdot \nabla f(x) \]

with \(((a_{ij}(x)) = \sigma(x)\sigma(x)^T.\]
Martingale problem for $A$

Assume $a_{ij}$ and $b_i$ are locally bounded, and let $\mathcal{D}(A) = C_c^2(\mathbb{R}^d)$. $X$ is a solution of the martingale problem for $A$ if there exists a filtration $\{\mathcal{F}_t\}$ such that $X$ is $\{\mathcal{F}_t\}$-adapted and

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds$$

is an $\{\mathcal{F}_t\}$-martingale for each $f \in \mathcal{D}(A)$.

Any solution of the SDE is a solution of the martingale problem.
General SDE

Let

\[ X(t) = X(0) + \int_{U_0 \times [0,t]} \sigma(X(s-), u) W(du \times ds) + \int_0^t \beta(X(s-)) ds \]
\[ + \int_{U_1 \times [0,t]} \alpha_1(X(s-), u) \tilde{\xi}_1(du \times ds) \]
\[ + \int_{U_2 \times [0,t]} \alpha_2(X(s-), u) \xi_2(du \times ds) . \]

Then

\[ f(X(t)) - f(X(0)) - \int_0^t Af(X(s)) ds \]
\[ = \int_{U_0 \times [0,t]} \nabla f(X(s)) \cdot \sigma(X(s), u) W(du \times ds) \]
\[ + \int_{U_1} (f(X(s-)) + \alpha_1(X(s-), u)) - f(X(s-)) \tilde{\xi}_1(du \times ds) \]
\[ + \int_{U_2} (f(X(s-)) + \alpha_2(X(s-), u)) - f(X(s-)) \tilde{\xi}_2(du \times ds) \]
Form of the generator

\[ Af(x) = \frac{1}{2} \sum a_{ij}(x) \partial_i \partial_j f(x) + b(x) \cdot \nabla f(x) \]

\[ + \int_{U_1} (f(x + \alpha_1(x,u)) - f(x) - \alpha_1(x,u) \cdot \nabla f(x)) \nu_1(du) \]

\[ + \int_{U_2} (f(x + \alpha_2(x,u)) - f(x)) \nu_2(du) \]

Let \( \mathcal{D}(A) \) be a collection of functions for which \( Af \) is bounded. Then a solution of the SDE is a solution of the martingale problem for \( A \).
The martingale problem for $A$

$X$ is a solution for the martingale problem for $(A, \nu_0)$, $\nu_0 \in \mathcal{P}(E)$, if $PX(0)^{-1} = \nu_0$ and there exists a filtration $\{\mathcal{F}_t\}$ such that

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds$$

is an $\{\mathcal{F}_t\}$-martingale for all $f \in \mathcal{D}(A)$.

**Theorem 2** If any two solutions of the martingale problem for $A$ satisfying $PX_1(0)^{-1} = PX_2(0)^{-1}$ also satisfy $PX_1(t)^{-1} = PX_2(t)^{-1}$ for all $t \geq 0$, then the f.d.d. of a solution $X$ are uniquely determined by $PX(0)^{-1}$.

If $X$ is a solution of the MGP for $A$ and $Y_a(t) = X(a + t)$, then $Y_a$ is a solution of the MGP for $A$.

**Theorem 3** If the conclusion of the above theorem holds, then any solution of the martingale problem for $A$ is a Markov process.

$A$ is called the *generator* for the Markov process.
Weak solutions of SDEs

\( \tilde{X} \) is a weak solution of the SDE if there exists a probability space on which are defined \( X, W, \xi_1, \) and \( \xi_2 \) satisfying the SDE and \( \tilde{X} \) has the same distribution as \( X. \)

**Theorem 4** Suppose that the \( a_{ij} \) and \( b_i \) are locally bounded and that for each \( f \in C^2_c(\mathbb{R}^d) \)

\[
\sup_x \int_{U_1} \left| f(x + \alpha_1(x,u)) - f(x) - \alpha_1(x,u) \cdot \nabla f(x) \right| \nu_1(du) < \infty
\]

and

\[
\sup_x \int_{U_2} \left| f(x + \alpha_2(x,u)) - f(x) \right| \nu_2(du) < \infty.
\]

Let \( D(A) = C^2_c(\mathbb{R}^d). \) Then any solution of the martingale problem for \( A \) is a weak solution of the SDE.
Nonsingular diffusions

Consider the SDE

\[ X(t) = X(0) + \int_0^t \sigma(X(s))dW(s) + b(X(s))ds \]

and assume that \( d = m \) (that is, \( \sigma \) is square). Let

\[ Af(x) = \frac{1}{2} \sum a_{ij}(x) \partial_i \partial_j f(x) + b(x) \cdot \nabla f(x). \]

Assume that \( \sigma(x) \) is invertible for each \( x \) and that \( |\sigma(x)^{-1}| \) is locally bounded. If \( \tilde{X} \) is a solution of the martingale problem for \( A \), then

\[ M(t) = \tilde{X}(t) - \int_0^t b(\tilde{X}(s))ds \]

is a local martingale and

\[ \tilde{W}(t) = \int_0^t \sigma(\tilde{X}(s))^{-1}dM(s) \]

is a standard Brownian motion compatible with \( \{\mathcal{F}_{\tilde{X}}^t\} \). It follows that

\[ \tilde{X}(t) = \tilde{X}(0) + \int_0^t \sigma(\tilde{X}(s))d\tilde{W}(s) + \int_0^t b(\tilde{X}(s))ds. \]
Natural form for the jump terms

Consider the generator for a simple pure jump process

\[ Af(x) = \lambda(x) \int_{\mathbb{R}^d} (f(z) - f(x)) \mu(x, dz), \]

where \( \lambda \geq 0 \) and \( \mu(x, \cdot) \) is a probability measure. There exists \( \gamma : \mathbb{R}^d \times [0, 1] \to \mathbb{R}^d \) such that

\[
\int_0^1 f(x + \gamma(x, u)) du = \int_{\mathbb{R}^d} f(z) \mu(x, dz).
\]

Let \( \xi \) be a Poisson random measure on \([0, \infty) \times [0, 1] \times [0, \infty)\) with Lebesgue mean measure. Then

\[
X(t) = X(0) + \int_{[0, \infty) \times [0, 1] \times [0, t]} I_{[0, \lambda(X(s-))]}(v) \gamma(X(s-), u) \xi(dv \times du \times ds)
\]

is a stochastic differential equation corresponding to \( A \). Note that this SDE is of the form above with \( U_2 = [0, \infty) \times [0, 1] \) and \( \alpha_2(x, u, v) = I_{[0, \lambda(x)]}(v) \gamma(x, u) \) and that

\[
\int_{[0, \infty) \times [0, 1]} |\alpha_2(x, u, v) - \alpha_2(y, u, v)| dvdu \leq |\lambda(x) - \lambda(y)| \int_{[0, 1]} \gamma(x, u) du
\]

\[ + \lambda(y) \int_{[0, 1]} |\gamma(x, u) - \gamma(y, u)| du \]
More general jump terms

\[ X(t) = X(0) + \int_{[0,\infty) \times U \times [0,t]} I_{[0,\lambda(X(s-),u)]}(v) \gamma(X(s-),u) \xi(dv \times du \times ds) \]

where \( \xi \) is a Poisson random measure with mean measure \( m \times \nu \times m \). The generator is of the form

\[ Af(x) = \int_U \lambda(x,u)(f(x + \gamma(x,u)) - f(x))\nu(du). \]

If \( \xi \) is replaced by \( \tilde{\xi} \), then

\[ Af(x) = \int_U \lambda(x,u)(f(x + \gamma(x,u)) - f(x) - \gamma(x,u) \cdot \nabla f(x))\nu(du). \]

Note that many different choices of \( \lambda, \gamma, \) and \( \nu \) will produce the same generator.
Reflecting diffusions

$X$ has values in $D$ and satisfies

$$X(t) = X(0) + \int_0^t \sigma(X(s))dW(s) + \int_0^t b(X(s))ds + \int_0^t \eta(X(s))d\lambda(s)$$

where $\lambda$ is nondecreasing and increases only when $X(t) \in \partial D$. By Itô's formula

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds - \int_0^t Bf(X(s))d\lambda(s) = \int_0^t \nabla f(X(s))^T \sigma(X(s))dW(s)$$

where

$$Af(x) = \frac{1}{2} \sum a_{ij}(x) \partial_i \partial_j f(x) + b(x) \cdot \nabla f(x) \quad Bf(x) = \eta(x) \cdot \nabla f(x)$$

Either take $\mathcal{D}(A) = \{f \in C_c^2(D) : Bf(x) = 0, x \in \partial D\}$ or formulate a constrained martingale problem with solution $(X, \lambda)$ by requiring $X$ to take values in $D$, $\lambda$ to be nondecreasing and increase only when $X \in \partial D$, and

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds - \int_0^t Bf(X(s))d\lambda(s)$$

to be an $\mathcal{F}_{t}^{X,\lambda}$-martingale.
Instantaneous jump conditions

$X$ has values in $D$ and satisfies

$$X(t) = X(0) + \int_0^t \sigma(X(s))dW(s) + \int_0^t b(X(s))ds + \int_0^t \alpha(X(s-), \zeta_{N(s-)+1})dN(s)$$

where $\zeta_1, \zeta_2, \ldots$ are iid and independent of $X(0)$ and $W$ and $N(t)$ is the number of times $X$ has hit the boundary by time $t$. Then

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds - \int_0^t Bf(X(s-))dN(s)$$

$$= \int_0^t \nabla f(X(s))^T \sigma(X(s))dW(s)$$

$$+ \int_0^t (f(X(s-)) + \alpha(X(s-), \zeta_{N(s-)+1}))$$

$$- \int_\mathcal{U} f(X(s-)) + \alpha(X(s-), u))\nu(du))dN(s)$$

where

$$Af(x) = \frac{1}{2} \sum a_{ij}(x) \partial_i \partial_j f(x) + b(x) \cdot \nabla f(x)$$

and

$$Bf(x) = \int_\mathcal{U} (f(x + \alpha(x, u)) - f(x))\nu(du).$$
3. Weak convergence for stochastic processes

- General definition of weak convergence
- Prohorov’s theorem
- Skorohod representation theorem
- Skorohod topology
- Conditions for tightness in the Skorohod topology
Topological proof of convergence

$(S, d)$ metric space

$F_n : S \rightarrow \mathbb{R}$

$F_n \rightarrow F$ in some sense (e.g., $x_n \rightarrow x$ implies $F_n(x_n) \rightarrow F(x)$)

$F_n(x_n) = 0$

1. Show that $\{x_n\}$ is compact
2. Show that any limit point of $\{x_n\}$ satisfies $F(x) = 0$
3. Show that the equation $F(x) = 0$ has a unique solution $x_0$
4. Conclude that $x_n \rightarrow x_0$
Convergence in distribution

$(S,d)$ complete, separable metric space

$X_n$ $S$-valued random variable

$\{X_n\}$ converges in distribution to $X$ ($\{P_{X_n}\}$ converges weakly to $P_X$) if for each $f \in C(S)$

$$\lim_{n \to \infty} E[f(X_n)] = E[f(X)].$$

Denote convergence in distribution by $X_n \Rightarrow X$.

Equivalent statements

$\{X_n\}$ converges in distribution to $X$ if and only if

$$\liminf_{n \to \infty} P\{X_n \in A\} \geq P\{X \in A\}, \text{ each open } A,$$

or equivalently

$$\limsup_{n \to \infty} P\{X_n \in B\} \leq P\{X \in B\}, \text{ each closed } B,$$
Tightness and Prohorov’s theorem

A sequence \( \{X_n\} \) is tight if for each \( \epsilon > 0 \), there exists a compact set \( K_\epsilon \subset S \) such that

\[
\sup_n P\{X_n \notin K_\epsilon\} \leq \epsilon.
\]

Theorem 5 Suppose that \( \{X_n\} \) is tight. Then there exists a subsequence \( \{n(k)\} \) along which the sequence converges in distribution.
Skorohod topology on $D_E[0,\infty)$

$(E,r)$ complete, separable metric space

$D_E[0,\infty)$ space of cadlag, $E$-valued functions

$x_n \to x \in D_E[0,\infty)$ in the Skorohod ($J_1$) topology if and only if there exist strictly increasing $\lambda_n$ mapping $[0,\infty)$ onto $[0,\infty)$ such that for each $T > 0$,

$$\lim_{n \to \infty} \sup_{t \leq T} (|\lambda_n(t) - t| + r(x_n \circ \lambda_n(t), x(t))) = 0.$$

The Skorohod topology is metrizable so that $D_E[0,\infty)$ is a complete, separable metric space.

Note that $I_{[1+\frac{1}{n},\infty)} \to I_{[1,\infty)}$ in $D_{\mathbb{R}}[0,\infty)$, but $(I_{[1+\frac{1}{n},\infty)}, I_{[1,\infty)})$ does not converge in $D_{\mathbb{R}^2}[0,\infty)$. 
Conditions for tightness

\( S^n_0(T) \) collection of discrete \( \{F^n_t\}\)-stopping times \( q(x, y) = 1 \land r(x, y) \)

**Theorem 6** Suppose that for \( t \in T_0, \) a dense subset of \([0, \infty), \) \( \{X_n(t)\} \) is tight. Then the following are equivalent.

a) \( \{X_n\} \) is tight in \( D_E[0, \infty) \).

b) (Kurtz) For \( T > 0, \) there exist \( \beta > 0 \) and random variables \( \gamma_n(\delta, T) \) such that for \( 0 \leq t \leq T, \) \( 0 \leq u \leq \delta, \) and \( 0 \leq v \leq t \land \delta \)

\[
E[q^\beta(X_n(t + u), X_n(t)) \land q^\beta(X_n(t), X_n(t - v))]|F^n_t] \leq E[\gamma_n(\delta, T)|F^n_t]
\]

\[
\lim_{\delta \to 0} \limsup_{n \to \infty} E[\gamma_n(\delta, T)] = 0,
\]

and

\[
\lim_{\delta \to 0} \limsup_{n \to \infty} E[q^\beta(X_n(\delta), X_n(0))] = 0. \tag{2}
\]

c) (Aldous) Condition (2) holds, and for each \( T > 0, \) there exists \( \beta > 0 \) such that

\[
C_n(\delta, T) \equiv \sup_{\tau \in S^n_0(T)} \sup_{u \leq \delta} \sup_{v \leq \delta \land \tau} E[q^\beta(X_n(\tau + u), X_n(\tau)) \land q^\beta(X_n(\tau), X_n(\tau - v))]
\]

satisfies \( \lim_{\delta \to 0} \limsup_{n \to \infty} C_n(\delta, T) = 0. \)
Example

$\eta_1, \eta_2, \ldots$ iid, $E[\eta_i] = 0$, $\sigma^2 = E[\eta_i^2] < \infty$

$$X_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \eta_i$$

Then

$$E[(X_n(t + u) - X_n(t))^2 | \mathcal{F}_t^{X_n}] = \frac{[n(t + u)] - [nt]}{n} \sigma^2 \leq (\delta + \frac{1}{n})\sigma^2$$

for $u \leq \delta$. 
Uniqueness of limit

Theorem 7 If \( \{X_n\} \) is tight in \( D_E[0, \infty) \) and

\[
(X_n(t_1), \ldots, X_n(t_k)) \Rightarrow (X(t_1), \ldots, X(t_k))
\]

for \( t_1, \ldots, t_k \in \mathcal{T}_0, \mathcal{T}_0 \) dense in \( [0, \infty) \), then \( X_n \Rightarrow X \).

For the example, this condition follows from the central limit theorem.
Skorohod representation theorem

Theorem 8 Suppose that $X_n \Rightarrow X$. Then there exists a probability space $(\Omega, \mathcal{F}, P)$ and random variables, $\tilde{X}_n$ and $\tilde{X}$, such that $\tilde{X}_n$ has the same distribution as $X_n$, $\tilde{X}$ has the same distribution as $X$, and $\tilde{X}_n \rightarrow \tilde{X}$ a.s.

Continuous mapping theorem

Corollary 9 Let $G(X) : S \rightarrow E$ and define $C_G = \{x \in S : G \text{ is continuous at } x\}$. Suppose $X_n \Rightarrow X$ and that $P\{X \in C_G\} = 1$. Then $G(X_n) \Rightarrow G(X)$. 

Some mappings on $D_E[0, \infty)$

$\pi_t : D_E[0, \infty) \to E \quad \pi_t(x) = x(t)$

$C_{\pi_t} = \{x \in D_E[0, \infty) : x(t) = x(t-)\}$

$G_t : D_R[0, \infty) \to \mathbb{R} \quad G_t(x) = \sup_{s \leq t} x(s)$

$C_{G_t} = \{x \in D_R[0, \infty) : \lim_{s \to t-} G_s(x) = G_t(x)\} \supset \{x \in D_R[0, \infty) : x(t) = x(t-)\}$

$G : D_R[0, \infty) \to D_R[0, \infty), \quad G(x)(t) = G_t(x), \quad \text{is continuous}$

$H_t : D_E[0, \infty) \to \mathbb{R} \quad H_t(x) = \sup_{s \leq t} r(x(s), x(s-))$

$C_{H_t} = \{x \in D_E[0, \infty) : \lim_{s \to t-} H_s(x) = H_t(x)\} \supset \{x \in D_E[0, \infty) : x(t) = x(t-)\}$

$H : D_E[0, \infty) \to D_R[0, \infty), \quad H(x)(t) = H_t(x), \quad \text{is continuous}$
Level crossing times

\[ \tau_c : D_{\mathbb{R}}[0, \infty) \rightarrow [0, \infty) \quad \tau_c(x) = \inf \{ t : x(t) > c \} \]

\[ \tau_c^- : D_{\mathbb{R}}[0, \infty) \rightarrow [0, \infty) \quad \tau_c^-(x) = \inf \{ t : x(t) \geq c \text{ or } x(t^-) \geq c \} \]

\[ G_{\tau_c} = G_{\tau_c^-} = \{ x : \tau_c(x) = \tau_c^-(x) \} \]

Note that \( \tau_c^-(x) \leq \tau_c(x) \) and that \( x_n \rightarrow x \) implies

\[ \tau_c^-(x) \leq \liminf_{n \rightarrow \infty} \tau_c^-(x_n) \leq \limsup_{n \rightarrow \infty} \tau_c(x_n) \leq \tau_c(x) \]
Localization

**Theorem 10** Suppose that for each $\alpha > 0$, $\tau_n^\alpha$ satisfies

$$P\{\tau_n^\alpha > \alpha\} \leq \alpha^{-1}$$

and $\{X_n(\cdot \wedge \tau_n^\alpha)\}$ is relatively compact. Then $\{X_n\}$ is relatively compact.

Compactification of the state space

**Theorem 11** Let $E \subset E_0$ where $E_0$ is compact and the topology on $E$ is the restriction of the topology on $E_0$. Suppose that for each $n$, $X_n$ is a process with sample paths in $D_E[0, \infty)$ and that $X_n \Rightarrow X$ in $D_{E_0}[0, \infty)$. If $X$ has sample paths in $D_E[0, \infty)$, then $X_n \Rightarrow X$ in $D_E[0, \infty)$. 
4. Convergence for Markov processes characterized by stochastic equations

- Martingale central limit theorem
- Convergence for stochastic integrals
- Convergence for SDEs driven by semimartingales
- Diffusion approximations for Markov chains
- Limit theorems involving Poisson random measures and Gaussian white noise
Martingale central limit theorem

Let $M_n$ be a martingale such that

$$\lim_{n \to \infty} E[\sup_{s \leq t} |M_n(s) - M_n(s-)|] = 0$$

and

$$[M_n]_t \to ct$$

in probability.

Then $M_n \Rightarrow \sqrt{c} W$.

**Vector-valued version:** If for each $1 \leq i \leq d$

$$\lim_{n \to \infty} E[\sup_{s \leq t} |M^i_n(s) - M^i_n(s-)|] = 0$$

and for each $1 \leq i, j \leq d$,

$$[M^i_n, M^j_n]_t \to c_{ij} t,$$

then $M_n \Rightarrow \sigma W$, where $W$ is $d$-dimensional standard Brownian motion and $\sigma$ is a symmetric $d \times d$-matrix satisfying $\sigma^2 = c = ((c_{ij}))$. 
**Example: Products of random matrices**

Let $A^{(1)}, A^{(2)}, \ldots$ be iid random matrices with $E[A^{(k)}] = 0$ and $E[|A^{(k)}|^2] < \infty$. Set $X_0 = I$

$$X(k+1) = (I + \frac{1}{\sqrt{n}} A^{(k+1)}) X(k) = (I + \frac{1}{\sqrt{n}} A^{(k+1)}) \cdots (I + \frac{1}{\sqrt{n}} A^{(1)})$$

$$X_n(t) = X([nt])$$

and

$$M_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} A^{(k)}.$$

Then

$$X_n(t) = X_n(0) + \int_0^t dM_n(s) X_n(s-).$$

$M_n \Rightarrow M$ where $M$ is a Brownian motion with

$$E[M_{ij}(t)M_{kl}(t)] = E[A_{ij}A_{kl}]t,$$

Can we conclude $X_n \Rightarrow X$ satisfying

$$X(t) = X(0) + \int_0^t dM(s) X(s)?$$
Example: Markov chains

\[ X_{k+1} = H(X_k, \xi_{k+1}) \text{ where } \xi_1, \xi_2 \ldots \text{ are iid} \]
\[ P\{X_{k+1} \in B \mid X_0, \xi_1, \ldots, \xi_k\} = P\{X_{k+1} \in B \mid X_k\} \]

Example:

\[ X_{n+1}^n = X_n^n + \sigma(X_n^n) \frac{1}{\sqrt{n}} \xi_{k+1} + b(X_{n+1}^n) \frac{1}{n} \]

Assume \( E[\xi_k] = 0 \) and \( Var(\xi_k) = 1. \)

Define \( X_n(t) = X_{\lfloor nt \rfloor}^n \quad A_n(t) = \frac{\lfloor nt \rfloor}{n} \quad W_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} \xi_k. \)

\[ X_n(t) = X_n(0) + \int_0^t \sigma(X_n(s-))dW_n(s) \]
\[ + \int_0^t b(X_n(s-))dA_n(s) \]

Can we conclude \( X_n \Rightarrow X \) satisfying

\[ X(t) = X(0) + \int_0^t \sigma(X(s))dW(s) + \int_0^t b(X(s))ds? \]
Stochastic integration

**Definition:** For cadlag processes $X$, $Y$,

$$\int_0^t X(s-)dY(s) = \lim_{\max |t_{i+1} - t_i| \to 0} \sum X(t_i)(Y(t_{i+1} \wedge t) - Y(t_i \wedge t))$$

whenever the limit exists in probability.

**Sample paths of bounded variation:** If $Y$ is a finite variation process, the stochastic integral exists (apply dominated convergence theorem) and

$$\int_0^t X(s-)dY(s) = \int_{(0,t]} X(s-)\alpha_Y(ds)$$

$\alpha_Y$ is the signed measure with

$$\alpha_Y(0, t] = Y(t) - Y(0)$$
Existence for square integrable martingales

If $M$ is a square integrable martingale, then

$$E[(M(t + s) - M(t))^2 | \mathcal{F}_t] = E[[M]_{t+s} - [M]_t | \mathcal{F}_t]$$

Let $X$ be bounded, cadlag and adapted. Then For partitions $\{t_i\}$ and $\{r_i\}$

$$E[(\sum X(t_i)(M(t_{i+1} \land t) - M(t_i \land t)) - \sum X(r_i)(M(r_{i+1} \land t) - M(r_i \land t)))^2]$$

$$= E\left[\int_0^t (X(t(s-)) - X(r(s-)))^2 d[M]_s\right]$$

$$= E\left[\int_{(0,t]} (X(t(s-)) - X(r(s-)))^2 \alpha[M](ds)\right]$$

$t(s) = t_i$ for $s \in [t_i, t_{i+1})$  $r(s) = r_i$ for $s \in [r_i, r_{i+1})$

Let $\sigma_c = \inf\{t : |X(t)| \geq c\}$ and

$$X^c(t) = \begin{cases} X(t) & t < \sigma_c \\ X(c-) & t \geq \sigma_c \end{cases}$$

Then for

$$\int_0^{t\land \tau_c} X(s-)dY(s) = \int_0^{t\land \tau_c} X^c(s-)dY(s)$$
Semimartingales

Definition: A cadlag, \( \{ \mathcal{F}_t \} \)-adapted process \( Y \) is an \( \{ \mathcal{F}_t \} \)-semimartingale if:

\[
Y = M + V
\]

\( M \) a local, square integrable \( \{ \mathcal{F}_t \} \)-martingale

\( V \) an \( \{ \mathcal{F}_t \} \)-adapted, finite variation process

Total variation: For a finite variation process

\[
T_t(Z) \equiv \sup_{\{ t_i \}} \sum |Z(t_{i+1} \wedge t) - Z(t_i \wedge t)| < \infty
\]

Quadratic variation: For cadlag semimartingale

\[
[Y]_t = \lim_{\max|t_{i+1} - t_i| \to 0} \sum (Y(t_{i+1} \wedge t) - Y(t_i \wedge t))^2
\]

Covariation: For cadlag semimartingales

\[
[Y, Z]_t = \lim \sum (Y(t_{i+1} \wedge t) - Y(t_i \wedge t))(Z(t_{i+1} \wedge t) - Z(t_i \wedge t))
\]
Probability estimates for SIs

\[ Y = M + V \]

\( M \) local square-integrable martingale

\( V \) finite variation process

Assume \(|X| \leq 1\).

\[
P\{ \sup_{s \leq t} | \int_0^s X(r-)dY(r)| > K \} \]

\[
\leq P\{\sigma \leq t\} + P\left\{ \sup_{s \leq t \wedge \sigma} | \int_0^s X(r-)dM(r)| > K/2 \right\} 
+ P\{\sup_{s \leq t} | \int_0^s X(r-)dV(r)| > K/2 \} 
\]

\[
\leq P\{\sigma \leq t\} + \frac{16E[\int_0^{t\wedge \sigma} X^2(s-)d[M]_s]}{K^2} 
+ P\left\{ \int_0^t |X(s-)dT_s(V) \geq K/2 \right\} 
\]

\[
\leq P\{\sigma \leq t\} + \frac{16E[M_{t \wedge \sigma}]}{K^2} 
+ P\{T_t(V) \geq K/2 \} 
\]
Good integrator condition

Let $\mathcal{S}^0$ be the collection of $X = \sum \xi_i I_{[\tau_i, \tau_{i+1})}$, where $\tau_1 < \cdots < \tau_m$ are stopping times and $\xi_i$ is $\mathcal{F}_{\tau_i}$-measurable. Then

$$\int_0^t X(s-)dY(s) = \sum \xi_i (Y(t \wedge \tau_{i+1}) - Y(t \wedge \tau_i)).$$

If $Y$ is a semimartingale, then

$$\{\int_0^t X(s-)dY(s) : X \in \mathcal{S}^0, |X| \leq 1\}$$

is stochastically bounded.

$Y$ satisfying this stochastic boundedness condition is a good integrator.

Bichteler-Dellacherie: $Y$ is a good integrator if and only if $Y$ is a semimartingale. (See, for example, Protter (1990), Theorem III.22.)
Uniformity conditions

\( Y_n = M_n + A_n \), a semimartingale adapted to \( \{ \mathcal{F}_t^n \} \)

\( T_t(A_n) \), the total variation of \( A_n \) on \([0, t]\)

\([M_n]_t\), the quadratic variation of \( M_n \) on \([0, t]\)

**Uniform tightness (UT) [JMP]:**

\[
\mathcal{H}_t^0 = \bigcup_{n=1}^{\infty} \{ \int_0^t Z(s-)dY_n(s) : Z \in S_0^n, \sup_{s \leq t} |Z(s)| \leq 1 \}
\]

is stochastically bounded.

**Uniformly controlled variations (UCV) [KP]:** \( \{T_t(A_n), n = 1, 2, \ldots\} \) is stochastically bounded, and there exist stopping times \( \{\tau_n^\alpha\} \) such that \( P\{\tau_n^\alpha \leq \alpha\} \leq \alpha^{-1} \) and \( \sup_n E[[M_n]_{t \wedge \tau_n^\alpha}] < \infty \)

A sequence of semimartingales \( \{Y_n\} \) that converges in distribution and satisfies either UT or UCV will be called *good*. 
Basic convergence theorem

Theorem. (Jakubowski, Mémin and Pagès; Kurtz & Protter)

\((X_n, Y_n) \{F^n_t\}\)-adapted in \(D_{\mathbb{M}^{km} \times \mathbb{R}^m}[0, \infty)\).

\(Y_n = M_n + A_n \) an \(\{F^n_t\}\)-semimartingale

Assume that \(\{Y_n\}\) satisfies either UT or UCV.

If \((X_n, Y_n) \Rightarrow (X, Y)\) in \(D_{\mathbb{M}^{km} \times \mathbb{R}^m}[0, \infty)\) with the Skorohod topology.

THEN

\[(X_n, Y_n, \int X_n(s-)dY_n(s)) \Rightarrow (X, Y, \int X(s-)dY(s))\]

in \(D_{\mathbb{M}^{km} \times \mathbb{R}^m \times \mathbb{R}^k}[0, \infty)\)

If \((X_n, Y_n) \rightarrow (X, Y)\) in probability in the Skorohod topology on \(D_{\mathbb{M}^{km} \times \mathbb{R}^m}[0, \infty)\)

THEN

\[(X_n, Y_n, \int X_n(s-)dY_n(s)) \rightarrow (X, Y, \int X(s-)dY(s))\]

in probability in \(D_{\mathbb{M}^{km} \times \mathbb{R}^m \times \mathbb{R}^k}[0, \infty)\)

"IN PROBABILITY" CANNOT BE REPLACED BY "A.S."
Stochastic differential equations

Suppose

$$F : D_{\mathbb{R}^d}[0, \infty) \rightarrow D_{\mathbb{M}^{d \times m}}[0, \infty)$$

is nonanticipating in the sense that

$$F(x, t) = F(x^t, t)$$

where $x^t(s) = x(s \wedge t)$. 

$U$ cadlag, adapted, with values in $\mathbb{R}^d$

$Y$ an $\mathbb{R}^m$-valued semimartingale.

Consider

$$X(t) = U(t) + \int_0^t F(X, s-)dY(s) \quad (3)$$

$(X, \tau)$ is a local solution of (3) if $X$ is adapted to a filtration $\{\mathcal{F}_t\}$ such that $Y$ is an $\{\mathcal{F}_t\}$-semimartingale, $\tau$ is an $\{\mathcal{F}_t\}$-stopping time, and

$$X(t \wedge \tau) = U(t \wedge \tau) + \int_0^{t \wedge \tau} F(X, s-)dY(s), \quad t \geq 0.$$ 

Strong local uniqueness holds if for any two local solutions $(X_1, \tau_1)$, $(X_2, \tau_2)$ with respect to a common filtration, we have $X_1(\cdot \wedge \tau_1 \wedge \tau_2) = X_2(\cdot \wedge \tau_1 \wedge \tau_2)$. 
Sequences of SDE’s

\[ X_n(t) = U_n(t) + \int_0^t F_n(X_n, s-)dY_n(s). \]

Structure conditions

\[ T_1[0, \infty) = \{ \gamma : \gamma \text{ nondecr. and maps } [0, \infty) \text{ onto } [0, \infty), \gamma(t + h) - \gamma(t) \leq h \} \]

C.a \textit{F}_n\textit{ behaves well under time changes: } If \( \{x_n\} \subset D_{\mathbb{R}^d}[0, \infty), \{\gamma_n\} \subset T_1[0, \infty), \text{ and } \{x_n \circ \gamma_n\} \text{ is relatively compact in } D_{\mathbb{R}^d}[0, \infty), \text{ then } \{F_n(x_n) \circ \gamma_n\} \text{ is relatively compact in } D_{\mathbb{M}^{d\times m}}[0, \infty).\]

C.b \textit{F}_n\textit{ converges to } F: \text{ If } (x_n, y_n) \rightarrow (x, y) \text{ in } D_{\mathbb{R}^d \times \mathbb{R}^m}[0, \infty), \text{ then } (x_n, y_n, F_n(x_n)) \rightarrow (x, y, F(x)) \text{ in } D_{\mathbb{R}^d \times \mathbb{R}^m \times \mathbb{M}^{d\times m}}[0, \infty).\]

Note that C.b implies continuity of \( F \), that is, \( (x_n, y_n) \rightarrow (x, y) \) implies \( (x_n, y_n, F(x_n)) \rightarrow (x, y, F(x)) \).
Examples

\[ F(x, t) = f(x(t)), \quad f \text{ continuous} \]

\[ F(x, t) = f(\int_0^t h(x(s))ds) \quad f, h \text{ continuous} \]

\[ F(x, t) = \int_0^t h(t - s, s, x(s))ds, \quad h \text{ continuous} \]
Convergence theorem

**Theorem 12** Suppose

- \((U_n, X_n, Y_n)\) satisfies
  \[
  X_n(t) = U_n(t) + \int_0^t F_n(X_n, s-)dY_n(s).
  \]
- \((U_n, Y_n)\) ⇒ \((U, Y)\)
- \(\{Y_n\}\) is good
- \(\{F_n\}, F\) satisfy structure conditions
- \(\sup_n \sup_x \|F_n(x, \cdot)\| < \infty.\)

Then \(\{(U_n, X_n, Y_n)\}\) is relatively compact and any limit point satisfies

\[
X(t) = U(t) + \int_0^t F(X, s-)dY(s)
\]

If, in addition, \((U_n, Y_n)\) → \((U, Y)\) in probability and strong uniqueness holds, then \((U_n, X_n, Y_n)\) → \((U, X, Y)\) in probability.
Euler scheme

Let \( 0 = t_0 < t_1 < \cdots \). The Euler approximation for

\[
X(t) = U(t) + \int_0^t F(X, s-)dY(s)
\]

is given by

\[
\hat{X}(t_{k+1}) = \hat{X}(t_k) + U(t_{k+1}) - U(t_k) + F(\hat{X}, t_k)(Y(t_{k+1}) - Y(t_k)).
\]

Consistency

Let \( \{t^n_k\} \) satisfy \( \max_k |t^n_{k+1} - t^n_k| \to 0 \), and define

\[
\begin{pmatrix}
Y_n(t) \\
U_n(t)
\end{pmatrix} = \begin{pmatrix}
Y(t^n_k) \\
U(t^n_k)
\end{pmatrix}, \quad t^n_k \leq t < t^n_{k+1}.
\]

Then \((U_n, Y_n) \Rightarrow (U, Y)\) and \(\{Y_n\}\) is good.

The Euler scheme corresponding to \(\{t^n_k\}\) satisfies

\[
\hat{X}_n(t) = U_n(t) + \int_0^t F(\hat{X}_n, s-)dY_n(s)
\]

If \(F\) satisfies the structure conditions and strong uniqueness holds, then \(\hat{X}_n \to X\) in probability. (In the diffusion case, Maruyama (1955))
Sequences of Poisson random measures

$\xi_n$ Poisson random measures with mean measures $n\nu \times m$.

$h$ measurable

For $A \in \mathcal{B}(U)$ satisfying $\int_A h^2(u)\nu(du) < \infty$, define

$$M_n(A, t) = \frac{1}{\sqrt{n}} \int_A h(u)(\xi_n(du \times [0, t]) - nt\nu(du)).$$

$M_n$ is an orthogonal martingale random measure with

$$[M_n(A), M_n(B)]_t = \frac{1}{n} \int_{A \cap B} h(u)^2 \xi_n(du \times [0, t])$$

$$\langle M_n(A), M_n(B) \rangle_t = t \int_{A \cap B} h(u)^2 \nu(du).$$

$M_n$ converges to Gaussian white noise $W$ with

$$E[W(A, t)W(B, s)] = t \wedge s \int_{A \cap B} h(u)^2 \nu(du)$$

$$E[W(\varphi_1, t)W(\varphi_2, s)] = t \wedge s \int \varphi_1(u)\varphi_2(u)h(u)^2\nu(du).$$
Continuous-time Markov chains

\[ X_n(t) = X_n(0) + \frac{1}{\sqrt{n}} \int_{U \times [0,t]} \alpha_1(X_n(s^-), u) \xi_n(du \times ds) \]
\[ + \frac{1}{n} \int_{U \times [0,t]} \alpha_2(X_n(s^-), u) \xi_n(du \times ds) \]

Assume \( \int_U \alpha_1(x, u) \nu(du) = 0 \). Then

\[ X_n(t) = X_n(0) + \frac{1}{\sqrt{n}} \int_{U \times [0,t]} \alpha_1(X_n(s^-), u) \tilde{\xi}_n(du \times ds) \]
\[ + \frac{1}{n} \int_{U \times [0,t]} \alpha_2(X_n(s^-), u) \xi_n(du \times ds) \]

Can we conclude \( X_n \Rightarrow X \) satisfying

\[ X(t) = X(0) + \int_{U \times [0,t]} \alpha_1(X(s), u) W(du \times ds) \]
\[ + \int_0^t \int_U \alpha_2(X(s^-), u) \nu(du) ds \]
Discrete-time Markov chains

Consider

\[ X_k^{n+1} = X_k^n + \sigma(X_k^n, \xi_k+1) \frac{1}{\sqrt{n}} + b(X_k^n, \zeta_k+1) \frac{1}{n} \]

where \( \{(\xi_k, \zeta_k)\} \) is iid in \( U_1 \times U_2 \).

\( \mu \) the distribution of \( \xi_k \)

\( \nu \) the distribution of \( \zeta_k \)

Assume \( \int_{U_1} \sigma(x, u_1) \mu(du_1) = 0 \)

Define

\[ M_n(A, t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} (I_A(\xi_k) - \mu(A)) \]

\[ V_n(B, t) = \frac{1}{n} \sum_{k=1}^{[nt]} I_B(\zeta_k) \]
Stochastic equation driven by random measure

Then $X_n(t) \equiv X_{[nt]}$ satisfies

$$X_n(t) = X_n(0) + \int_0^t \int_{U_1} \sigma_n(X_n(s), u) M_n(du \times ds)$$
$$+ \int_0^t \int_{U_2} b_n(X_n(s), u) V_n(du \times ds)$$

$$V_n(A, t) \rightarrow t\nu(A) \quad M_n(A, t) \Rightarrow M(A, t)$$

$M$ is Gaussian with covariance

$$E[M(A, t)M(B, s)] = t \wedge s(\mu(A \cap B) - \mu(A)\mu(B))$$

Can we conclude that $X_n \Rightarrow X$ satisfying

$$X(t) = X(0) + \int_0^t \int_{U_1} \sigma(X(s), u) M(du \times ds)$$
$$+ \int_0^t \int_{U_2} b(X(s), u) \nu(du)ds ?$$
Good integrator condition

$H$ a separable Banach space ($H = L_2(\nu), L_1(\nu), L_2(\mu)$, etc.)

$Y(\varphi, t)$ a semimartingale for each $\varphi \in H$

$Y(\sum a_k \varphi_k, t) = \sum_k a_k Y(\varphi_k, t)$

Let $\mathcal{S}$ be the collection of cadlag, adapted processes of the form $Z(t) = \sum_{k=1}^{m} \xi_k(t) \varphi_k$, $\varphi_k \in H$.

Define

$$I_Y(Z, t) = \int_{U \times [0,t]} Z(u, s-) Y(du \times ds) = \sum_k \int_0^t \xi_k(s-) dY(\varphi_k, s).$$

Basic assumption:

$$\mathcal{H}_t = \{ \sup_{s \leq t} |I_Y(Z, s)| : Z \in \mathcal{S}, \sup_{s \leq t} \|Z(s)\|_H \leq 1 \}$$

is stochastically bounded. (Call $Y$ a good $H^\#$-semimartingale.)

The integral extends to all cadlag, adapted $H$-valued processes.
Convergence for $H^\#$-semimartingales

$H$ a separable Banach space of functions on $U$

$Y_n$ an $\{F^n_t\}$-$H^\#$-semimartingale (for each $\varphi \in H$, $Y(\varphi, \cdot)$ is an $\{F^n_t\}$-semimartingale)

$\{X_n\}$ cadlag, $H$-valued processes

$$(X_n, Y_n) \Rightarrow (X, Y), \text{ if}$$

$$(X_n, Y_n(\varphi_1, \cdot), \ldots, Y_n(\varphi_m, \cdot)) \Rightarrow (X, Y(\varphi_1, \cdot), \ldots, Y(\varphi_m, \cdot))$$

in $D_{H \times \mathbb{R}^m}[0, \infty)$ for each choice of $\varphi_1, \ldots, \varphi_m \in H$. 
Convergence for Stochastic Integrals

Let
\[ \mathcal{H}_{n,t} = \{ \sup_{s \leq t} |I_{Y_n}(Z, s)| : Z \in S_n, \sup_{s \leq t} \|Z(s)\|_H \leq 1 \} . \]

**Definition:** \( \{Y_n\} \) is *uniformly tight* if \( \bigcup_n \mathcal{H}_{n,t} \) is stochastically bounded for each \( t \).

**Theorem 13** Protter and Kurtz (1996). Assume that \( \{Y_n\} \) is uniformly tight. If \( (X_n, Y_n) \Rightarrow (X, Y) \), then there is a filtration \( \{\mathcal{F}_t\} \), such that \( Y \) is an \( \{\mathcal{F}_t\} \)-adapted, good, \( H^\# \)-semimartingale, \( X \) is \( \{\mathcal{F}_t\} \)-adapted and
\[
(X_n, Y_n, I_{Y_n}(X_n)) \Rightarrow (X, Y, I_Y(X)) .
\]
If \( (X_n, Y_n) \rightarrow (X, Y) \) in probability, then \( (X_n, Y_n, I_{Y_n}(X_n)) \rightarrow (X, Y, I_Y(X)) \) in probability.

Cho (1994) for distribution-valued martingales

Sequences of SDE’s

\[ X_n(t) = U_n(t) + \int_{U \times [0,t]} F_n(X_n, s-, u)Y_n(du \times ds). \]

Structure conditions

\[ T_1[0,\infty) = \{ \gamma : \gamma \text{ nondecreasing and maps } [0,\infty) \text{ onto } [0,\infty), \gamma(t+h) - \gamma(t) \leq h \} \]

C.a \( F_n \) behaves well under time changes: If \( \{x_n\} \subset D_{\mathbb{R}^d}[0,\infty), \{\gamma_n\} \subset T_1[0,\infty), \) and \( \{x_n \circ \gamma_n\} \) is relatively compact in \( D_{\mathbb{R}^d}[0,\infty) \), then \( \{F_n(x_n) \circ \gamma_n\} \) is relatively compact in \( D_{H^d}[0,\infty) \).

C.b \( F_n \) converges to \( F \): If \( (x_n, y_n) \rightarrow (x, y) \) in \( D_{\mathbb{R}^d \times \mathbb{R}^m}[0,\infty) \), then \( (x_n, y_n, F_n(x_n)) \rightarrow (x, y, F(x)) \) in \( D_{\mathbb{R}^d \times \mathbb{R}^m \times H^d}[0,\infty) \).

Note that C.b implies continuity of \( F \), that is, \( (x_n, y_n) \rightarrow (x, y) \) implies \( (x_n, y_n, F(x_n)) \rightarrow (x, y, F(x)) \).
SDE convergence theorem

**Theorem 14** Suppose that \((U_n, X_n, Y_n)\) satisfies

\[
X_n(t) = U_n(t) + \int_{U \times [0, t]} F_n(X_n, s-, u)Y_n(du \times ds),
\]

that \((U_n, Y_n) \Rightarrow (U, Y)\), and that \(\{Y_n\}\) is uniformly tight. Assume that \(\{F_n\}\) and \(F\) satisfy the structure condition and that \(\sup_n \sup_x \|F_n(x, \cdot)\|_{H^d} < \infty\). Then \(\{(U_n, X_n, Y_n)\}\) is relatively compact and any limit point satisfies

\[
X(t) = U(t) + \int_{U \times [0, t]} F(X, s-, u)Y(du \times ds).
\]
5. Convergence for Markov processes characterized by martingale problems

- Tightness estimates based on generators
- Convergence of processes based on convergence of generators
- Averaging
- A measure-valued limit
Compactness conditions based on compactness of real functions

**Compact containment condition:** For each $T > 0$ and $\epsilon > 0$, there exists a compact $K \subset E$ such that

$$\lim_{n \to \infty} \inf P\{X_n(s) \in K, s \leq T\} \geq 1 - \epsilon.$$

**Theorem 15** Let $D$ be dense in $\overline{C}(E)$ in the compact uniform topology. $\{X_n\}$ is relatively compact (in distribution in $D_E[0, \infty)$) iff $\{X_n\}$ satisfies the compact containment conditions and $\{f \circ X_n\}$ is relatively compact for each $f \in D$.

Note that

$$E[(f(X_n(t + u)) - f(X_n(t)))^2|\mathcal{F}_t^n]$$

$$= E[f^2(X_n(t + u)) - f^2(X_n(t))|\mathcal{F}_t^n]$$

$$- 2f(X_n(t))E[f(X_n(t + u)) - f(X_n(t))|\mathcal{F}_t^n]$$
Limits of generators

\[ X_n(t) = \frac{1}{\sqrt{n}}(N_b(nt) - N_d(nt)), \]

where \( N_b, N_d \) are independent, unit Poisson processes.

For \( f \in C_c^3(\mathbb{R}) \)

\[ A_n f(x) = n \left( \frac{f(x + \frac{1}{\sqrt{n}}) + f(x - \frac{1}{\sqrt{n}})}{2} - f(x) \right) = \frac{1}{2} f''(x) + O\left( \frac{1}{\sqrt{n}} \right). \]

Set \( A f = \frac{1}{2} f''. \)

\[
\begin{align*}
E[(f(X_n(t + u)) - f(X_n(t)))^2 | \mathcal{F}^n_t] & = E[f^2(X_n(t + u)) - f^2(X_n(t)) | \mathcal{F}^n_t] \\
& \quad - 2f(X_n(t)) E[f(X_n(t + u)) - f(X_n(t)) | \mathcal{F}^n_t] \\
& \leq u(\|A_n f^2\| + 2\|f\|\|A_n f\|). 
\end{align*}
\]

It follows that \( \{X_n\} \) is relatively compact in \( D_{\mathbb{R}^\Delta}[0, \infty) \), or using the fact that

\[
P\{\sup_{s \leq t} |X_n(s)| \geq k\} \leq \frac{4E[X_n^2(t)]}{k^2} = \frac{4t}{k^2},
\]

the compact containment condition holds and \( \{X_n\} \) is relatively compact in \( D_{\mathbb{R}}[0, \infty) \).
Limits of martingales

Lemma 16  For each \( n = 1, 2, \ldots \), let \( M_n \) and \( Z_n \) be cadlag stochastic processes and let \( M_n \) be a \( \{ \mathcal{F}_t^{(M_n,Z_n)} \} \)-martingale. Suppose that \( (M_n, Z_n) \Rightarrow (M, Z) \). If for each \( t \geq 0 \), \( \{ M_n(t) \} \) is uniformly integrable, then \( M \) is a \( \{ \mathcal{F}_t^{(M,Z)} \} \)-martingale.

Recall that if a sequence of real-valued random variables \( \{ \psi_n \} \) is uniformly integrable and \( \psi_n \Rightarrow \psi \), then \( E[\psi_n] \to E[\psi] \).

It follows that if \( X \) is the limit of a subsequence of \( \{ X_n \} \), then

\[
f(X_n(t)) - \int_0^t A_n f(X_n(s)) ds \Rightarrow f(X(t)) - \int_0^t A f(X(s)) ds
\]

(along the subsequence) and \( X \) is a solution of the martingale problem for \( A \).
Elementary convergence theorem

$E$ compact, $E_n$, $n = 1, 2, \ldots$ complete, separable metric space.

$\eta_n : E_n \to E$

$Y_n$ Markov in $E_n$ with generator $A_n$, $X_n = \eta_n(Y_n)$ cadlag

$A \subset C(E) \times C(E)$ (for simplicity, we write $Af = g$ if $(f, g) \in A$).

$\mathcal{D}(A) = \{f : (f, g) \in A\}$

For each $(f, g) \in A$, there exist $f_n \in \mathcal{D}(A_n)$ such that

$$\sup_{x \in E_n} (|f_n(x) - f \circ \eta_n(x)| + |A_n f_n(x) - g \circ \eta_n(x)|) \to 0.$$  

THEN $\{X_n\}$ is relatively compact and any limit point is a solution of the martingale problem for $A$.  

Reflecting random walk

$E_n = \{0, \frac{1}{\sqrt{n}}, \frac{2}{\sqrt{n}}, \ldots\}$

\[ A_n f(x) = n\lambda(f(x + \frac{1}{\sqrt{n}}) - f(x)) + n\lambda I_{\{x>0\}}(f(x - \frac{1}{\sqrt{n}}) - f(x)) \]

Let $f \in C^3_c[0, \infty)$. Then

\[ A_n f(x) = \lambda f''(x) + O(\frac{1}{\sqrt{n}}) \quad x > 0 \]

\[ A_n f(0) = \sqrt{n}\lambda f'(0) + \frac{\lambda}{2} f''(0) + O(\frac{1}{\sqrt{n}}) \]

Assume $f'(0) = 0$, but still have discontinuity at 0.

Let $f_n = f + \frac{1}{\sqrt{n}} h$, $f \in \{f \in C^3_c[0, \infty) : f'(0) = 0\}$, $h \in C^3_c[0, \infty)$. Then

\[ A_n f_n(x) = \lambda f''(x) + \frac{1}{\sqrt{n}} \lambda h''(x) + O(\frac{1}{\sqrt{n}}) \quad x > 0 \]

\[ A_n f_n(0) = \frac{\lambda}{2} f''(0) + \lambda h'(0) + O(\frac{1}{\sqrt{n}}) \]

Assume that $h'(0) = \frac{1}{2} f''(0)$. Then, noting that $\eta_n(x) = x$, $x \in E_n$,

\[ \sup_{x \in E_n} (|f_n(x) - f(x)| + |A_n f_n(x) - Af(x)|) \to 0 \]
Averaging

$Y$ stationary with marginal distribution $\nu$, ergodic, and independent of $W$

$Y_n(t) = Y(\beta_n t)$, $\beta_n \to \infty$

$$X_n(t) = X(0) + \int_0^t \sigma(X_n(s), Y_n(s))dW(s) + \int_0^t b(X_n(s), Y_n(s))ds$$

Lemma 17 Let $\{\mu_n\}$ be measures on $U \times [0, \infty)$ satisfying $\mu_n(U \times [0, t]) = t$. Suppose

$$\int_U \varphi(u)\mu_n(du \times [0, t]) \to \int_U \varphi(u)\mu(du \times [0, t]),$$

for each $\varphi \in \overline{C}(U)$ and $t \geq 0$, and that $x_n \to x$ in $D_E[0, \infty)$. Then

$$\int_{U \times [0, t]} h(u, x_n(s))\mu_n(du \times ds) \to \int_{U \times [0, t]} h(u, x(s))\mu(du \times ds)$$

for each $h \in \overline{C}(U \times E)$ and $t \geq 0$. 
Convergence of averaged generator

Assume that $\sigma$ and $b$ are bounded and continuous. Then for $f \in C_c^2(\mathbb{R})$,

$$f(X_n(t)) - f(X_n(0)) - \int_0^t Af(X_n(s), Y_n(s))ds = \int_0^t \sigma(X_n(s), Y_n(s)) f'(X_n(s))dW(s)$$

$$Af(x, y) = \frac{1}{2} \sigma^2(x, y)f''(x) + b(x, y)f'(x)$$

is a martingale, $\{X_n\}$ is relatively compact,

$$\int_0^t \varphi(Y_n(s))ds = \frac{1}{\beta_n} \int_0^{\beta_n t} \varphi(Y(s))ds \to t \int_U \varphi(u)\nu(du),$$

so any limit point of $\{X_n\}$ is a solution of the martingale problem for

$$\overline{Af}(x) = \int_U Af(x, u)\nu(du).$$

Khas’minskii (1966), Kurtz (1992), Pinsky (1991)
Coupled system

\[ X_n(t) = X(0) + \int_0^t \sigma(X_n(s), Y_n(s))dW_1(s) + \int_0^t b(X_n(s), Y_n(s))ds \]

\[ Y_n(t) = Y(0) + \int_0^t \sqrt{n} \alpha(X_n(s), Y_n(s))dW_2(s) + \int_0^t n\beta(X_n(s), Y_n(s))ds \]

Consequently,

\[ f(X_n(t)) - f(X_n(0)) - \int_0^t Af(X_n(s), Y_n(s))ds \]

and

\[ g(Y_n(t)) - g(Y(0)) - \int_0^t nBg(X_n(s), Y_n(s))ds \]

\[ Bg(x, y) = \frac{1}{2} \alpha^2(x, y)g''(y) + \beta(x, y)g'(y) \]

are martingales.
Estimates

Suppose that for $g \in C_c^2(\mathbb{R})$, $B g(x, y)$ is bounded, and that, taking $g(y) = y^2$,

$$B g(x, y) \leq K_1 - K_2 y^2.$$  

Then, assuming $E[Y(0)^2] < \infty$,

$$E[Y_n(t)^2] \leq E[Y(0)^2] + \int_0^t (K_1 - K_2 E[Y_n(s)^2]) ds$$

which implies

$$E[Y_n(t)^2] \leq E[Y(0)^2] e^{-K_2 t} + \frac{K_1}{K_2} (1 - e^{-K_2 t}).$$

The sequence of measures defined by

$$\int_{\mathbb{R} \times [0,t]} \varphi(y) \Gamma_n(dy \times ds) = \int_0^t \varphi(Y_n(s)) ds$$

is relatively compact.
Convergence of the averaged process

If \( \sigma \) and \( b \) are bounded, then \( \{X_n\} \) is relatively compact and any limit point of \( \{(X_n, \Gamma_n)\} \) must satisfy:

\[
f(X(t)) - f(X(0)) - \int_{\mathbb{R} \times [0,t]} A f(X(s), y) \Gamma(dy \times ds)
\]

is a martingale for each \( f \in C^2_c(\mathbb{R}) \) and

\[
\int_{\mathbb{R} \times [0,t]} B g(X(s), y) \Gamma(dy \times ds)
\]

is a martingale for each \( g \in C^2_c(\mathbb{R}) \).

But (4) is continuous and of finite variation. Therefore

\[
\int_{\mathbb{R} \times [0,t]} B g(X(s), y) \Gamma(dy \times ds) = \int_0^t \int_{\mathbb{R}} B g(X(s), y) \gamma_s(dy) ds = 0.
\]
Characterizing $\gamma_s$

For almost every $s$

$$\int_{\mathbb{R}} Bg(X(s), y)\gamma_s(dy) = 0, \quad g \in C^2_c(\mathbb{R})$$

But, fixing $x$, and setting $B_x g(y) = Bg(x, y)$

$$\int_{\mathbb{R}} B_x g(y)\pi(dy) = 0, \quad g \in C_2(\mathbb{R}),$$

implies $\pi$ is a stationary distribution for the diffusion with generator

$$B_x g(y) = \frac{1}{2} \alpha_x''(y)g''(y) + \beta_x(y)g'(y), \quad \alpha_x(y) = \alpha(x, y), \quad \beta_x(y) = \beta(x, y)$$

If $\alpha_x(y) > 0$ for all $y$, then the stationary distribution $\pi_x$ is uniquely determined. If uniqueness hold for all $x$, $\Gamma(dy \times ds) = \pi_{X(s)}(dy)ds$ and $X$ is a solution of the martingale problem for

$$\bar{A}f(x) = \int_{\mathbb{R}} Af(x, y)\pi_x(dy).$$
Moran models in population genetics

\( E \) type space

\( B \) generator of mutation process

\( \sigma \) selection coefficient

Generator with state space \( E^n \)

\[
A^n f(x) = \sum_{i=1}^{n} B_i f(x) + \frac{1}{2(n-2)} \sum_{1 \leq i \neq j \neq k \leq n} (1 + \frac{2}{n} \sigma(x_i, x_j))(f(\eta_k(x|x_i)) - f(x))
\]

\( \eta_k(x|z) = (x_1, \ldots, x_{k-1}, z, x_{k+1}, \ldots x_n) \)

Note that if \((X_1, \ldots, X_n)\) is a solution of the martingale problem for \(A\), then for any permutation \(\sigma\), \((X_{\sigma_1}, \ldots, X_{\sigma_n})\) is a solution of the martingale problem for \(A\).
Conditioned martingale lemma

**Lemma 18** Suppose $U$ and $V$ are $\{F_t\}$-adapted,

$$U(t) - \int_0^t V(s)ds$$

is an $\{F_t\}$-martingale, and $G_t \subset F_t$. Then

$$E[U(t)|G_t] - \int_0^t E[V(s)|G_s]ds$$

is a $\{G_t\}$-martingale.
Generator for measure-valued process

\[ \mathcal{P}^n(E) = \{ \mu \in \mathcal{P}(E) : \mu = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i} \} \]

\[ Z(t) = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i(t)} \]

For \( f \in B(E^m) \) and \( \mu \in \mathcal{P}^n(E) \)

\[ \langle f, \mu^{(m)} \rangle = \frac{1}{n \cdots (n-m+1)} \sum_{i_1 \neq \cdots \neq i_m} f(x_{i_1}, \ldots, x_{i_m}). \]

Symmetry and the conditioned martingale lemma imply

\[ \langle f, Z^{(n)}(t) \rangle - \int_0^t \langle A^n f, Z^{(n)}(s) \rangle ds \]

is a \( \{ \mathcal{F}_t^Z \} \)-martingale.

Define \( F(\mu) = \langle f, \mu^{(n)} \rangle \)

\[ A^n F(\mu) = \langle A^n f, \mu^{(n)} \rangle \]
Convergence of the generator

If $f$ depends on $m$ variables ($m < n$)

$$A^n f(x) = \sum_{i=1}^{m} B_i f(x)$$

$$+ \frac{1}{2(n-2)} \sum_{k=1}^{m} \sum_{1 \leq i \neq k \leq m} \sum_{1 \leq j \neq k, i \leq n} (1 + \frac{2}{n} \sigma(x_i, x_j))(f(\eta_k(x|x_i)) - f(x))$$

$$+ \frac{1}{2(n-2)} \sum_{k=1}^{m} \sum_{i=m+1}^{n} \sum_{1 \leq j \neq k, i \leq n} (1 + \frac{2}{n} \sigma(x_i, x_j))(f(\eta_k(x|x_i)) - f(x))$$

$$\langle A^n f, \mu^{(n)} \rangle = \sum_{i=1}^{m} \langle B_i f, \mu^{(m)} \rangle + \frac{1}{2} \sum_{1 \leq i \neq k \leq m} (\langle \Phi_{ik} f, \mu^{(m-1)} \rangle - \langle f, \mu^{(m)} \rangle) + O\left(\frac{1}{n}\right)$$

$$+ \sum_{k=1}^{m} (\langle \sigma_k f, \mu^{(m+1)} \rangle - \langle \sigma f, \mu^{(m+2)} \rangle) + O\left(\frac{1}{n}\right)$$
Conclusions

- If $E$ is compact, compact containment condition is immediate ($\mathcal{P}(E)$ is compact).
- $\mathbb{A}^n F$ is bounded as long as $B_i f$ is bounded.
- Limit of uniformly integrable martingales is a martingale.
- $Z^n \Rightarrow Z$ if uniqueness holds for limiting martingale problem.
6. The lecturer’s whims

- Wong-Zakai corrections
- Processes with boundary conditions
- Averaging for stochastic equations
Not good (evil?) sequences

**Markov chains:** Let \( \{\xi_k\} \) be an irreducible finite Markov chain with stationary distribution \( \pi(x) \), \( \sum g(x)\pi(x) = 0 \), and let \( h \) satisfy \( Ph-h = g \) \( (Ph(x) = \sum p(x,y)h(y)) \). Define

\[
V_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} g(\xi_k).
\]

Then \( V_n \Rightarrow \sigma W \), where

\[
\sigma^2 = \sum_{x,y} \pi(x)p(x,y) (Ph(x) - h(y))^2.
\]

\( \{V_n\} \) is not “good” but

\[
V_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} (Ph(\xi_{k-1}) - h(\xi_k)) + \frac{1}{\sqrt{n}} (Ph(\xi_{[nt]}) - Ph(\xi_0)) = M_n(t) + Z_n(t)
\]

where \( \{M_n\} \) is good sequence of martingales and \( Z_n \Rightarrow 0 \).
Piecewise linear interpolation of $W$:

$$W_n(t) = W\left(\frac{\lfloor nt \rfloor}{n}\right) + (t - \frac{\lfloor nt \rfloor}{n})n \left(W\left(\frac{\lfloor nt \rfloor + 1}{n}\right) - W\left(\frac{\lfloor nt \rfloor}{n}\right)\right)$$

(Classical Wong-Zakai example.)

$$W_n(t) = W\left(\frac{\lfloor nt \rfloor + 1}{n}\right) - (\frac{\lfloor nt \rfloor + 1}{n} - t)n \left(W\left(\frac{\lfloor nt \rfloor + 1}{n}\right) - W\left(\frac{\lfloor nt \rfloor}{n}\right)\right)
= M_n(t) + Z_n(t)$$

where $\{M_n\}$ is good (take the filtration to be $\mathcal{F}^n_t = \mathcal{F}_{\frac{\lfloor nt \rfloor + 1}{n}}$) and $Z_n \Rightarrow 0$.

**Renewal processes:** $N(t) = \max\{k : \sum_{i=1}^{k} \xi_i \leq t\}$, $\{\xi_i\}$ iid, positive, $E[\xi_k] = \mu$, $\text{Var} (\xi_k) = \sigma^2$.

$$V_n(t) = \frac{N(n t) - n t / \mu}{\sqrt{n}}.$$

Then $V_n \Rightarrow \alpha W$, $\alpha = \sigma / \mu^{3/2}$.

$$V_n(t) = \frac{(N(nt) + 1)\mu - S_{N(nt)+1}}{\mu \sqrt{n}} + \frac{S_{N(nt)+1} - nt}{\mu \sqrt{n}}
= M_n(t) + Z_n(t)$$
Not so evil after all

Assume $V_n(t) = Y_n(t) + Z_n(t)$ where $\{Y_n\}$ is good and $Z_n \Rightarrow 0$. In addition, assume $\{\int Z_n dZ_n\}$, $\{[Y_n, Z_n]\}$, and $\{[Z_n]\}$ are good.

\[
X_n(t) = X_n(0) + \int_0^t F(X_n(s-))dV_n(s)
\]

\[
= X_n(0) + \int_0^t F(X_n(s-))dY_n(s) + \int_0^t F(X_n(s-))dZ_n(s)
\]

Integrate by parts using

\[
F(X_n(t)) = F(X_n(0)) + \int_0^t F'(X_n(s-))F(X_n(s-))dV_n(s) + R_n(t)
\]

where $R_n$ can be estimated in terms of $[V_n] = [Y_n] + 2[Y_n, Z_n] + [Z_n]$. 
Integration by parts

\[
\int_{0}^{t} F(X_n(s-))dZ_n(s)
\]

\[
= F(X_n(t))Z_n(t) - F(X_n(0))Z_n(0) - \int_{0}^{t} Z_n(s-)dF(X_n(s)) - [F \circ X_n, Z_n]_t
\]

\[
\approx - \int_{0}^{t} Z_n(s-)F'(X_n(s-))F(X_n(s-))dY_n(s) - \int_{0}^{t} F'(X_n(s-))F(X_n(s-))Z_n(s-)dZ_n(s)
\]

\[
- \int_{0}^{t} Z_n(s-)dR_n(s) - \int_{0}^{t} F'(X_n(s-))F(X_n(s-))d([Y_n, Z_n]_s + [Z_n]_s) - [R_n, Z_n]_t
\]

**Theorem 19** Assume that \(V_n = Y_n + Z_n\) where \(\{Y_n\}\) is good, \(Z_n \Rightarrow 0\), and \(\{\int Z_n dZ_n\}\) is good. If \((X_n(0), Y_n, Z_n, \int Z_n dZ_n, [Y_n, Z_n]) \Rightarrow (X(0), Y, 0, H, K)\), then \(\{X_n\}\) is relatively compact and any limit point satisfies

\[
X(t) = X(0) + \int_{0}^{t} F(X(s-))dY(s) + \int_{0}^{t} F'(X(s-))F(X(s-))d(H(s) - K(s))
\]

**Note:** For all the examples, \(H(t) - K(t) = ct\) for some \(c\).
Reflecting diffusions

$X$ has values in $D$ and satisfies 

$$X(t) = X(0) + \int_0^t \sigma(X(s))dW(s) + \int_0^t b(X(s))ds + \int_0^t \eta(X(s))d\lambda(s)$$

where $\lambda$ is nondecreasing and increases only when $X(t) \in \partial D$. By Itô’s formula 

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds - \int_0^t Bf(X(s))d\lambda(s)$$

$$= \int_0^t \nabla f(X(s))^T \sigma(X(s))dW(s)$$

where 

$$Af(x) = \frac{1}{2} \sum a_{ij}(x) \partial_i \partial_j f(x) + b(x) \cdot \nabla f(x) \quad Bf(x) = \eta(x) \cdot \nabla f(x)$$

Either take $\mathcal{D}(A) = \{ f \in C^2_c(D) : Bf(x) = 0, x \in \partial D \}$ or formulate a constrained martingale problem with solution $(X, \lambda)$ by requiring $X$ to take values in $D$, $\lambda$ to be nondecreasing and increase only when $X \in \partial D$, and 

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds - \int_0^t Bf(X(s))d\lambda(s)$$

to be an $\{\mathcal{F}_t^{x,\lambda}\}$-martingale.
Instantaneous jump conditions

$X$ has values in $D$ and satisfies

$$X(t) = X(0) + \int_0^t \sigma(X(s))dW(s) + \int_0^t b(X(s))ds + \int_0^t \alpha(X(s-), \zeta_{N(s-)+1})dN(s)$$

where $\zeta_1, \zeta_2, \ldots$ are iid and independent of $X(0)$ and $W$ and $N(t)$ is the number of times $X$ has hit the boundary by time $t$. Then

$$f(X(t)) - f(X(0)) - \int_0^t A f(X(s))ds - \int_0^t B f(X(s-))dN(s)$$

$$= \int_0^t \nabla f(X(s))^T \sigma(X(s))dW(s)$$

$$+ \int_0^t (f(X(s-)) + \alpha(X(s-), \zeta_{N(s-)+1}))$$

$$- \int_U (f(X(s-)) + \alpha(X(s-), u))\nu(du))dN(s)$$

where

$$A f(x) = \frac{1}{2} \sum a_{ij}(x) \partial_i \partial_j f(x) + b(x) \cdot \nabla f(x)$$

and

$$B f(x) = \int_U (f(x + \alpha(x, u)) - f(x))\nu(du).$$
Approximation of reflecting diffusion

$X$ has values in $D$ and satisfies

$$X_n(t) = X(0) + \int_0^t \sigma(X_n(s))dW(s) + \int_0^t b(X_n(s))ds + \int_0^t \frac{1}{n} \eta(X_n(s-))dN_n(s)$$

$$\lambda_n(t) \equiv \frac{N_n(t)}{n}$$

Assume there exists $\varphi \in C^2$ such that $\varphi$, $A\varphi$ and $\nabla \varphi \cdot \sigma$ are bounded on $D$ and $\inf_{x \in \partial D} \eta(x) \cdot \nabla \varphi(x) \geq \epsilon > 0$. Then

$$\inf_{x \in \partial D} n(\varphi(x + \frac{1}{n} \eta(x)) - \varphi(x))\lambda_n(t)$$

$$\leq 2\|\varphi\| + 2\|A\varphi\|t - \int_0^t \nabla \varphi(X_n(s))^T \sigma(X_n(s))dW(s)$$

and $\{\lambda_n(t)\}$ is stochastically bouned
Convergence of time changed sequence

\[ \gamma_n(t) = \inf\{s : s + \lambda_n(s) > t\} \quad t \leq \gamma_n(t) + \lambda_n \circ \gamma_n(t) \leq t + \frac{1}{n} \]

\[ \hat{X}_n(t) \equiv X_n \circ \gamma_n(t) \]

\[ \hat{X}_n(t) = X(0) + \int_0^t \sigma(\hat{X}_n(s))dW \circ \gamma_n(s) + \int_0^t b(\hat{X}_n(s))d\gamma_n(s) \]

\[ + \int_0^t \eta(\hat{X}_n(s-))d\lambda_n \circ \gamma_n(s) \]

Check relative compactness and goodness of \{(W \circ \gamma_n, \gamma_n, \lambda_n \circ \gamma_n)\}.

\( \gamma_n \) is Lipschitz with Lipschitz constant 1, \( \lambda_n \circ \gamma_n \) is finite variation, nondecreasing and bounded by \( t + \frac{1}{n} \), \( M_n = W \circ \gamma_n \) is a martingale with \([M_n]_t = \gamma_n(t)\).

If \( \sigma \) and \( b \) are bounded and continuous, then (by the general theorem) \{(\hat{X}_n, W \circ \gamma_n, \gamma_n, \lambda_n \circ \gamma_n)\} is relatively compact and any limit point will satisfy

\[ \hat{X}(t) = X(0) + \int_0^t \sigma(\hat{X}(s))dW \circ \gamma(s) + \int_0^t b(\hat{X}(s))d\gamma(s) + \int_0^t \eta(\hat{X}(s))d\hat{\lambda}(s) \]

where \( \gamma(t) + \hat{\lambda}(t) = t \). (Note that \( \gamma \) and \( \hat{\lambda} \) are continuous.)
Convergence of original sequence

\[ \hat{X}(t) = X(0) + \int_0^t \sigma(\hat{X}(s))dW \circ \gamma(s) + \int_0^t b(\hat{X}(s))d\gamma(s) + \int_0^t \eta(\hat{X}(s))d\hat{\lambda}(s) \]

If \( \gamma \) is constant on \([a, b]\), then for \( a \leq t \leq b \), \( \hat{X}(t) \in \partial D \) and

\[ \hat{X}(t) = \hat{X}(a) + \int_a^t \eta(\hat{X}(s))ds. \]

Under appropriate conditions on \( \eta \), no such interval can exist with \( a < b \) and \( \gamma \) must be strictly increasing.

Then (at least along a subsequence) \( X_n \Rightarrow X \equiv \hat{X} \circ \gamma^{-1} \) and setting \( \lambda = \hat{\lambda} \circ \gamma^{-1} \),

\[ X(t) = X(0) + \int_0^t \sigma(X(s))dW(s) + \int_0^t b(X(s))ds + \int_0^t \eta(X(s))d\lambda(s) \]
Rapidly varying components

• $W$  a standard Brownian motion in $\mathbb{R}$

• $X_n(0)$ independent of $W$

• $\zeta$ a stochastic process with state space $U$, independent of $W$ and $X_n(0)$

Define $\zeta_n(t) = \zeta(nt)$.

$$X_n(t) = X_n(0) + \int_0^t \sigma(X_n(s), \zeta_n(s))dW(s) + \int_0^t b(X_n(s), \zeta_n(s))ds$$

Define

$$M_n(A, t) = \int_0^t I_A(\zeta_n(s))dW(s)$$

and

$$V_n(A, t) = \int_0^t I_A(\zeta_n(s))ds,$$

so that

$$X_n(t) = X_n(0) + \int_{U \times [0, t]} \sigma(X_n(s), u)M_n(du \times ds) + \int_{U \times [0, t]} b(X_n(s), u)V_n(du \times ds)$$
Convergence of driving processes

Define

\[
M_n(\varphi, t) = \int_0^t \varphi(\zeta_n(s))dW(s)
\]

\[
V_n(\varphi, t) = \int_0^t \varphi(\zeta_n(s))ds
\]

Assume that

\[
\frac{1}{t} \int_0^t \varphi(\zeta(s))ds \to \int_U \varphi(u)\nu(du)
\]

in probability for each \(\varphi \in \overline{C}(U)\).

Observe that

\[
[M_n(\varphi_1, \cdot), M_n(\varphi_2, \cdot)]_t = \int_0^t \varphi_1(\zeta_n(s))\varphi_2(\zeta_n(s))ds
\]

\[
\to t \int_U \varphi_1(u)\varphi_2(u)\nu(du)
\]
Limiting equation

The functional central limit theorem for martingales implies $M_n(\varphi, t) \Rightarrow M(\varphi, t)$ where $M$ is Gaussian with

$$E[M(\varphi_1, t)M(\varphi_2, s)] = t \land s \int_U \varphi_1(u)\varphi_2(u)\nu(du)$$

and

$$V_n(\varphi, t) \rightarrow t \int_U \varphi(u)\nu(du)$$

If $\{(M_n, V_n)\}$ is good, then $X_n \Rightarrow X$ satisfying

$$X(t) = X(0) + \int_0^t \int_U \sigma(X(s), u)M(du \times ds) + \int_0^t \int_U b(X(s), u)\nu(du)ds$$
Estimates for averaging problem

Recall

\[ M_n(\varphi, t) = \int_0^t \varphi(\zeta_n(s))dW(s) \]

If

\[ Z(t, u) = \sum_{k=1}^m \xi_k(t)\varphi_k(u) \]

then

\[ \int_{U \times [0, t]} Z(s, u)M_n(du \times ds) = \sum_{k=1}^m \int_0^t \xi_k(s-)dM_n(\varphi_k, s) \]

and

\[ E[\left( \int_{U \times [0, t]} Z(s, u)M_n(du \times ds) \right)^2] = E \left[ \int_0^t \sum_{k,l} \xi_k(s)\xi_l(s)\varphi_k(\zeta_n(s))\varphi_l(\zeta_n(s))ds \right] \]

\[ = E \left[ \int_0^t Z(s, \zeta_n(s))^2ds \right] \]

Assume \( U \) is locally compact, \( \psi \) is strictly positive and vanishes at \( \infty \), \( \|\varphi\|_H = \sup |\psi(u)\varphi(u)| \), and \( \sup_T E\left[ \frac{1}{T} \int_0^T \frac{1}{\psi(\zeta(s))}ds \right] < \infty. \)