Abstract

Let $E$ be a complete, separable metric space and $A$ be an operator on $C_b^\infty(E)$. We give an abstract definition of viscosity sub/supersolution of the resolvent equation $\lambda u - Au = h$ and show that, if the comparison principle holds, then the martingale problem for $A$ has a unique solution. Our proofs work also under two alternative definitions of viscosity sub/supersolution which might be useful, in particular, in infinite dimensional spaces, for instance to study measure-valued processes.

We prove the analogous result for stochastic processes that must satisfy boundary conditions, modeled as solutions of constrained martingale problems. In the case of reflecting diffusions in $D \subset \mathbb{R}^d$, our assumptions allow $D$ to be nonsmooth and the direction of reflection to be degenerate.

Two examples are presented: A diffusion with degenerate oblique direction of reflection and a class of jump diffusion processes with infinite variation jump component and possibly degenerate diffusion matrix.

Key words: martingale problem, uniqueness, metric space, viscosity solution, boundary conditions, constrained martingale problem

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1 Introduction

There are many ways of specifying Markov processes, the most popular being as solutions of stochastic equations, as solutions of martingale problems, or in terms of solutions of the
Kolmogorov forward equation (the Fokker-Planck equation or the master equation depending on context). The solution of a stochastic equation explicitly gives a process while a solution of a martingale problem gives the distribution of a process and a solution of a forward equation gives the one dimensional distributions of a process. Typically, these approaches are equivalent (assuming that there is a stochastic equation formulation) in the sense that existence of a solution specified by one method implies existence of corresponding solutions to the other two (weak existence for the stochastic equation) and hence uniqueness for one method implies uniqueness for the other two (distributional uniqueness for the stochastic equation).

One approach to proving uniqueness for a forward equation and hence for the corresponding martingale problem is to verify a condition on the generator similar to the range condition of the Hille-Yosida theorem. See Theorem 2.6. The strategy that we will follow in the proof of our main theorems is to show that we can extend the original generator $A$ of our martingale problem (or a restriction of the original generator $A$ in the case of martingale problems with boundary conditions) to a generator $\hat{A}$ such that every solution of the martingale problem for $A$ is a solution for $\hat{A}$ and $\hat{A}$ satisfies the range condition of Theorem 2.6. Our argument is based on the fact that the extensions we construct give viscosity solutions of a resolvent equation

$$\lambda u - Au = h.$$  

Viscosity solutions have been used to study value functions in stochastic optimal control and optimal stopping theory since the very beginning (see the classical references Crandall, Ishii, and Lions (1992), Pham (1998), as well as Fleming and Soner (2006)). The methodology is also important for related problems in finance (for example Soner and Touzi (2000), Benth, Karlsen, and Reikvam (2001), Kabanov and Klüppelberg (2004), Bardi, Cesaroni, and Manca (2010), Costantini, Papi, and D’Ippoliti (2012) and many others).

Viscosity solutions have also been used to study the partial differential equations associated with forward-backward stochastic differential equations (Ma and Zhang (2011), Ekren, Keller, Touzi, and Zhang (2014)) and in the theory of large deviations (Feng and Kurtz (2006)).

The basic data for our work is an operator $A \subset C_b(E) \times C_b(E)$ on a complete, separable metric space $E$. We offer an abstract definition of viscosity sub/supersolution for (1.1) (which for integro-differential operators in $\mathbb{R}^d$ is equivalent to the usual one) and prove, under very general conditions, that the martingale problem for $A$ has a unique solution if the comparison principle for (1.1) holds.

We believe the interest of this result is twofold: on one hand it clarifies the general connection between viscosity solutions and martingale problems; on the other, there are still many martingale problems, for instance for measure valued processes, for which uniqueness is an open question.

We also discuss two alternative abstract definitions of viscosity sub/supersolution that might be especially useful in infinite dimensional spaces. The first is a modification of a definition suggested to us by Nizar Touzi and used in Ekren, Keller, Touzi, and Zhang (2014), while the second appears in Feng and Kurtz (2006). All our proofs work under these alternative definitions as well.

Next we consider stochastic processes that must satisfy some boundary conditions, for
example, reflecting diffusions. Boundary conditions are expressed in terms of an operator $B$ which enters into the formulation of a constrained martingale problem (see Kurtz (1990)). We restrict our attention to models in which the boundary term in the constrained martingale problem is expressed as an integral against a local time. Then it still holds that uniqueness of the solution of the constrained martingale problem follows from the comparison principle between viscosity sub and supersolutions of (1.1) with the corresponding boundary conditions. Notice that, as for the standard martingale problem, uniqueness for the constrained martingale problem implies that the solution is Markovian (see Kurtz (1990), Proposition 2.6).

In the presence of boundary conditions, even for $\mathbb{R}^d$-valued diffusions, there are examples for which uniqueness of the martingale problem is not known. Processes in domains with boundaries that are only piecewise smooth or with boundary operators that are second order or with directions of reflection that are tangential on some part of the boundary continue to be a challenge. In this last case, as an example of application of our results, we use the comparison principle proved in Popivanov and Kutev (2005) to obtain uniqueness.

The strategy of our proofs is inspired by the proof of Krylov’s selection theorem for martingale problems that appears in Ethier and Kurtz (1986) and originally appeared in unpublished work of Gray and Griffeath (1977). In that proof the generator is recursively extended in such a way that there are always solutions of the martingale problem for the extended generator, but eventually only one. If uniqueness fails for the original martingale problem, there is more than one way to do the extension. Conversely if, at each stage of the recursion, there is only one way to do the extension and all solutions of the martingale problem for the original generator remain solutions for the extended generator, then uniqueness must hold for the original generator.

Our proofs short circuit this argument. Both for ordinary martingale problems and martingale problems with boundary conditions, we directly consider an operator $\hat{A}$ such that all solutions of the martingale problem (constrained martingale problem) for $\hat{A}$ are solutions of the martingale problem (constrained martingale problem, with the same boundary conditions) for $A$. We show that for every $h$ such that the comparison principle for (1.1) (with the appropriate boundary conditions) holds, the viscosity solution of (1.1) belongs to the domain of $\hat{A}$. Then, if the comparison principle holds for a large enough class of $h$’s, $\hat{A}$ satisfies a range condition similar to the range condition of the Hille-Yosida theorem (Theorem 2.6) and uniqueness holds for $\hat{A}$, and hence for $A$.

For diffusions in $\mathbb{R}^d$, Bayraktar and Sîrbu (2012) use similar arguments based on the backward equation rather than the resolvent equation (1.1). Assuming a comparison principle exists, they show that the backward equation has a unique viscosity solution, and it follows that the corresponding stochastic differential equation has a unique weak solution. For Markovian forward-backward stochastic differential equations Ma and Zhang (2011) also derive uniqueness of the weak solution from existence of a comparison principle for the corresponding partial differential equation. In the non-Markovian case, the associated partial differential equation becomes path dependent. Ekren, Keller, Touzi, and Zhang (2014) propose the notion of viscosity solution of (semilinear) path dependent partial differential equations on the space of continuous paths already mentioned above and prove a comparison principle.
The rest of this paper is organized as follows: Section 2 contains some background material on martingale problems and on viscosity solutions; Section 3 deals with martingale problems; the alternative definitions of viscosity solution are discussed in Section 4; Section 5 deals with martingale problems with boundary conditions; finally in Section 6 we present two examples, including the application to diffusions with degenerate direction of reflection.

2 Background

2.1 Martingale problems

Throughout this paper we will assume that \((E,r)\) is a complete separable metric space, \(D_E[0,\infty)\) is the space of cadlag, \(E\)-valued functions with the Skorohod topology, and

\[ A \subset C_b(E) \times C_b(E), \]

where \(C_b(E)\) denotes the space of bounded, continuous functions on \((E,r)\), while \(B(E)\) will denote the space of bounded, measurable functions on \((E,r)\) and \(P(E)\) will denote the space of probability measures on \((E,r)\).

Because linear combinations of martingales are martingales, without loss of generality, we can assume that \(A\) is a linear operator and that \((1,0) \in A\).

**Definition 2.1** A stochastic process \(X\) with sample paths in \(D_E[0,\infty)\) is a solution of the martingale problem for \(A\) provided there exists a filtration \(\{\mathcal{F}_t\}\) such that \(X\) is \(\{\mathcal{F}_t\}\)-adapted and

\[ M_f(t) = f(X(t)) - f(X(0)) - \int_0^t g(X(s)) ds \]

is a \(\{\mathcal{F}_t\}\)-martingale for each \((f,g) \in A\). If \(X(0)\) has distribution \(\mu\), we say \(X\) is a solution of the martingale problem for \((A,\mu)\).

For \(\mu \in P(E)\) and \(f \in B(E)\) we will use the notation

\[ \mu f = \int_E f(x) \mu(dx). \]

**Lemma 2.2** Let \(X\) be a \(\{\mathcal{F}_t\}\)-adapted stochastic process with sample paths in \(D_E[0,\infty)\), with initial distribution \(\mu\), and \(f,g \in B(E)\). If \(\mu f\) is a \(\{\mathcal{F}_t\}\)-martingale then, for \(\lambda > 0\),

\[ M^\lambda_f(t) = e^{-\lambda t} f(X(t)) - f(X(0)) + \int_0^t e^{-\lambda s} (\lambda f(X(s)) - g(X(s))) ds \]

is a \(\{\mathcal{F}_t\}\)-martingale. In particular

\[ \mu f = E[\int_0^\infty e^{-\lambda s} (\lambda f(X(s)) - g(X(s))) ds]. \]
Proof. By applying Itô’s formula for semimartingales to \( e^{-\lambda t}f(X(t)) \), we obtain
\[
e^{-\lambda t}f(X(t)) - f(X(0)) = \int_0^t (-f(X(s))\lambda e^{-\lambda s} + e^{-\lambda s}g(X(s)))ds + \int_0^t e^{-\lambda s}dM_f(s).
\]
In particular
\[
E[f(X(0)) - e^{-\lambda t}f(X(t))] = E[\int_0^t e^{-\lambda s}(\lambda f(X(s)) - g(X(s)))ds]
\]
and the second statement follows by taking \( t \to \infty \).

The following lemma is a simple consequence of the optional sampling theorem. (See the proof of Lemma 5.5.)

**Lemma 2.3** Let \( X \) be a solution of the martingale problem for \( A \) with respect to a filtration \( \{\mathcal{F}_t\} \). Let \( \tau \geq 0 \) be a finite \( \{\mathcal{F}_t\} \)-stopping time and \( H \geq 0 \) be a \( \mathcal{F}_\tau \)-measurable random variable such that \( 0 < E[H] < \infty \). Define \( P^{\tau,H} \) by
\[
P^{\tau,H}(C) = \frac{E[H1_{C}(X(\tau + \cdot))]}{E[H]}, \quad C \in \mathcal{B}(D_E[0,\infty)).
\]
Then \( P^{\tau,H} \) is the distribution of a solution of the martingale problem for \( A \).

**Definition 2.4** A linear operator \( A \subset B(E) \times B(E) \) is dissipative provided
\[
\|\lambda f - g\| \geq \lambda\|f\|
\]
for each \((f, g) \in A \) and each \( \lambda > 0 \).

**Lemma 2.5** Suppose for each \( x \in E \), there exists a solution of the martingale problem for \((A, \delta_x)\). Then \( A \) is dissipative.

**Proof.** By Lemma 2.2
\[
|f(x)| \leq E[\int_0^\infty e^{-\lambda s}|\lambda f(X(s)) - g(X(s))|ds] \leq \frac{1}{\lambda}\|\lambda f - g\|
\]
\( \Box \)

**Theorem 2.6** Let \( A \) be dissipative. Suppose that for each \( \lambda > 0 \), \( \mathcal{R}(\lambda - A) \) is separating. Then for each initial distribution \( \mu \in \mathcal{P}(E) \), any two solutions of the martingale problem for \((A, \mu)\) have the same distribution on \( D_E[0,\infty) \).

**Proof.** This result is essentially Corollary 4.4.4 of Ethier and Kurtz (1986). The assumption regarding the \( D(A) \) in the statement of that Corollary was not used in the proof. \( \Box \)
2.2 Viscosity solutions

Let $A \subset C_b(E) \times C_b(E)$. Theorem 2.6 states that if the equation
\begin{equation}
\lambda u - g = h
\end{equation}
has a solution $(u, g) \in A$ for every $\lambda > 0$ and for every $h$ in a class of functions $M \subseteq C_b(E)$ that is separating, then for each initial distribution $\mu$, the martingale problem for $(A, \mu)$ has at most one solution. Unfortunately in many situations it is hard to verify that (2.3) has a solution in $A$. Thus one is lead to consider a weaker notion of solution, namely the notion of viscosity solution.

**Definition 2.7** Let $A$ be as above, $\lambda > 0$, and $h \in C_b(E)$.

a) $u \in B(E)$ is a viscosity subsolution of (2.3) if and only if $u$ is upper semicontinuous and if $(f, g) \in A$ and $x_0 \in E$ satisfy
\begin{equation}
\sup_x (u - f)(x) = (u - f)(x_0),
\end{equation}
then
\begin{equation}
\lambda u(x_0) - g(x_0) \leq h(x_0).
\end{equation}

b) $u \in B(E)$ is a viscosity supersolution of (2.3) if and only if $u$ is lower semicontinuous and if $(f, g) \in A$ and $x_0 \in E$ satisfy
\begin{equation}
\inf_x (u - f)(x) = (u - f)(x_0),
\end{equation}
then
\begin{equation}
\lambda u(x_0) - g(x_0) \geq h(x_0).
\end{equation}

A function $u \in C_b(E)$ is a viscosity solution of (2.3) if it is both a subsolution and a supersolution.

In the theory of viscosity solutions, usually existence of a viscosity solution follows by existence of a viscosity subsolution and a viscosity supersolution, together with the following comparison principle, which obviously also yields uniqueness of the viscosity solution.

**Definition 2.8** The comparison principle holds for (2.3) when every subsolution is pointwise less than or equal to every supersolution.

**Remark 2.9** To better motivate the notion of viscosity solution in the context of martingale problems, suppose that there exists $v \in C_b(E)$ such that
\begin{equation}
e^{-\lambda t}v(X(t)) + \int_0^t e^{-\lambda s}h(X(s))ds
\end{equation}
is an $\{F_t^X\}$-martingale for every solution $X$ of the martingale problem for $A$. Let $(f, g) \in A$ and $x_0$ satisfy
\begin{equation}
\sup_x (v - f)(x) = (v - f)(x_0).
\end{equation}
Let \( X \) be a solution of the martingale problem for \( A \) with \( X(0) = x_0 \). Then
\[
e^{-\lambda t}(v(X(t)) - f(X(t))) + \int_0^t e^{-\lambda s}(h(X(s)) - \lambda f(X(s)) + g(X(s))) ds
\]
is an \( \mathcal{F}_t^X \)-martingale, and
\[
E\left[ \int_0^t e^{-\lambda s}(\lambda v(X(s)) - g(X(s)) - \lambda f(X(s))) ds \right]
= E\left[ \int_0^t e^{-\lambda s}(v(X(s)) - f(X(s))) ds \right] + E[\lambda e^{-\lambda t}(v(X(t)) - f(X(t))) - (v(x_0) - f(x_0))]
\leq 0.
\]
Dividing by \( t \) and letting \( t \to 0 \), we see that
\[
\lambda v(x_0) - g(x_0) \leq h(x_0),
\]
so \( v \) is a subsolution for (2.3). A similar argument shows that it is also a supersolution and hence a viscosity solution. We will give conditions such that if the comparison principle holds for some \( h \), then a viscosity solution \( v \) exists and (2.8) is a martingale for every solution of the martingale problem for \( A \).

In the case of a domain with boundary, in order to uniquely determine the solution of the martingale problem for \( A \) one usually must specify some boundary conditions, by means of boundary operators \( B_1, \ldots, B_m \).

Let \( E_0 \subseteq E \) be an open set and let
\[
\partial E_0 = \bigcup_{k=1}^m E_k,
\]
for disjoint, nonempty Borel sets \( E_1, \ldots, E_m \).

Let \( A \subseteq C_b(\overline{E}_0) \times C_b(\overline{E}_0) \), \( B_k \subseteq C_b(\overline{E}_0) \times C_b(\overline{E}_0) \), \( k = 1, \ldots, m \), be linear operators with a common domain \( D \). For simplicity we will assume that \( E \) is compact (hence the subscript \( b \) will be dropped) and that \( A, B_1, \ldots, B_m \) are single valued.

**Definition 2.10** Let \( A, B_1 \ldots, B_m \) be as above and let \( \lambda > 0 \). For \( h \in C_b(\overline{E}_0) \), consider the equation
\[
\lambda u - Au = h, \quad \text{on } E_0
\]
\[
-B_k u = 0, \quad \text{on } E_k, \quad k = 1, \ldots, m.
\]

a) \( u \in B(\overline{E}_0) \) is a viscosity subsolution of (2.9) if and only if \( u \) is upper semicontinuous and if \( f \in D \) and \( x_0 \in \overline{E}_0 \) satisfy
\[
\sup_x (u - f)(x) = (u - f)(x_0),
\]
then
\[
\lambda u(x_0) - Af(x_0) \leq h(x_0), \quad \text{if } x_0 \in E_0, \quad (2.11)
\]
\[
(\lambda u(x_0) - Af(x_0) - h(x_0)) \wedge \min_{k: x_0 \in E_k} (-B_k f(x_0)) \leq 0, \quad \text{if } x_0 \in \partial E_0. \quad (2.12)
\]
b) \( u \in B(\overline{E}_0) \) is a viscosity supersolution of (2.9) if and only if \( u \) is lower semicontinuous and if \( f \in \mathcal{D} \) and \( x_0 \in \overline{E}_0 \) satisfy

\[
\inf_x (u - f)(x) = (u - f)(x_0),
\]

then

\[
\lambda u(x_0) - Af(x_0) \geq h(x_0), \quad \text{if } x_0 \in E_0, \quad (2.14)
\]

\[
(\lambda u(x_0) - Af(x_0) - h(x_0)) \lor \max_{k: x_0 \in E_k} (-B_k f(x_0)) \geq 0, \quad \text{if } x_0 \in \partial E_0.
\]

A function \( u \in C_b(\overline{E}_0) \) is a viscosity solution of (2.9) if it is both a subsolution and a supersolution.

**Remark 2.11** The above definition, with the ‘relaxed’ requirement that on the boundary either the interior inequality or the boundary inequality be satisfied by at least one among \(-B_1 f, \ldots, -B_m f\) is the standard one in the theory of viscosity solutions where it is used in particular because it is stable under limit operations and because it can be localized. As will be clear in Section 5, it suits perfectly our approach to martingale problems with boundary conditions.

### 3 Comparison principle and uniqueness for martingale problems

Let \( \Pi \subset \mathcal{P}(D_E[0, \infty)) \) denote the collection of distributions of solutions of the martingale problem for \( A \), and for \( \mu \in \mathcal{P}(E) \), let \( \Pi_\mu \subset \Pi \) denote the subcollection with initial distribution \( \mu \). If \( \mu = \delta_x \), we will write \( \Pi_x \) for \( \Pi_{\delta_x} \). In this section and in the next one, \( X \) will be the canonical process on \( D_E[0, \infty) \). Assume the following condition.

**Condition 3.1**

a) \( \mathcal{D}(A) \) is dense in \( C_b(E) \) in the topology of uniform convergence on compact sets.

b) For each \( \mu \in \mathcal{P}(E) \), \( \Pi_\mu \neq \emptyset \).

c) If \( K \subset \mathcal{P}(E) \) is compact, then \( \cup_{\mu \in K} \Pi_\mu \) is compact. (See Proposition 3.3 below.)

**Remark 3.2** In working with these conditions, it is simplest to take the usual Skorohod topology on \( D_E[0, \infty) \). (See, for example, Sections 3.5-3.9 of [Ethier and Kurtz (1986)].) The results of this paper also hold if we take the Jakubowski topology ([Jakubowski (1997)]). The \( \sigma \)-algebra of Borel sets \( \mathcal{B}(D_E[0, \infty)) \) is the same for both topologies and, in fact, is simply the smallest \( \sigma \)-algebra under which all mappings of the form \( x \in D_E[0, \infty) \to x(t), t \geq 0, \) are measurable.

It is also relevant to note that mappings of the form

\[
x \in D_E[0, \infty) \to \int_0^\infty e^{-\lambda t} h(x(t)) dt, \quad h \in C_b(E), \lambda > 0,
\]
are continuous under both topologies. The Jakubowski topology could be particularly useful for extensions of the results of Section 5 to constrained martingale problems in which the boundary terms are not local-time integrals.

**Proposition 3.3** In addition to Condition 3.1(b), assume that for each compact \( K \subset E \), \( \varepsilon > 0 \), and \( T > 0 \), there exists a compact \( K' \subset E \) such that

\[
P\{X(t) \in K', t \leq T, X(0) \in K\} \geq (1 - \varepsilon)P\{X(0) \in K\}, \quad \forall P \in \Pi.
\]

Then Condition 3.1(c) holds.

**Proof.** The result follows by Theorem 4.5.11 (b) of Ethier and Kurtz (1986). \( \square \)

**Remark 3.4** The assumption in Proposition 3.3 above and Condition 3.1(a) imply, in particular, that every progressive process such that (2.1) is a martingale has a modification with sample paths in \( D_E[0, \infty) \) (See Theorem 4.3.6 of Ethier and Kurtz (1986).)

Let \( \lambda > 0 \), and for \( h \in C^b(E) \) define

\[
u_+(x) = u_+(x, h) = \sup_{P \in \Pi_x} E^P[\int_0^\infty e^{-\lambda t}h(X(t))dt], \tag{3.1}
\]

\[
u_-(x) = u_-(x, h) = \inf_{P \in \Pi_x} E^P[\int_0^\infty e^{-\lambda t}h(X(t))dt]. \tag{3.2}
\]

**Lemma 3.5** Assume Condition 3.1, and for \( h \in C^b(E) \), define

\[
\pi_+(\Pi_\mu, h) = \sup_{P \in \Pi_\mu} E^P[\int_0^\infty e^{-\lambda t}h(X(t))dt], \tag{3.3}
\]

\[
\pi_-(\Pi_\mu, h) = \inf_{P \in \Pi_\mu} E^P[\int_0^\infty e^{-\lambda t}h(X(t))dt]. \tag{3.4}
\]

Then \( u_+(x, h) \) is upper semicontinuous (hence measurable), and

\[
\pi_+(\Pi_\mu, h) = \int_E u_+(x, h)\mu(dx). \tag{3.5}
\]

The analogous result holds for \( u_- \) and \( \pi_- \).

**Proof.** By the compactness of \( \Pi_x \), the supremum in (3.1) is achieved. Suppose \( x_n \in E \to x \). Then by Condition 3.1(c), \( \Pi_x \cup \cup_n \Pi_{x_n} \) is compact and hence \( \limsup u_+(x_n, h) \leq u_+(x, h) \) giving the upper semicontinuity. By Theorem 4.5.11(a) and Lemmas 4.5.8 and 4.5.10 of Ethier and Kurtz (1986), (3.5) holds for \( h \geq 0 \). For an arbitrary \( h \in C^b(E) \), note that \( \pi_+ (\Pi_\mu, h) = \pi_+ (\Pi_\mu, h - \inf_x h) + \inf_x h \) and \( u_+(x, h) = u_+(x, h - \inf_x h) + \inf_x h \). To prove the assertion for \( \pi_- \) use the fact that \( \pi_-(\Pi_\mu, h) = -\pi_+(\Pi_\mu, -h) \). \( \square \)
Lemma 3.6 Assume that Condition \[3.1\] holds. Then \(u_+\) is a viscosity subsolution of (2.3) and \(u_-\) is a viscosity supersolution of the same equation.

Proof. Since \(u_-(x,h) = -u_+(x,-h)\) it is enough to consider \(u_+\). Let \((f,g) \in A\). Suppose \(x_0\) is a point such that \(u_+(x_0) - f(x_0) = \sup_x (u_+(x) - f(x))\). Since we can always add a constant to \(f\), we can assume \(u_+(x_0) - f(x_0) = 0\). By compactness (Condition \[3.1(c)\]), we have

\[
u_+(x_0) = E^P \left[ \int_{0}^{\infty} e^{-\lambda t} h(X(t)) dt \right]
\]

for some \(P \in \Pi_{x_0}\).

For \(\epsilon > 0\), define

\[
\tau_\epsilon = \epsilon \wedge \inf \{t > 0: r(X(t), x_0) \geq \epsilon \text{ or } r(X(t^-), x_0) \geq \epsilon \}
\]

(3.6) and let \(H_\epsilon = e^{-\lambda \tau_\epsilon}\). Then, by Lemma \[2.2\]

\[
0 = u_+(x_0) - f(x_0)
\]

\[
= E^P \left[ \int_{0}^{\infty} e^{-\lambda t} (h(X(t)) - \lambda f(X(t)) + g(X(t))) dt \right]
\]

\[
= E^P \left[ \int_{0}^{\tau_\epsilon} e^{-\lambda t} (h(X(t)) - \lambda f(X(t)) + g(X(t))) dt \right]
\]

\[
+ E^P \left[ e^{-\lambda \tau_\epsilon} \int_{0}^{\infty} e^{-\lambda t} (h(X(t + \tau_\epsilon)) - \lambda f(X(t + \tau_\epsilon)) + g(X(t + \tau_\epsilon))) dt \right]
\]

\[
= E^P \left[ \int_{0}^{\tau_\epsilon} e^{-\lambda t} (h(X(t)) - \lambda f(X(t)) + g(X(t))) dt \right]
\]

\[
+ E^P[H_\epsilon] E^{\tau_\epsilon, H_\epsilon} \left[ \int_{0}^{\infty} e^{-\lambda t} (h(X(t)) - \lambda f(X(t)) + g(X(t))) dt \right].
\]

Setting \(\mu_\epsilon(\cdot) = P^{\tau_\epsilon, H_\epsilon}(X(0) \in \cdot)\), by Lemma \[2.3\] and Lemma \[2.2\] the above chain of equalities can be continued as (with the notation (2.2))

\[
\leq E^P \left[ \int_{0}^{\tau_\epsilon} e^{-\lambda t} (h(X(t)) - \lambda f(X(t)) + g(X(t))) dt \right] + E^P[H_\epsilon](\pi_+(\Pi_{\mu_\epsilon}, h) - \mu_\epsilon f),
\]

and, by Lemma \[3.5\]

\[
= E^P \left[ \int_{0}^{\tau_\epsilon} e^{-\lambda t} (h(X(t)) - \lambda f(X(t)) + g(X(t))) dt \right] + E^P[H_\epsilon] (\mu_\epsilon u_+ - \mu_\epsilon f)
\]

\[
= E^P \left[ \int_{0}^{\tau_\epsilon} e^{-\lambda t} (h(X(t)) - \lambda f(X(t)) + g(X(t))) dt \right] + E^P \left[ e^{-\lambda \tau_\epsilon} (u_+(X(\tau_\epsilon)) - f(X(\tau_\epsilon))) \right]
\]

\[
\leq E^P \left[ \int_{0}^{\tau_\epsilon} e^{-\lambda t} (h(X(t)) - \lambda f(X(t)) + g(X(t))) dt \right],
\]

where the last inequality uses the fact that \(u_- - f \leq 0\). Therefore

\[
0 \leq \lim_{\epsilon \to 0} \frac{E^P \left[ \int_{0}^{\tau_\epsilon} e^{-\lambda t} (h(X(t)) - \lambda f(X(t)) + g(X(t))) dt \right]}{E^P[\tau_\epsilon]}
\]

\[
= h(x_0) - \lambda f(x_0) + g(x_0)
\]

\[
= h(x_0) - \lambda u_+(x_0) + g(x_0).
\]
Corollary 3.7 Let \( h \in C_b(E) \). If in addition to Condition 3.1, the comparison principle holds for equation (2.3), then \( u = u_+ = u_- \) is the unique viscosity solution of equation (2.3).

An essential ingredient in the proof of our main result is Lemma 4.5.18 of Ethier and Kurtz [1986], which we restate here for the convenience of the reader.

Lemma 3.8 Assume Condition 3.1(b) holds. Suppose \( u, h \in B(E) \) satisfy
\[
\mu u = E^P \left[ \int_0^\infty e^{-\lambda t} h(X(t))dt \right],
\]
for every \( P \in \Pi_{\mu} \) and \( \mu \in \mathcal{P}(E) \). Then
\[
u(X(t)) - \int_0^t (\lambda u(X(s)) - h(X(s)))ds
\]
is a \( \{ F^X_t \} \)-martingale under \( P \) for every \( P \in \Pi \).

Remark 3.9 By Lemmas 3.5 and 3.6, if the comparison principle holds for some \( h \) then (3.7) is satisfied by \( u = u_+ = u_- \) and hence (3.8) holds. However, even if the comparison principle does not hold, under Condition 3.1, by Lemma 4.5.18 of Ethier and Kurtz [1986], for each \( \mu \in \mathcal{P}(E) \), there exists \( P \in \Pi_{\mu} \) such that under \( P \)
\[
u(X(t)) - \int_0^t (\lambda u(X(s)) - h(X(s)))ds
\]
is a \( \{ F^X_t \} \)-martingale.

Theorem 3.10 Assume that Condition 3.1 holds. For \( \lambda > 0 \), let \( M_\lambda \) be the set of \( h \in C_b(E) \), such that the comparison principle holds for (2.3). If for each \( \lambda > 0 \) \( M_\lambda \) is separating, then uniqueness holds for the martingale problem for \( A \).

Proof. The proof is a simplified version of the proof of the Krylov extension theorem as it appears in Ethier and Kurtz [1986] (Theorem 4.5.19). Let \( \hat{A} \) be the collection of \( (f, g) \in B(E) \times B(E) \) such that \( f(X(t)) - \int_0^t g(X(s))ds \) is an \( \{ F^X_t \} \)-martingale for all \( P \in \Pi \). Then \( \hat{A} \supseteq A \) and all solutions of the martingale problem for \( \hat{A} \) are solutions of the martingale problem for \( A \); conversely, by definition of \( \hat{A} \), all solutions of the martingale problem for \( A \) are solutions of the martingale problem for \( \hat{A} \), that is \( \hat{A} \) is an extension of \( A \) that has the same solutions of the martingale problem as \( A \). \( \hat{A} \) is linear and, by Lemma 2.5 dissipative. By the comparison principle, Lemma 3.5 and Lemma 3.8 for each \( h \in M_\lambda \) and \( u = u_+ = u_- \) given by (3.1) (or equivalently (3.2)), \( (u, \lambda u - h) \) belongs to \( \hat{A} \), or equivalently the pair \( (u, h) \) belongs to \( \lambda - \hat{A} \). Consequently \( \mathcal{R}(\lambda - \hat{A}) \supseteq M_\lambda \) is separating and the theorem follows from Theorem 2.6. \( \square \)

Remark 3.11 If, for some \( h \), there exists \((u, g) \in \hat{A} \) such that \( \lambda u - g = h \) (essentially the analog of a stochastic solution as defined in Stroock and Varadhan [1972]) then, by Lemma 2.2 and Remark 2.4, \( u \) is a viscosity solution of (2.3).
4 Alternative definitions of viscosity solution

Different definitions of viscosity solution may be useful, depending on the setting. Here we discuss two other possibilities. As mentioned in the Introduction, the first, which is stated in terms of solutions of the martingale problem, is a modification of a definition suggested to us by Nizar Touzi and used in Ekren, Keller, Touzi, and Zhang (2014), while the second appears in Feng and Kurtz (2006). We show that Lemma 3.6 still holds under these alternative definitions and hence all the results of Section 3 carry over. $\mathcal{T}$ will denote the set of $\{\mathcal{F}^X_t\}$-stopping times.

**Definition 4.1** Let $A \subset C_b(E) \times C_b(E)$, $\lambda > 0$, and $h \in C_b(E)$.

a) $u \in B(E)$ is a viscosity subsolution of (2.3) if and only if $u$ is upper semicontinuous and if $(f, g) \in A$, $x_0 \in E$, and there exists a strictly positive $\tau_0 \in \mathcal{T}$ such that

$$\sup_{P \in \Pi_{x_0}} \frac{E^P[e^{-\lambda \tau \wedge \tau_0}(u - f)(X(\tau \wedge \tau_0))]}{E^P[e^{-\lambda \tau \wedge \tau_0}]} = (u - f)(x_0), \quad (4.1)$$

then

$$\lambda u(x_0) - g(x_0) \leq h(x_0). \quad (4.2)$$

b) $u \in B(E)$ is a viscosity supersolution of (2.3) if and only if $u$ is lower semicontinuous and if $(f, g) \in A$, $x_0 \in E$, and there exists a strictly positive $\tau_0 \in \mathcal{T}$ such that

$$\inf_{P \in \Pi_{x_0}} \frac{E^P[e^{-\lambda \tau \wedge \tau_0}(u - f)(X(\tau \wedge \tau_0))]}{E^P[e^{-\lambda \tau \wedge \tau_0}]} = (u - f)(x_0), \quad (4.3)$$

then

$$\lambda u(x_0) - g(x_0) \geq h(x_0). \quad (4.4)$$

A function $u \in C_b(E)$ is a viscosity solution of (2.3) if it is both a subsolution and a supersolution.

**Remark 4.2** If $(u - f)(x_0) = \sup_x(u - f)(x)$, then (4.1) is satisfied. Consequently, every sub/supersolution in the sense of Definition 4.1 is a sub/supersolution in the sense of Definition 2.7.

**Remark 4.3** Definition 4.1 requires (4.2) (or (4.4)) to hold only at points $x_0$ for which (4.1) (or (4.3)) is verified for some $\tau_0$. Note that, as in Definition 2.7, such an $x_0$ might not exist.

Definition 4.1 essentially requires a local maximum principle and is related to the notion of characteristic operator as given in Dynkin (1965).

For Definition 4.1, we have the following analog of Lemma 3.6

**Lemma 4.4** Assume that Condition 3.1 holds. Then, in the sense of Definition 4.1, $u_+$ given by (3.1) is a viscosity subsolution of (2.3) and $u_-$ given by (3.2) is a viscosity supersolution of the same equation.
Proof. Let \((f, g) \in A\). Suppose \(x_0\) is a point such that (4.1) holds for \(u_+\) for some \(\tau_0 \in \mathcal{T}\), \(\tau_0 > 0\). Since we can always add a constant to \(f\), we can assume \(u_+(x_0) - f(x_0) = 0\). By the same arguments used in the proof of Lemma 3.6, defining \(\tau_{\epsilon}\) and \(H_\epsilon\) in the same way, we obtain, for some \(P \in \Pi_{x_0}\) (independent of \(\epsilon\)),

\[
0 = u_+(x_0) - f(x_0) \
\leq E^P \left[ \int_0^{\tau_{\epsilon} \wedge \tau_0} e^{-\lambda t} (h(X(t)) - \lambda f(X(t)) + g(X(t))) dt \right] \
+ E^P \left[ e^{-\lambda \tau_{\epsilon} \wedge \tau_0} (u_+(X(\tau_{\epsilon} \wedge \tau_0)) - f(X(\tau_{\epsilon} \wedge \tau_0))) \right] \
\leq E^P \left[ \int_0^{\tau_{\epsilon} \wedge \tau_0} e^{-\lambda t} (h(X(t)) - \lambda f(X(t)) + g(X(t))) dt \right],
\]

where the last inequality uses (4.1) and the fact that \(u_+(x_0) - f(x_0) = 0\). Then the result follows as in Lemma 3.6. \qed

The following is essentially Definition 7.1 of Feng and Kurtz (2006).

**Definition 4.5** Let \(A \subset C_b(E) \times C_b(E)\), \(\lambda > 0\), and \(h \in C_b(E)\).

a) \(u \in B(E)\) is a viscosity subsolution of (2.3) if and only if \(u\) is upper semicontinuous and for each \((f, g) \in A\), there exist \(y_n \in E\) such that

\[
\lim_{n \to \infty} (u - f)(y_n) = \sup_x (u - f)(x), \tag{4.5}
\]

and

\[
\limsup_{n \to \infty} (\lambda u(y_n) - g(y_n) - h(y_n)) \leq 0. \tag{4.6}
\]

b) \(u \in B(E)\) is a viscosity supersolution of (2.3) if and only if \(u\) is lower semicontinuous and for each \((f, g) \in A\), there exist \(y_n \in E\) such that

\[
\lim_{n \to \infty} (u - f)(y_n) = \inf_x (u - f)(x), \tag{4.7}
\]

and

\[
\liminf_{n \to \infty} (\lambda u(y_n) - g(y_n) - h(y_n)) \geq 0. \tag{4.8}
\]

A function \(u \in C_b(E)\) is a viscosity solution of (2.3) if it is both a subsolution and a supersolution.

**Remark 4.6** For \(E\) compact, every sub/supersolution in the sense of Definition 2.7 is a sub/supersolution in the sense of Definition 4.5.

For Definition 4.5, we have the following analog of Lemma 3.6, \(C_{b,u}(E)\) denotes the space of bounded, uniformly continuous functions on \(E\).
Lemma 4.7 For $\epsilon > 0$, define

$$\tau_\epsilon = \epsilon \wedge \inf\{t > 0 : r(X(t), X(0)) \geq \epsilon \text{ or } r(X(t-), X(0)) \geq \epsilon\}.$$ 

Assume $A \subset C_{b,u}(E) \times C_{b,u}(E)$, for each $\epsilon > 0$, $\inf_{P \in \Pi} E^P[\tau_\epsilon] > 0$, and that Condition 3.1 holds. Then, in the sense of Definition 4.5, for $h \in C_{b,u}(E)$, $u_+$ given by (3.1) is a viscosity subsolution of (2.3) and $u_-$ given by (3.2) is a viscosity supersolution of the same equation.

Proof. Let $(f, g) \in A$. Suppose $\{y_n\}$ is a sequence such that (4.5) holds for $u_+$. Since we can always add a constant to $f$, we can assume $\sup_x (u_+ - f)(x) = 0$. Let $H_\epsilon = e^{-\lambda \tau_\epsilon}$. Then, by the same arguments as in Lemma 3.6, we have, for some $P_n \in \Pi y_n$ (independent of $\epsilon$),

$$(u_+ - f)(y_n) \leq E^{P_n} \left[ \int_0^{\tau_\epsilon} e^{-\lambda t} (h(X(t)) - \lambda f(X(t)) + g(X(t))) \, dt \right],$$

where we have used the fact that $\sup_x (u_+ - f)(x) = 0$. Therefore

$$\frac{(u_+ - f)(y_n)}{E^{P_n}[\tau_\epsilon]} \leq \frac{E^{P_n} \left[ \int_0^{\tau_\epsilon} e^{-\lambda t} (h(X(t)) - \lambda f(X(t)) + g(X(t))) \, dt \right]}{E^{P_n}[\tau_\epsilon]}.$$

Replacing $\epsilon$ by $\epsilon_n$ going to zero sufficiently slowly so that the left side converges to zero, the uniform continuity of $f$, $g$, and $h$ implies the right side is asymptotic to $h(y_n) - \lambda f(y_n) + g(y_n)$ giving

$$0 \leq \liminf_{n \to \infty} (h(y_n) - \lambda f(y_n) + g(y_n)) = \liminf_{n \to \infty} (h(y_n) - \lambda u_+(y_n) + g(y_n)).$$

Remark 4.8 In Lemma 4.7, we have actually proved that for each $(f, g) \in A$ and for each $\{y_n\} \subseteq E$ such that (4.5) holds, $u_+$ given by (3.1) satisfies (4.6). Analogously, for each $\{y_n\} \subseteq E$ such that (4.7) holds, $u_-$ given by (3.2) satisfies (4.8). These conditions could also be used as definitions of sub/supersolutions, but we prefer to refer to the more general ones of Definition (4.5).

The following lemma is essentially Lemma 7.4 of Feng and Kurtz (2006). It gives the intuitively natural result that if $h \in \overline{R}(\lambda - A)$ (where the closure is taken under uniform convergence), then, under Definition 4.5, the comparison principle holds for $\lambda u - Au = h$. If $E$ is compact, the same result holds for Definition 2.7, by Remark 4.6.

Lemma 4.9 Suppose $h \in C_b(E)$ and there exist $(f_n, g_n) \in A$ satisfying $\sup_x |\lambda f_n(x) - g_n(x) - h| \to 0$. Then, under Definition 4.5 the comparison principle holds for $\lambda u - Au = h$.

Proof. Suppose $u$ is a viscosity subsolution. Set $h_n = \lambda f_n - g_n$. For $\epsilon_n > 0$, $\epsilon_n \to 0$, there exist $y_n \in E$ satisfying $u(y_n) - f_n(y_n) \geq \sup_x (u(x) - f_n(x)) - \epsilon_n$ and $\lambda u(y_n) - g_n(y_n) - h(y_n) \leq \epsilon_n$ for all $n$. Therefore, for $\epsilon_n$ sufficiently small, we have

$$u(y_n) - f_n(y_n) \geq \sup_x (u(x) - f_n(x)) - \epsilon_n \geq u(y_n) - f_n(y_n) - \epsilon_n.$$
Then
\[ \sup_x (\lambda u(x) - \lambda f_n(x)) \leq \lambda u(y_n) - \lambda f_n(y_n) + \epsilon_n \]
\[ \leq h(y_n) + g_n(y_n) - \lambda f_n(y_n) + 2\epsilon_n \]
\[ = h(y_n) - h_n(y_n) + 2\epsilon_n \]
\[ \rightarrow 0. \]

Similarly, if $\overline{u}$ is a supersolution of $\lambda u - Au = h$,
\[ \liminf_{n \to \infty} \inf_x (\overline{u}(x) - f_n(x)) \geq 0, \]
and it follows that $u \leq \overline{u}$. \qed

\section{Martingale problems with boundary conditions}

The study of stochastic processes that are constrained to some set $E_0$ and must satisfy some boundary condition on $\partial E_0$, described by one or more boundary operators $B_1, \ldots, B_m$, is typically carried out by incorporating the boundary condition in the definition of the domain $\mathcal{D}(A)$ (see Remark 5.11 below). However, this approach restricts the problems that can be dealt with to fairly regular ones, so we follow the formulation of a constrained martingale problem given in Kurtz (1990). (See also Kurtz (1991); Kurtz and Stockbridge (2001)).

Let $E_0 \subseteq E$ be an open set and let
\[ \partial E_0 = \bigcup_{k=1}^m E_k, \]
for disjoint, nonempty Borel sets $E_1, \ldots, E_m$. Let $A \subseteq C_b(E_0) \times C_b(E_0)$, $B_k \subseteq C_b(E_0) \times C_b(E_0)$, $k = 1, \ldots, m$, be linear operators with a common domain $\mathcal{D}$ such that $(1,0) \in A$, $(1,0) \in B_k$, $k = 1, \ldots, m$. For simplicity we will assume that $E$ is compact (hence the subscript $b$ will be dropped) and that $A, B_1, \ldots, B_m$ are single-valued.

\textbf{Definition 5.1} A stochastic process $X$ with sample paths in $D_{\overline{E}_0}[0, \infty)$ is a solution of the constrained martingale problem for $(A, E_0; B_1, E_1; \ldots; B_m, E_m)$ provided there exist a filtration $\{\mathcal{F}_t\}$ and continuous, nondecreasing processes $\gamma_1, \ldots, \gamma_m$ such that $X, \gamma_1, \ldots, \gamma_m$ are $\{\mathcal{F}_t\}$-adapted,
\[ \gamma_k(t) = \int_0^t 1_{E_k}(X(s^-))d\gamma_k(s), \]
and for each $f \in \mathcal{D}$,
\[ M_f(t) = f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds - \sum_{k=1}^m \int_0^t B_k f(X(s^-))d\gamma_k(s) \quad (5.1) \]
is a $\{\mathcal{F}_t\}$-martingale.
Remark 5.2 \( \gamma_1, \ldots, \gamma_m \) will be called local times since \( \gamma_k \) increases only when \( X \) is in \( E_k \).

Without loss of generality, we can assume that the \( \gamma_k \) are \( \{F^X_t\} \)-adapted. (Replace \( \gamma_k \) by its dual, predictable projection on \( \{F^X_t\} \).) Definition 5.1 does not require that the \( \gamma_k \) be uniquely determined by the distribution of \( X \), but if \( \gamma_1^k, \ldots, \gamma_m^k \), \( k = 1, \ldots, m \), are continuous and satisfy the martingale requirement with the same filtration, we must have

\[
\sum_{k=1}^m \int_0^t B_k f(X(s-))d\gamma_k^1(s) - \sum_{k=1}^m \int_0^t B_k f(X(s-))d\gamma_k^2(s) = 0,
\]

since this expression will be a continuous martingale with finite variation paths.

Remark 5.3 The main example of a constrained martingale problem in the sense of the above definition is the constrained martingale problem that describes a reflected diffusion process. In this case, \( A \) is a second order elliptic operator and the \( B_k \) are first order differential operators. Although there is a vast literature on this topic, there are still relevant cases of reflected diffusions that have not been uniquely characterized as solutions of martingale problems or stochastic differential equations. In Section 6, the results of this section are used in one of these cases. More general constrained diffusions where the \( B_k \) are second order elliptic operators, for instance diffusions with sticky reflection, also satisfy Definition 5.1. Definition 5.1 is a special case of a more general definition of constrained martingale problem given in Kurtz and Stockbridge (2001). This broader definition allows for more general boundary behavior, such as models considered in Costantini and Kurtz (2006).

Lemma 5.4 Let \( X \) be a stochastic process with sample paths in \( D_{E_0}[0, \infty) \), \( \gamma_1, \ldots, \gamma_m \) be continuous, nondecreasing processes such that \( X, \gamma_1, \ldots, \gamma_m \) are \( \{F_t\} \)-adapted. Then for \( f \in \mathcal{D} \) such that \( (5.1) \) is a \( \{F_t\} \)-martingale and \( \lambda > 0 \),

\[
M^\lambda_f(t) = e^{-\lambda t} f(X(t)) - f(X(0)) + \int_0^t e^{-\lambda s}(\lambda f(X(s)) - Af(X(s)))ds - \sum_{k=1}^m \int_0^t e^{-\lambda s}B_k f(X(s-))d\gamma_k(s)
\]

is a \( \{F_t\} \)-martingale.

Proof. By applying Itô’s formula for semimartingales to \( e^{-\lambda t} f(X(t)) \), we obtain

\[
e^{-\lambda t} f(X(t)) - f(X(0)) = \int_0^t (-f(X(s))\lambda e^{-\lambda s} + e^{-\lambda s} Af(X(s)))ds + \sum_{k=1}^m \int_0^t e^{-\lambda s}B_k f(X(s-))d\gamma_k(s) + \int_0^t e^{-\lambda s}dM_f(s).
\]

Lemma 5.5 a) The set of distributions of solutions of the constrained martingale problem for \( (A, E_0; B_1, E_1; \ldots; B_m, E_m) \) is convex.
b) Let $X, \gamma_1, \ldots, \gamma_m$ satisfy Definition 5.1. Let $\tau \geq 0$ be a bounded $\{\mathcal{F}_k\}$-stopping time and $H \geq 0$ be a $\mathcal{F}_\tau$-measurable random variable such that $0 < E[H] < \infty$. Then the measure $P^{\tau,H} \in \mathcal{P}(D_E[0,\infty])$ defined by

$$P^{\tau,H}(C) = \frac{E[H1_C(X(\tau + \cdot))]}{E[H]}, \quad C \in \mathcal{B}(D_E[0,\infty]),$$

is the distribution of a solution of the constrained martingale problem for $(A, E_0; B_1, E_1; \ldots; B_m, E_m)$.

**Proof.** Part (a) is immediate. For Part (b), let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space on which $X, \gamma_1, \ldots, \gamma_m$ are defined, and define $\mathbb{P}^H$ by

$$\mathbb{P}^H(C) = \frac{E^\mathbb{P}[H1_C]}{E^\mathbb{P}[H]}, \quad C \in \mathcal{F}.$$  

Define $X^\tau$ and $\gamma_k^\tau$ by $X^\tau(t) = X(\tau + t)$ and $\gamma_k^\tau(t) = \gamma_k(\tau + t) - \gamma_k(\tau)$. $X^\tau$ and the $\gamma_k^\tau$ are adapted to the filtration $\{\mathcal{F}_{\tau+t}\}$ and for $0 \leq t_1 < \cdots < t_n < t_{n+1}$ and $f_1, \cdots, f_n \in \mathcal{B}(E)$,

$$E^{\mathbb{P}^H}\left[\{f(X^\tau(t_{n+1})) - f(X^\tau(t_n)) - \int_{t_n}^{t_{n+1}} Af(X^\tau(s))ds - \sum_{k=1}^{m} \int_{t_n}^{t_{n+1}} B_k f(X^\tau(s-))d\gamma_k^\tau(s)\}\right]$$

$$\Pi_{i=1}^n f_i(X^\tau(t_i))
\frac{1}{E^{\mathbb{P}[H]}E^{\mathbb{P}}[H]}E^{\mathbb{P}}\left[H\{f(X(\tau + t_{n+1})) - f(X(\tau + t_n)) - \int_{\tau+t_n}^{\tau+t_{n+1}} Af(X(s))ds - \sum_{k=1}^{m} \int_{\tau+t_n}^{\tau+t_{n+1}} B_k f(X(s-))d\gamma_k(s)\}\right]
\Pi_{i=1}^n f_i(X(\tau + t_i))
= 0$$

by the optional sampling theorem. Therefore, under $\mathbb{P}^H$, $X^\tau$ is a solution of the constrained martingale problem with local times $\gamma_1^\tau, \ldots, \gamma_m^\tau$. $P^{\tau,H}$, given by (5.2), is the distribution of $X^\tau$ on $D_E[0,\infty)$. \qed

As in Section 3, let $\Pi$ denote the set of distributions of solutions of the constrained martingale problem and $\Pi_\mu$ denote the set of distributions of solutions with initial condition $\mu$. We assume that the following conditions hold. See Section 5.1 below for settings in which these conditions are valid.

**Condition 5.6**

a) $\mathcal{D}$ is dense in $C(E_0)$ in the topology of uniform convergence.

b) For each $\mu \in \mathcal{P}(E_0)$, $\Pi_\mu \neq \emptyset$.

c) For any compact $K \subseteq \mathcal{P}(E_0)$, $\cup_{\mu \in K} \Pi_\mu$ is compact.

d) For each $P \in \Pi$ and $\lambda > 0$, there exist $\gamma_1, \ldots, \gamma_m$ satisfying the requirements of Definition 5.1 such that $E^P\left[\int_0^\infty e^{-\lambda t}d\gamma_k(t)\right] < \infty$, $k = 1, \cdots, m$.  

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Remark 5.7 For $P \in \Pi$, Condition 5.6(d) and Lemma 5.4 give
\[
\mu f = E[\int_0^\infty e^{-\lambda s}(\lambda f(X(s)) - Af(X(s)))\,ds - \sum_{k=1}^m \int_0^\infty e^{-\lambda s}B_k f(X(s-))\,d\gamma_k(s)].
\] (5.3)

Remark 5.8 We can take the topology on $D_E[0, \infty)$ to be either the Skorohod topology or the Jakubowski topology (see Remark 3.4).

The definitions of $u_+, u_-, \pi_+$ and $\pi_-$ are still given by (3.1), (3.2), (3.3) and (3.4). Lemma 3.5 and Lemma 3.8, with Condition 3.1 replaced by Condition 5.6, still hold in the present setting as long as we can apply Lemmas 4.5.8 and 4.5.10 in Ethier and Kurtz (1986), which is granted by Lemma 5.5 above. In the rest of this section $X$ is the canonical process on $D_E[0, \infty)$ and $\gamma_1, \ldots, \gamma_m$ are a set of $\{\mathcal{F}_t^X\}$-adapted local times (see Remark 5.2).

Lemma 5.9 Assume Condition 5.6 holds. Then $u_+$ is a viscosity subsolution of (2.9) and $u_-$ is a viscosity supersolution of the same equation.

Proof. The proof is similar to the proof of Lemma 3.6, so we will only sketch the argument. For $f \in \mathcal{D}$, let $x_0$ satisfy $\sup_{x \in E_0}(u_+ - f)(x) = u_+(x_0) - f(x_0)$. By adding a constant to $f$ if necessary, we can assume that $u_+(x_0) - f(x_0) = 0$.

With $\tau_\epsilon$ as in the proof of Lemma 3.6 by compactness of $\Pi_{x_0}$ (Condition 5.6(c)), for some $P \in \Pi_{x_0}$ (independent of $\epsilon$) we have
\[
0 \leq E^P\left[\int_0^{\tau_\epsilon} e^{-\lambda t}(h(X(t)) - \lambda f(X(t)) + Af(X(t)))\,dt + \sum_{k=1}^m \int_0^{\tau_\epsilon} e^{-\lambda t}B_k f(X(t-))d\gamma_k(t)\right].
\]
Dividing the expectations by $E^P[\lambda^{-1}(1 - e^{-\lambda \tau_\epsilon}) + \sum_{k=1}^m \int_0^{\tau_\epsilon} e^{-\lambda t}d\gamma_k(t)]$ and letting $\epsilon \to 0$, we must have
\[
0 \leq (h(x_0) - \lambda f(x_0) + Af(x_0)) \vee \max_{k: x_0 \in E_k} B_k f(x_0),
\]
which, since $f(x_0) = u_+(x_0)$, implies (2.12), if $x_0 \in \partial E_0$, and (2.11), if $x_0 \in E_0$. \hfill \Box

The following theorem is the analog of Theorem 3.10.

Theorem 5.10 Assume Condition 5.6 holds. For $\lambda > 0$, let $M_\lambda$ be the set of $h \in C(\overline{E_0})$ such that the comparison principle holds for (2.9). If, for every $\lambda > 0$, $M_\lambda$ is separating, then the distribution of the solution $X$ of the constrained martingale problem for $(A, E_0; B_1, E_1; \ldots; B_m, E_m)$ is uniquely determined.

Proof. Let $\hat{A}$ be the collection of $(f, g) \in B(\overline{E_0}) \times B(\overline{E_0})$ such that $f(X(t)) - \int_0^t g(X(s))\,ds$ is a $\{\mathcal{F}_t^X\}$-martingale for all $P \in \Pi$. Denote by $\hat{\Pi}$ the set of the distributions of solutions of the martingale problem for $\hat{A}$, and by $\hat{\Pi}_\mu$ the set of solutions with initial distribution $\mu$. Then, by construction, for each $\mu \in \mathcal{P}(\overline{E_0})$, $\Pi_\mu \subseteq \hat{\Pi}_\mu$. $\hat{A}$ is linear and, by Lemma 2.5 dissipative. By the comparison principle, Lemma 3.5(d) and Lemma 3.8 for each $h \in M_\lambda$ and $u = u_+ = u_-$ given by (3.1) (or equivalently, (3.2)), $(u, \lambda u - h)$ belongs to $\hat{A}$, or equivalently the pair $(u, h)$ belongs to $\lambda - \hat{A}$. Consequently $\mathcal{R}(\lambda - \hat{A}) \supseteq M_\lambda$ is separating and the theorem follows from Theorem 2.6. \hfill \Box
Remark 5.11 In the proof of Theorem 3.10, the operator \( \hat{A} \) is an extension of the original operator \( A \). In the constrained martingale problem case \( \hat{A} \) is not an extension of \( A \) as an operator on the domain \( D \), but it is an extension of \( A \) restricted to the domain \( D_0 = \{ f \in D : B_k f(x) = 0 \text{ } \forall x \in E_k, k = 1, \ldots, m \} \). The distribution of the solution \( X \) of the constrained martingale problem for \((A, E_0; B_1, E_1; \ldots; B_m, E_m, \mu)\) is uniquely determined even though the same might not hold for the solution of the martingale problem for \((A|_{D_0}, \mu)\).

5.1 Sufficient conditions for the validity of Condition 5.6

In what follows, we assume that \( E - E_0 = \bigcup_{k=1}^{m} \bar{E}_k \), where the \( \bar{E}_k \) are disjoint Borel sets satisfying \( \bar{E}_k \supseteq E_k, k = 1, \ldots, m \).

Proposition 5.12 Assume Condition 5.6(a) and that the following hold:

i) There exist linear operators \( \tilde{A}, \tilde{B}_1, \ldots, \tilde{B}_m: \tilde{D} \subseteq C(E) \rightarrow C(E) \) with \( \tilde{D} \) dense in \( C(E) \), \((1, 0) \in \tilde{A}, (1, 0) \in \tilde{B}_k, k = 1, \ldots, m \), that are extensions of \( A, B_1, \ldots, B_m \) in the sense that for every \( f \in D \) there exists \( \tilde{f} \in \tilde{D} \) such that \( f = \tilde{f}|_{E_0}, A f = \tilde{A} \tilde{f}|_{E_0} \), and \( B_k f = \tilde{B}_k \tilde{f}|_{E_0} \), \( k = 1, \ldots, m \), and such that the martingale problem for each of \( \tilde{A}, \tilde{B}_1, \ldots, \tilde{B}_m \) with initial condition \( \delta_x \) has a solution for every \( x \in E \).

ii) If \( E \neq \overline{E}_0 \), there exists \( \varphi \in \tilde{D} \) such that \( \varphi = 0 \) on \( \overline{E}_0 \), \( \varphi > 0 \) on \( E - \overline{E}_0 \) and \( \tilde{A} \varphi = 0 \) on \( \overline{E}_0 \), \( \tilde{B}_k \varphi \leq 0 \) on \( \overline{E}_k \), \( k = 1, \ldots, m \).

iii) There exist \( \{ \varphi_n \} \), \( \varphi_n \in D \), such that \( \sup_{n,x} |\varphi_n(x)| < \infty \) and \( B_k \varphi_n(x) \geq n \) on \( \overline{E}_k \) for all \( k = 1, \ldots, m \).

Then b) c) and d) in Condition 5.6 are verified.

Proof. Condition 5.6(b). We will obtain a solution of the constrained martingale problem for \((A, E_0; B_1, E_1; \ldots; B_m, E_m)\) by constructing a solution of the constrained martingale problem for \((\tilde{A}, E_0; \tilde{B}_1, \tilde{E}_1; \ldots; \tilde{B}_m, \tilde{E}_m)\) and showing that any such solution that starts in \( \overline{E}_0 \) stays in \( \overline{E}_0 \) for all times. Following [Kurtz (1990)], we will construct a solution of the constrained martingale problem for \((\tilde{A}, E_0; \tilde{B}_1, \tilde{E}_1; \ldots; \tilde{B}_m, \tilde{E}_m)\) from a solution of the corresponding patchwork martingale problem.

\( \tilde{A}, \tilde{B}_1, \ldots, \tilde{B}_m \) are dissipative operators by i) and Lemma 2.5. Then, by Lemma 1.1 in [Kurtz (1990)], for each initial distribution on \( E \), there exists a solution of the patchwork martingale problem for \((\tilde{A}, E_0; \tilde{B}_1, \tilde{E}_1; \ldots; \tilde{B}_m, \tilde{E}_m)\). In addition, if \( E \neq \overline{E}_0 \), by ii) and the same argument used in the proof of Lemma 1.4 in [Kurtz (1990)], for every solution \( Y \) of the patchwork martingale problem for \((\tilde{A}, E_0; \tilde{B}_1, \tilde{E}_1; \ldots; \tilde{B}_m, \tilde{E}_m)\) with initial distribution concentrated on \( \overline{E}_0 \), \( Y(t) \in \overline{E}_0 \) for all \( t \geq 0 \). Therefore \( Y \) is also a solution of the patchwork martingale problem for \((A, E_0; B_1, E_1; \ldots; B_m, E_m)\). By iii) and Lemma 1.8, Lemma 1.9, Proposition 2.2, and Proposition 2.3 in [Kurtz (1990)], from \( Y \), a solution \( X \) of the constrained martingale problem for \((A, E_0; B_1, E_1; \ldots; B_m, E_m)\) can be constructed.
Condition 5.6(c). If $X$ is a solution of the constrained martingale problem for $(A,E; B_1,E_1; \ldots; B_m,E_m)$ and $\gamma_1, \ldots, \gamma_m$ are associated local times, then $\eta_0(t) = \inf\{s: s + \gamma_1(s) + \ldots + \gamma_m(s) > t\}$ is strictly increasing and diverging to infinity as $t$ goes to infinity, with probability one, and $Y = X \circ \eta_0$ is a solution of the patchwork martingale problem for $(A,E; B_1,E_1; \ldots; B_m,E_m)$. Let $\eta_0, \eta_1 = \gamma_1 \circ \eta_0, \ldots, \eta_m = \gamma_m \circ \eta_0$ are associated increasing processes (see the proof of Corollary 2.5 of Kurtz (1990)). Let $\{(X^n, \eta^n_0, \eta^n_1, \ldots, \eta^n_m)\}$ be a sequence of solutions of the constrained martingale problem for $(A,E; B_1,E_1; \ldots; B_m,E_m)$ with initial conditions $\mu^n \in \mathcal{K}$, with associated local times. Since $\mathcal{K}$ is compact, we may assume, without loss of generality, that $\{\mu^n\}$ converges to $\mu \in \mathcal{K}$. Let $\{(Y^n, \eta^n_0, \eta^n_1, \ldots, \eta^n_m)\}$ be the sequence of the corresponding solutions of the patchwork martingale problem and associated increasing processes. Then by the density of $\mathcal{D}$ and Theorems 3.9.1 and 3.9.4 of Ethier and Kurtz (1986), $\{(Y^n, \eta^n_0, \eta^n_1, \ldots, \eta^n_m)\}$ is relatively compact taking the Skorohod topology on $D_{\mathbb{R}^{\mathbb{R}^{n+1}}[0,\infty)}$.

Let $\{(X^{n_k}, \eta^{n_k}_0, \eta^{n_k}_1, \ldots, \eta^{n_k}_m)\}$ be a subsequence converging to a limit $(Y, \eta_0, \eta_1, \ldots, \eta_m)$. Then $Y$ is a solution of the patchwork martingale problem for $(A,E; B_1,E_1; \ldots; B_m,E_m)$ with initial condition $\mu$ and $\eta_0, \eta_1, \ldots, \eta_m$ are associated increasing processes. By iii) and Lemma 1.8 and Lemma 1.9 in Kurtz (1990), $\eta_0$ is strictly increasing and diverging to infinity as $t$ goes to infinity, with probability one. It follows that $\{(\eta^n_0)^{-1}\}$ converges to $(\eta_0)^{-1}$ and hence $\{(X^{n_k}, \gamma^{n_k}_1, \ldots, \gamma^{n_k}_m)\} = \{(Y^n \circ (\eta^n_0)^{-1}, \eta^{n_k}_1 \circ (\eta^n_0)^{-1}, \ldots, \eta^{n_k}_m \circ (\eta^n_0)^{-1})\}$ converges to $(Y \circ (\eta_0)^{-1}, \eta_1 \circ (\eta_0)^{-1}, \ldots, \eta_m \circ (\eta_0)^{-1})$ and $Y \circ (\eta_0)^{-1}$ with associated local times $\eta_1 \circ (\eta_0)^{-1}, \ldots, \eta_m \circ (\eta_0)^{-1}$ is a solution of the constrained martingale problem for $(A,E; B_1,E_1; \ldots; B_m,E_m)$ with initial condition $\mu$.

Condition 5.6(d). Let $\varphi_1$ be the function of iii) for $n = 1$. By Lemma 5.4 and iii) we have

$$E\left[\sum_{k=1}^m \int_0^t e^{-\lambda s} \gamma_k(s)\right] \leq e^{-\lambda t} E[\varphi_1(X(t))] - E[\varphi_1(X(0))] + \int_0^t e^{-\lambda s} E[\lambda \varphi_1(X(s)) - A\varphi_1(X(s))] ds.$$

$\square$

6 Examples

Several examples of application of the results of the previous sections can be given by exploiting comparison principles proved in the literature. Here we will discuss in detail two examples.

The first example is a class of diffusion processes reflecting in a domain $D \subseteq \mathbb{R}^d$ according to an oblique direction of reflection which may become tangential. This case is not covered by the existing literature on reflecting diffusions, which assumes that the direction of reflection is uniformly bounded away from the tangent hyperplane.

The second example is a large class of jump diffusion processes with jump component of unbounded variation and possibly degenerate diffusion matrix. In this case uniqueness results are already available in the literature (see e.g. Ikeda and Watanabe (1989), Graham (1992), Kurtz and Protter (1996)) but we believe it is still a good benchmark to show how our method works.
6.1 Diffusions with degenerate oblique direction of reflection

Let $D \subseteq \mathbb{R}^d$, $d \geq 2$, be a bounded domain with $C^3$ boundary, i.e.

$$D = \{ x \in \mathbb{R}^d : \psi(x) > 0 \}, \quad \partial D = \{ x \in \mathbb{R}^d : \psi(x) = 0 \},$$

for some function $\psi \in C^3(\mathbb{R}^d)$, let $l : \overline{D} \to \mathbb{R}^d$ be a vector field in $C^2(\overline{D})$ such that

$$|l(x)| > 0 \text{ and } \langle l(x), \nu(x) \rangle \geq 0, \quad \forall x \in \partial D,$$

$\nu$ being the unit inward normal vector field, and let

$$\partial_0 D = \{ x \in \partial D : \langle l(x), \nu(x) \rangle = 0 \}.$$

We assume that $\partial_0 D$ has dimension $d - 2$. More precisely, for $d \geq 3$, we assume that $\partial_0 D$ has a finite number of connected components, each of the form

$$\{ x \in \partial D : \psi(x) = 0, \tilde{\psi}(x) = 0 \},$$

where $\psi$ is the function above and $\tilde{\psi}$ is another function in $C^2(\mathbb{R}^d)$ such that the level set $\{ x \in \partial D : \tilde{\psi}(x) = 0 \}$ is bounded and $|\nabla \tilde{\psi}(x)| > 0$ on it. For $d = 2$, we assume that $\partial_0 D$ consists of a finite number of points. In addition, we assume that $l(x)$ is never tangential to $\partial_0 D$.

Our goal is to prove uniqueness of the reflecting diffusion process with generator of the form

$$Af(x) = \frac{1}{2} tr \left( D^2 f(x) \sigma(x) \sigma(x)^T \right) + \nabla f(x) b(x),$$

where $\sigma$ and $b$ are Lipschitz continuous functions on $\overline{D}$, and direction of reflection $l$. We will characterize this reflecting diffusion process as the unique solution of the constrained martingale problem for $(A, B; \partial D)$, where $A$ is given by (6.4),

$$Bf(x) = \nabla f(x) l(x),$$

and the common domain of $A$ and $B$ is $D = C^2(\overline{D})$. Our tools will be the results of Section 5 and the comparison principle proved by Popivanov and Kutev (2005).

**Proposition 6.1** Condition [5.6] is verified.

**Proof.** Condition [5.6] is obviously verified. Therefore we only need to prove that the assumptions of Proposition 5.12 are satisfied. Let $0 < r < 1$ be small enough that for $d(x, \partial D) < \frac{4}{3}r$ the normal projection of $x$ on $\partial D$, $\pi_\nu(x)$, is well defined and $|\nabla \psi(x)| > 0$. Set $U(\overline{D}) = \{ x : d(x, \overline{D}) < r \}$. Let $\chi(c)$ be a nondecreasing function in $C^\infty(\mathbb{R})$ such that $0 \leq \chi(c) \leq 1$, $\chi(c) = 1$ for $c \geq \frac{2r}{3}$, $\chi(c) = 0$ for $c \leq \frac{r}{3}$. We can extend $l$ to a Lipschitz continuous vector field on $U(\overline{D})$ by setting, for $x \in U(\overline{D}) - \overline{D}$,

$$l(x) = (1 - \chi(d(x, \partial D))) l(\pi_\nu(x)).$$
We can also extend \( \sigma \) and \( b \) to Lipschitz continuous functions on \( \bar{U(D)} \) by setting, for \( x \in \bar{U(D)} - D \),
\[
\sigma(x) = (1 - \chi(d(x, \partial D))) \sigma(\pi_\nu(x)), \\
b(x) = (1 - \chi(d(x, \partial D))) b(\pi_\nu(x)).
\]

Clearly, both the martingale problem for \( A \), with domain \( C^2(\bar{U(D)}) \), and the martingale problem for \( B \), with the same domain, have a solution for every initial condition \( \delta_x, x \in \bar{U(D)} \). Since every \( f \in C^2(D) \) can be extended to a function \( \tilde{f} \in C^2(\bar{U(D)}) \) and
\[
Af = \left( A\tilde{f} \right)|_{\partial D}, \\
Bf = \left( B\tilde{f} \right)|_{\partial D},
\]
Condition (i) in Proposition 5.12 is verified.

Next, consider the function \( \varphi \) defined as
\[
\varphi(x) = \begin{cases} 
0, & \text{for } x \in \partial D, \\
\exp\left\{ \frac{-1}{d(x, \partial D)} \right\}, & \text{for } x \in \bar{U(D)} - D,
\end{cases}
\]
where \( U(D) \) is as above. Since \( \partial D \) is of class \( C^3 \), \( \varphi \in C^2(\bar{U(D)}) \). Moreover
\[
\nabla \varphi(x) = -|\nabla \varphi(x)| \nu(\pi_\nu(x)), \quad \text{for } x \in \bar{U(D)} - D.
\]
Therefore \( \varphi \) satisfies Condition (ii) in Proposition 5.12.

Finally, in order to verify (iii) of Proposition 5.12, we just need to modify slightly the proof of Lemma 3.1 in Popivanov and Kutev (2005). Suppose first that \( \partial_0 D \) is connected. Let \( \tilde{\psi} \) be the function in (6.3). Since \( l(x) \) is never tangent to \( \partial_0 D \), it must hold \( \nabla \tilde{\psi}(x) l(x) \neq 0 \) for each \( x \in \partial_0 D \), and hence, possibly replacing \( \psi \) by \( -\psi \), we can assume that
\[
\tilde{\psi}(x) = 0, \quad \nabla \tilde{\psi}(x) l(x) > 0, \quad \forall x \in \partial_0 D.
\]
Let \( U(\partial_0 D) \) be a neighborhood of \( \partial_0 D \) such that \( \inf_{\bar{U(\partial_0 D)}} \nabla \tilde{\psi}(x) l(x) > 0 \), and for each \( n \in \mathbb{N} \), set
\[
\partial_0^n D = \left\{ x \in \partial D \cap \bar{U(\partial_0 D)} : |\tilde{\psi}(x)| < \frac{1}{2^n} \right\}, \\
\tilde{C}_n = \frac{1}{\inf_{\partial_0^n D} \nabla \tilde{\psi}(x) l(x)^*}, \\
C_n = \frac{\tilde{C}_n \sup_{\partial_0^n D} |\nabla \tilde{\psi}(x) l(x)|^{1+1}}{\inf_{\partial_0^n D - \partial_0 D} \nabla \tilde{\psi}(x) l(x)^{1}}.
\]
Let \( \chi_n \) be a function in \( C_\infty(\mathbb{R}) \) such that \( \chi_n(c) = nc \) for \( |c| \leq \frac{1}{2n} \), \( \chi_n(c) = -1 \) for \( c \leq \frac{1}{n} \), \( \chi_n(c) = 1 \) for \( c \geq \frac{1}{n} \), \( 0 \leq \chi_n'(c) \leq n \) for every \( c \in \mathbb{R} \), and define
\[
\varphi_n(x) = \chi_n(C_n \tilde{\psi}(x)) + \tilde{C}_n \chi_n(\tilde{\psi}(x)).
\]
Then \( |\varphi_n(x)| \) is bounded by \( 1 + \frac{1}{\inf_{\partial_0 D} \nabla \tilde{\psi}(x) l(x)} \) and we have, for \( x \in \partial_0^n D \),
\[
\nabla \varphi_n(x) l(x) = n \left[ C_n \nabla \tilde{\psi}(x) l(x) + \tilde{C}_n \nabla \tilde{\psi}(x) l(x) \right] \geq n.
\]
and for $x \in \partial D - \partial_0^k D$, 
\[
\nabla \varphi_n(x)l(x) = nC_n\nabla \psi(x)l(x) + \tilde{C}_n\chi_n(\tilde{\psi}(x))\nabla \tilde{\psi}(x)l(x) \geq n.
\]

If $\partial D$ is not connected, there is a function $\tilde{\psi}^k$ satisfying (6.6) for each connected component $\partial_0^k D$. Let $U^k(\partial_0^k D)$ be neighborhoods such that $\inf_{U^k(\partial_0^k D)} \nabla \tilde{\psi}^k(x)l(x) > 0$. We can assume, without loss of generality, that $U^k(\partial_0^k D) \subseteq V^k(\partial_0^k D)$, where $V^k(\partial_0^k D)$ are pairwise disjoint and $\tilde{\psi}^k$ vanishes outside $V^k(\partial_0^k D)$. Then, defining $\partial_0^{k,n} D$ and $\tilde{C}_n^k$ as above,
\[
C_n^k = \frac{\tilde{C}_n^k \sup_{\partial D} |\nabla \tilde{\psi}^k(x)l(x)| + 1}{\inf_{\partial D - \cup \partial_0^{k,n} D} \nabla \psi(x)l(x)},
\]
and $\varphi_n^k$ as above, $\varphi_n(x) = \sum_k \varphi_n^k(x)$ verifies iii) of Proposition 5.12.

Theorem 2.6 of Popivanov and Kutev (2005) gives the comparison principle for a class of linear and nonlinear equations that includes, in particular, the partial differential equation with boundary conditions
\[
\begin{align*}
\lambda u(x) - Au(x) &= h(x), \quad \text{in } D, \\
-Bu(x) &= 0, \quad \text{on } \partial D,
\end{align*}
\]
where $h$ is a Lipschitz continuous function, and $A, B$ are given by (6.4), (6.5) and verify, in addition to the the assumptions formulated at the beginning of this section, the following local condition on $\partial D$.

**Condition 6.2** For every $x_0 \in \partial D$, let $\phi$ be a $C^2$ diffeomorphism from the closure of a suitable neighborhood $V$ of the origin into the closure of a suitable neighborhood of $x_0$, $U(x_0)$, such that $\phi(0) = x_0$ and the $d$th column of $J\phi(z)$, $J_d\phi(z)$, satisfies
\[
J_d\phi(z) = -l(\phi(z)), \quad \forall z \in \phi^{-1}\left(\partial D \cap U(x_0)\right).
\]

Let $\tilde{A}$,
\[
\tilde{A}f(z) = \frac{1}{2} \text{tr} \left(D^2 f(z)\tilde{\sigma}(z)\tilde{\sigma}^T(z)\right) + \nabla f(z)\tilde{b}(z),
\]
be the operator such that
\[
\tilde{A}(f \circ \phi)(z) = A f(\phi(z)), \quad \forall z \in \phi^{-1}\left(\partial D \cap U(x_0)\right).
\]

Assume
\begin{enumerate}
\item $\tilde{b}_i, \ i = 1, ..., d - 1$, is a function of the first $d - 1$ coordinates $(z^1, ..., z^{d-1})$ only, and $\tilde{b}_d$ is a function of $z_d$ only.
\item $\tilde{\sigma}^{ij}, \ i = 1, ..., d - 1, j = 1, ..., d$ is a function of the first $d - 1$ coordinates $(z^1, ..., z^{d-1})$ only.
\end{enumerate}
Remark 6.3 For every $x_0 \in \partial_0 D$, some coordinate of $l(x_0)$, say the $d$th coordinate, must be nonzero. Then in (6.8) we can choose $U(x_0)$ such that in $\overline{U(x_0)}$ $l^d(x) \neq 0$ and we can replace $l(x)$ by $l(x)/|l^d(x)|$, since this normalization does not change the boundary condition of (6.7) in $D \cap U(x_0)$ (i.e. any viscosity sub/supersolution of (6.7) in $D \cap U(x_0)$ is a viscosity sub/supersolution of (6.4) in $D \cap U(x_0)$ with the normalized vector field and conversely).

Moreover, since (6.8) must be verified only in $\phi^{-1}(\partial D \cap \overline{U(x_0)})$, in the construction of $\phi$ we can use any $C^2$ vector field $\tilde{l}$ that agrees with $l$, or the above normalization of $l$, on $\partial D \cap U(x_0)$.

Therefore, whenever Condition 6.2 is satisfied Theorem 5.10 applies and there exists one and only one diffusion process reflecting in $D$ according to the degenerate oblique direction of reflection $l$.

The following is a concrete example where Condition 6.2 is satisfied.

Example 6.4 Let

$$D = B_1(0) \subseteq \mathbb{R}^2.$$ 

and suppose the direction of reflection $l$ satisfies (6.1) with the strict inequality at every $x \in \partial D$ except at $x_0 = (1,0)$, where

$$l(1,0) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$ 

Of course, in a neighborhood of $x_0 = (1,0)$ we can always assume that $l$ depends only on the second coordinate $x_2$. In addition, by Remark 6.3 we can suppose

$$l_2(x) = -1.$$ 

Consider

$$\sigma(x) = \sigma = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

and a drift $b$ that, in a neighborhood of $x_0 = (1,0)$, depends only on the second coordinate, i.e.

$$b(x) = b(x_2).$$

Assume that, in a neighborhood of $x_0 = (1,0)$, the direction of reflection $l$ is parallel to $b$. Then we can find a nonlinear change of coordinates $\phi$ such that Condition 6.2 is verified, namely

$$\phi_2(z) = z_2$$

$$\phi_1(z) = -\int_0^{z_2} l_1(\zeta_2) d\zeta_2 + z_1 + 1,$$

which yields

$$\tilde{\sigma}(z) = \sigma, \quad \tilde{b}_1(z) = 0, \quad \tilde{b}_2(z) = b_2(z_2).$$
6.2 Jump diffusions with degenerate diffusion matrix

Consider the operator

\[ Af = Lf + Jf \]

\[ Lf(x) = \frac{1}{2} tr \left( a(x) D^2 f(x) \right) + b(x) \nabla f(x) \]  

(6.9)

\[ Jf(x) = \int_{\mathbb{R}^d - \{0\}} \left[ f(x + \eta(x, z)) - f(x) - \eta(x, z) \nabla f(x) I_{|z| < 1} \right] m(dz), \]

where \( b \) and \( \eta \) are viewed as row vectors. Assume:

**Condition 6.5**

a) \( a = \sigma \sigma^T \), \( \sigma \) and \( b \) are continuous.

b) \( \eta(\cdot, z) \) is continuous for every \( z \), \( \eta(x, \cdot) \) is Borel measurable for every \( x \), \( \sup_{|z| < 1} \eta(x, z) < +\infty \) and

\[ |\eta(x, z)| I_{|z| < 1} \leq \rho(z)(1 + |x|) I_{|z| < 1}, \]

for some positive, measurable function \( \rho \) such that \( \lim_{|z| \to 0} \rho(z) = 0 \).

c) \( m \) is a Borel measure such that

\[ \int_{\mathbb{R}^d - \{0\}} \left[ \rho(z)^2 I_{|z| < 1} + I_{|z| \geq 1} \right] m(dz) < +\infty. \]

Then, with \( D(A) = \{ f + c : f \in C^2_c(\mathbb{R}^d), c \in \mathbb{R} \} \), \( A \subset C_b(E) \times C_b(E) \) and \( A \) satisfies Condition 3.1.

A comparison principle for bounded subsolutions and supersolutions of the equation (2.3) when \( A \) is given by (6.9) is proven in Jakobsen and Karlsen (2006), as a special case of a more general result, under the following assumptions:

**Condition 6.6**

a) \( \sigma \) and \( b \) are Lipschitz continuous.

b) \( |\eta(x, z) - \eta(y, z)| I_{|z| < 1} \leq \rho(z)|x - y| I_{|z| < 1}. \)

c) \( h \) is uniformly continuous.

Then, under the above assumptions, our result of Theorem 3.10 applies and uniqueness of the solution of the martingale problem for \( A \) is granted.
References


