1. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{D}$ be a sub-σ-algebra of $\mathcal{F}$. An $(H, \mathcal{H})$-valued random variable X is independent of $\mathcal{D}$ if and only if $P(\{X \in \Gamma\} \cap D) = P\{X \in \Gamma\}P(D)$ for all $\Gamma \in \mathcal{H}$ and $D \in \mathcal{D}$. Prove that if X is independent of $\mathcal{D}$, then

$$E[f(X)|\mathcal{D}] = E[f(X)],$$

for all f satisfying $E[|f(X)|] < \infty$.

2. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{D}$ be a sub-σ-algebra of $\mathcal{F}$. Let $(H_i, \mathcal{H}_i)$, $i = 1, 2$, be measurable spaces, and suppose that X is an H_1-valued random variable and Y is an H_2-valued random variable define on $(\Omega, \mathcal{F}, P)$. Suppose that X is independent of $\mathcal{D}$ and Y is $\mathcal{D}$-measurable. Let μ_X denote the distribution of X. Let $\psi : H_1 \times H_2 \to \mathbb{R}$ be bounded and $\mathcal{H}_1 \times \mathcal{H}_2$-measurable, and define $\varphi(y) = \int_{H_1} \psi(x, y)\mu_X(dx)$. Show that

$$E[\psi(X, Y)|\mathcal{D}] = \varphi(Y).$$

3. Let $\{\mathcal{F}_t\}$ be a filtration. τ is an $\{\mathcal{F}_t\}$-stopping time if $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$, and the "information" available at the random time τ is $\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t, t \geq 0\}$.

   (a) Show that τ is $\mathcal{F}_\tau$-measurable.

   (b) Suppose $P\{\tau = a\} > 0$. Show that for an integrable random variable Z

   $$E[Z|\mathcal{F}_\tau]I_{\{\tau=a\}} = E[Z|\mathcal{F}_t]I_{\{\tau=a\}}.$$

   (c) Let τ be a discrete stopping time with range $\{t_1, t_2, \ldots\}$. Show that

   $$E[Z|\mathcal{F}_\tau] = \sum_{k=1}^{\infty} E[Z|\mathcal{F}_{t_k}]I_{\{\tau=t_k\}}\ .$$

4. Let $0 \leq \tau_1 \leq \tau_2 \leq \cdots$ be $\{\mathcal{F}_t\}$-stopping times, and for $k = 1, 2, \ldots$, let ξ_k be $\mathcal{F}_{\tau_k}$-measurable. Define

$$X(t) = \sum_{k=1}^{\infty} \xi_k I_{[\tau_k, \tau_{k+1})}(t).$$

Show that X is $\{\mathcal{F}_t\}$-adapted.

5. Let ξ be a Poisson process with mean measure $\nu \times m$, compatible with $\{\mathcal{F}_t\}$. Let Z be cadlag with values in $L_2(\nu)$ and adapted to $\{\mathcal{F}_t\}$, and define

$$X(t) = \int_{U\times[0,t]} Z(u, s-)\tilde{\xi}(du \times ds).$$

Let $f \in C^2(\mathbb{R})$. Represent $f(X(t)) - f(X(0))$ as an integral involving ξ. (In other words, apply Itô's formula to $f(X(t))$ and express the result interms of ξ.)

6. Let Y be cadlag and suppose $T_t(Y) < \infty$ for all $t > 0$. Describe $[Y]_t$.

7. Let $d = 1$ and

$$Af(x) = \frac{1}{2}a(x)f''(x) + b(x)f'(x).$$

Assume that $a(x) > 0$ for each x and that $1/a(x)$ is locally bounded. If $\tilde{X}$ is a solution of the martingale problem for A, then

$$M(t) = \tilde{X}(t) - \int_0^t b(\tilde{X}(s))ds$$

is a local martingale. Show that

$$\tilde{W}(t) = \int_0^t \frac{1}{\sqrt{a(\tilde{X}(s))}} dM(s)$$

is a standard Brownian motion.

8. The generator for a process with independent increments can be written as

$$\frac{1}{2}\sigma^2 f''(x) + bf'(x) + \int_{-\infty}^{\infty} (f(x+u) - f(x) - I_{\{|u|\leq 1\}} uf'(x))\nu(du),$$

where ν satisfies $\int_{-\infty}^{\infty} u^2 \wedge 1 \nu(du) < \infty$. Show how to represent the process in terms of a standard Brownian motion W and a Poisson random measure ξ on $(-\infty, \infty) \times [0, \infty)$ with mean measure $\nu \times m$.

9. For $i = 1, \ldots, m$, let X_i be a solution of the martingale problem for A_i. Suppose that $X_1, \ldots, X_m$ are independent. Show that $X = (X_1, \ldots, X_m)$ is a solution of the martingale problem for

$$\{(\prod_{i=1}^m f_i, \left(\prod_{i=1}^m f_i\right) \sum_{k=1}^m \frac{A_k f_k}{f_k}) : f_k \in \mathcal{D}(A_k)\}.$$

10. Let Y in E be a solution of the martingale problem for A, and for $\beta : E \to [0, \infty)$, let X satisfy

$$X(t) = Y\left(\int_0^t \beta(X(s))ds\right).$$

Show that X is a solution of the martingale problem for βA.

11. Let $\lambda : \mathbb{R}^d \to [0, \infty)$ be measurable, and let $\mu(x, dz)$ be a transition function on $\mathbb{R}^d$. There exists $\gamma : \mathbb{R}^d \times [0, 1] \to \mathbb{R}^d$ such that

$$\int_0^1 f(x + \gamma(x, u))du = \int_{\mathbb{R}^d} f(z)\mu(x, dz).$$

Let ξ be a Poisson random measure on $[0, \infty) \times [0, 1] \times [0, \infty)$ with Lebesgue mean measure. Show that

$$X(t) = X(0) + \int_{[0,\infty)\times[0,1]\times[0,t]} I_{[0,\lambda(X(s-))]}(v)\gamma(X(s-), u)\xi(dv \times du \times ds)$$

is a stochastic differential equation corresponding to A given by

$$Af(x) = \lambda(x) \int_{\mathbb{R}^d} (f(z) - f(x))\mu(x, dz).$$

12. Suppose X has values in D and satisfies

$$X(t) = X(0) + \int_0^t \sigma(X(s))dW(s) + \int_0^t b(X(s))ds + \int_0^t \alpha(X(s))dW_0 \circ \lambda(s) + \int_0^t \eta(X(s))d\lambda(s)$$

where λ is nondecreasing and increases only when $X(t) \in \partial D$ and W_0 is a standard Brownian motion independent of W. (If $n(x)$ is the inward normal vector at $x \in \partial D$, then we require $\eta(x) \cdot n(x) > 0$ and $n(x)^T \alpha(x) = 0$.) Derive the martingale problem satisfied by X.

13. Show that $I_{[1+\frac{1}{n},\infty)} \to I_{[1,\infty)}$ in $D_{\mathbb{R}}[0,\infty)$ but that $(I_{[1+\frac{1}{n},\infty)}, I_{[1,\infty)})$ does *not* converge in $D_{\mathbb{R}^2}[0,\infty)$. (It does converge in $D_{\mathbb{R}}[0,\infty) \times D_{\mathbb{R}}[0,\infty)$.)

14. For each of the following mappings, verify the stated continuity properties. (E, r) is a complete, separable metric space; $D_E[0,\infty)$ is the space of cadlag E-valued functions with the Skorohod topology; C_F denotes the set of continuity points of a mapping F.

    (a) $\pi_t : D_E[0,\infty) \to E$ is defined by $\pi_t(x) = x(t)$. Then

    $$C_{\pi_t} = \{x \in D_E[0,\infty) : x(t) = x(t-)\}$$

    (b) $G_t : D_{\mathbb{R}}[0,\infty) \to \mathbb{R}$ is defined by $G_t(x) = \sup_{s \le t} x(s)$. Then

    $$C_{G_t} = \{x \in D_{\mathbb{R}}[0,\infty) : \lim_{s \to t-} G_s(x) = G_t(x)\} \supset \{x \in D_{\mathbb{R}}[0,\infty) : x(t) = x(t-)\}$$

    (c) $G : D_{\mathbb{R}}[0,\infty) \to D_{\mathbb{R}}[0,\infty)$ is defined by $G(x)(t) = G_t(x)$. Then G is continous.

    (d) $H_t : D_E[0,\infty) \to \mathbb{R}$ is defined by $H_t(x) = \sup_{s \le t} r(x(s), x(s-))$. Then

    $$C_{H_t} = \{x \in D_E[0,\infty) : \lim_{s \to t-} H_s(x) = H_t(x)\} \supset \{x \in D_E[0,\infty) : x(t) = x(t-)\}$$

    (e) $H : D_E[0,\infty) \to D_{\mathbb{R}}[0,\infty)$ is defined by $H(x)(t) = H_t(x)$. Then H is continuous.

    (f) $\tau_c : D_{\mathbb{R}}[0,\infty) \to [0,\infty)$ is defined by $\tau_c(x) = \inf\{t : x(t) > c\}$, and $\tau_c^- : D_{\mathbb{R}}[0,\infty) \to [0,\infty)$ is defined by $\tau_c^-(x) = \inf\{t : x(t) \ge c \text{ or } x(t-) \ge c\}$. Then

    $$G_{\tau_c} = G_{\tau_c^-} = \{x : \tau_c(x) = \tau_c^-(x)\}.$$

    Note that $\tau_c^-(x) \le \tau_c(x)$ and that $x_n \to x$ implies

    $$\tau_c^-(x) \le \liminf_{n\to\infty} \tau_c^-(x_n) \le \limsup_{n\to\infty} \tau_c(x_n) \le \tau_c(x).$$

15. Suppose

$$X(t)=X(0)+\int_0^t \sigma(X(s))dW(s)+\int_0^t b(X(s))ds$$

where σ and b are bounded. Estimate

$$E[(X(t+h)-X(t))^2|\mathcal{F}_t^X].$$

16. Let Y be a semimartingale, and define

$$Y_n(t)=Y(\frac{k}{n}) \qquad \frac{k}{n}\le t<\frac{k+1}{n}.$$

Show that $\{Y_n\}$ is a good sequence.

17. Let ξ be a Poisson random measure on $U\times[0,\infty)$ with mean measure $\nu\times m$, and let $U_n\subset U_{n+1}\subset U$ with $U=\cup_n U_n$. Define

$$\xi_n(\varphi,t)=\int_{U_n}\varphi(u)\xi(du\times ds) \quad \varphi\in L_1(\nu)$$

and

$$\tilde{\xi}_n(\varphi,t)=\int_{U_n}\varphi(u)\tilde{\xi}(du\times ds).$$

Show that $\{\xi_n\}$ is uniformly tight for $H=L_1(\nu)$ and that $\{\tilde{\xi}_n\}$ is uniformly tight for $H=L_2(\nu)$.

18. Prove the conditioned martingale lemma.

19. Let N be a unit Poisson process and let

$$W_n(t)=\frac{1}{\sqrt{n}}\int_0^{nt}(-1)^{N(s)}ds$$

Show that there exist martingales M_n such that $W_n=M_n+V_n$ and $V_n\to 0$, but $T_t(V_n)\to\infty$. Apply the martingale central limit theorem to show that $W_n\Rightarrow W$ where W is standard Brownian motion.

20. Let W_n be as in Problem 19. Let σ have a bounded, continuous derivative, and let

$$X_n(t)=\int_0^t \sigma(X_n(s))dW_n(s).$$

Show that $X_n\Rightarrow X$ for some X and identify the stochastic differential equation satisfied by X. Hint: Write

$$X_n(t)=\int_0^t \sigma(X_n(s-))dM_n(s)+\int_0^t \sigma(X_n(s-))dV_n(s). \tag{1}$$

Integrate the second term on the right of (1) by parts, and show that the sequence of equations that results, does satisfies the conditions of the SDE convergence theorem.

Central limit theorem for Markov chains. (Problems 21-28.) Let $\xi_0, \xi_1, \ldots$ be an irreducible Markov chain on a finite state space $\{1, ..., d\}$, let $P = ((p_{ij}))$ denote its transition matrix, and let π be its stationary distribution. For any function h on the state space, let πh denote $\sum_i \pi_i h(i)$.

21. Show that

$$f(\xi_n) - \sum_{k=0}^{n-1} (Pf(\xi_k) - f(\xi_k))$$

is a martingale.

22. Show that for any function h, there exists a solution to the equation $Pg = h - \pi h$, that is, to the system

$$\sum_j p_{ij} g(j) - g(i) = h(i) - \pi h.$$

23. The ergodic theorem for Markov chains states that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} h(\xi_k) = \pi h.$$

Use the martingale central limit theorem to prove convergence in distribution for

$$W_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} (h(\xi_k) - \pi h) \; .$$

24. Use the martingale central limit theorem to prove the analogue of Problem 23 for a continuous time finite Markov chain $\{\xi(t), t \geq 0\}$. In particular, use the multidimensional theorem to prove convergence for the vector-valued process $U_n = (U_n^1, \ldots, U_n^d)$ defined by

$$U_n^k(t) = \frac{1}{\sqrt{n}} \int_0^{nt} (I_{\{\xi(s)=k\}} - \pi_k) ds$$

25. Explore extensions of Problems 23 and 24 to infinite state spaces.

Limit theorems for stochastic differential equations driven by Markov chains

26. Show that W_n defined in Problem 23 and U_n defined in Problem 24 are not "good" sequences of semimartingales. (The easiest approach is probably to show that the conclusion is not valid.)

27. Show that W_n and U_n can be written as $M_n + Z_n$ where $\{M_n\}$ is a "good" sequence and $Z_n \Rightarrow 0$.

28. (Random evolutions) Let ξ be as in Problem 24, and let X_n satisfy

$$\dot{X}_n(t) = \sqrt{n} F(X_n(s), \xi(ns)).$$

Suppose $\sum_i F(x,i)\pi_i = 0$. Write X_n as a stochastic differential equations driven by U_n, give conditions under which X_n converges in distribution to a limit X, and identify the limit.

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