

Viscosity methods giving uniqueness for martingale problems*

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Abstract

Let E be a complete, separable metric space and A be an operator on $C_b(E)$. We give an abstract definition of viscosity sub/supersolution of the resolvent equation $\lambda u - Au = h$ and show that, if the comparison principle holds, then the martingale problem for A has a unique solution. Our proofs work also under two alternative definitions of viscosity sub/supersolution which might be useful, in particular, in infinite dimensional spaces, for instance to study measure-valued processes.

We prove the analogous result for stochastic processes that must satisfy boundary conditions, modeled as solutions of constrained martingale problems. In the case of reflecting diffusions in $D \subset \mathbf{R}^d$, our assumptions allow D to be nonsmooth and the direction of reflection to be degenerate.

Two examples are presented: A diffusion with degenerate oblique direction of reflection and a class of jump diffusion processes with infinite variation jump component and possibly degenerate diffusion matrix.

Keywords: martingale problem; uniqueness; metric space; viscosity solution; boundary conditions; constrained martingale problem.

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1 Introduction

There are many ways of specifying Markov processes, the most popular being as solutions of stochastic equations, as solutions of martingale problems, or in terms of solutions of the Kolmogorov forward equation (the Fokker-Planck equation or the master equation depending on context). The solution of a stochastic equation explicitly gives a process while a solution of a martingale problem gives the distribution of a process and a solution of a forward equation gives the one dimensional distributions of a process.

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Typically, these approaches are equivalent (assuming that there is a stochastic equation formulation) in the sense that existence of a solution specified by one method implies existence of corresponding solutions to the other two (weak existence for the stochastic equation) and hence uniqueness for one method implies uniqueness for the other two (distributional uniqueness for the stochastic equation).

One approach to proving uniqueness for a forward equation and hence for the corresponding martingale problem is to verify a condition on the generator similar to the range condition of the Hille-Yosida theorem. (See Corollary 2.14.) We show that the original generator A of our martingale problem (or a restriction of the original generator A , in the case of martingale problems with boundary conditions) can be extended to a generator \widehat{A} such that every solution of the martingale problem for A is a solution for \widehat{A} and \widehat{A} satisfies the range condition of Corollary 2.14. Our extension is constructed by including viscosity solutions of the resolvent equation

$$\lambda u - Au = h. \quad (1.1)$$

Viscosity solutions have been used to study value functions in stochastic optimal control and optimal stopping theory since the very beginning (see the classical references [6], [24], as well as [12]). It may be interesting to note that, in the context of Hamilton-Jacobi equations, the idea of studying a parabolic equation by solving a resolvent equation in the viscosity sense appears already in [7], Section VI.3, where it is applied to a model problem. The methodology is also important for related problems in finance (for example [26], [3], [18], [1], [5] and many others).

Viscosity solutions have also been used to study the partial differential equations associated with forward-backward stochastic differential equations ([23],[9]) and in the theory of large deviations ([11]).

The basic data for our work is an operator $A \subset C_b(E) \times C_b(E)$ on a complete, separable metric space E . We offer an abstract definition of viscosity sub/supersolution for (1.1) (which for integro-differential operators in \mathbf{R}^d is equivalent to the usual one) and prove, under very general conditions, that the martingale problem for A has a unique solution if the comparison principle for (1.1) holds.

We believe the interest of this result is twofold: on one hand it clarifies the general connection between viscosity solutions and martingale problems; on the other, there are still many martingale problems, for instance in infinite dimension, for which uniqueness is an open question.

We also discuss two alternative abstract definitions of viscosity sub/supersolution that might be especially useful in infinite dimensional spaces. All our proofs work under these alternative definitions as well.

The first alternative definition is a modification of a definition suggested to us by Nizar Touzi and used in [9]. Being a stronger definition (it allows for more test functions), it should be easier to prove comparison results under this definition.

The second alternative definition appears in [11] and is a stronger definition too. Under this definition, a sort of converse of our main result holds, namely if h belongs to $\overline{\mathcal{R}(\lambda - A)}$ (under uniform convergence), then the comparison principle for semisolutions of (1.1) holds (Theorem 4.8). When E is compact, this definition is equivalent to our main definition, hence the comparison principle holds for semisolutions in that sense as well.

Next we consider stochastic processes that must satisfy some boundary conditions, for example, reflecting diffusions. Boundary conditions are expressed in terms of an operator B which enters into the formulation of a *constrained martingale problem* (see [19]). We restrict our attention to models in which the boundary term in the constrained martingale problem is expressed as an integral against a local time. Then it still holds that uniqueness of the solution of the constrained martingale problem follows from

the comparison principle between viscosity sub and supersolutions of (1.1) with the appropriate boundary conditions. Notice that, as for the standard martingale problem, uniqueness for the constrained martingale problem implies that the solution is Markovian (see [19], Proposition 2.6).

In the presence of boundary conditions, even for \mathbf{R}^d -valued diffusions, there are examples for which uniqueness of the martingale problem is not known. Processes in domains with boundaries that are only piecewise smooth or with boundary operators that are second order or with directions of reflection that are tangential on some part of the boundary continue to be a challenge. In this last case, as an example of an application of our results, we use the comparison principle proved in [25] to obtain uniqueness.

The strategy of our proofs has been initially inspired by the proof of Krylov's selection theorem for martingale problems that appears in [10] and originally appeared in unpublished work of [14]. In that proof the generator is recursively extended in such a way that there are always solutions of the martingale problem for the extended generator, but eventually only one. If uniqueness fails for the original martingale problem, there is more than one way to do the extension. Conversely if, at each stage of the recursion, there is only one way to do the extension and all solutions of the martingale problem for the original generator remain solutions for the extended generator, then uniqueness must hold for the original generator.

Analogously, assuming the comparison principle for (1.1) (or (1.1) with the appropriate boundary conditions) holds for a large enough class of functions h , we construct an extension \hat{A} of the original operator A (of a restriction of the original operator A , in the case of constrained martingale problems) such that all solutions of the martingale problem (the constrained martingale problem) for A are solutions of the martingale problem for \hat{A} , and such that uniqueness holds for \hat{A} . Actually, in the case of ordinary martingale problems the extension, although possible, is not needed, because the comparison principle for (1.1) directly yields a condition ((2.9)) that, if valid for a large enough class of functions h , ensures uniqueness of the one-dimensional distributions of solutions to the martingale problem, and hence uniqueness of the solution. The extension is needed, instead, for constrained martingale problems.

A few works on viscosity solutions of partial differential equations and weak solutions of stochastic differential equations have appeared in recent years. For diffusions in \mathbf{R}^d , [2], assuming a comparison principle exists, show that the backward equation has a unique viscosity solution, and it follows that the corresponding stochastic differential equation has a unique weak solution. For Markovian forward-backward stochastic differential equations, [23] also derive uniqueness of the weak solution from existence of a comparison principle for the corresponding partial differential equation. In the non-Markovian case, the associated partial differential equation becomes path dependent. [9] propose the notion of viscosity solution of (semilinear) path dependent partial differential equations on the space of continuous paths already mentioned above and prove a comparison principle.

The rest of this paper is organized as follows: Section 2 contains some background material on martingale problems and on viscosity solutions; Section 3 deals with martingale problems; the alternative definitions of viscosity solution are discussed in Section 4; Section 5 deals with martingale problems with boundary conditions; finally in Section 6, we present two examples, including the application to diffusions with degenerate direction of reflection.

2 Background

2.1 Martingale problems

Throughout this paper we will assume that (E, r) is a complete separable metric space, $D_E[0, \infty)$ is the space of cadlag, E -valued functions endowed with the Skorohod topology, $\mathcal{B}(D_E[0, \infty))$ is the σ -algebra of Borel sets of $D_E[0, \infty)$, $C_b(E)$ denotes the space of bounded, continuous functions on (E, r) , $B(E)$ denotes the space of bounded, measurable functions on (E, r) , and $\mathcal{P}(E)$ denotes the space of probability measures on (E, r) . $\|\cdot\|$ will denote the supremum norm on $C_b(E)$ or $B(E)$.

Definition 2.1. A measurable stochastic process X defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a solution of the martingale problem for

$$A \subset B(E) \times B(E),$$

provided there exists a filtration $\{\mathcal{F}_t\}$ such that X and $\int_0^t g(X(s))ds$ are $\{\mathcal{F}_t\}$ -adapted, for every $g \in B(E)$, and

$$M_f(t) = f(X(t)) - f(X(0)) - \int_0^t g(X(s))ds \tag{2.1}$$

is a $\{\mathcal{F}_t\}$ -martingale for each $(f, g) \in A$. If $X(0)$ has distribution μ , we say X is a solution of the martingale problem for (A, μ) .

Remark 2.2. Because linear combinations of martingales are martingales, without loss of generality, we can, but need not, assume that A is linear and that $(1, 0) \in A$.

We do not, and cannot for our purposes, require A to be single valued. In particular, the operator \hat{A} defined in the proof of Theorem 5.11 will typically not be single valued.

In the next sections we will restrict our attention to processes X with sample paths in $D_E[0, \infty)$.

Definition 2.3. A stochastic process X , defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with sample paths in $D_E[0, \infty)$ is a solution of the martingale problem for

$$A \subset B(E) \times B(E),$$

in $D_E[0, \infty)$ provided there exists a filtration $\{\mathcal{F}_t\}$ such that X is $\{\mathcal{F}_t\}$ -adapted and

$$M_f(t) = f(X(t)) - f(X(0)) - \int_0^t g(X(s))ds \tag{2.2}$$

is a $\{\mathcal{F}_t\}$ -martingale for each $(f, g) \in A$. If $X(0)$ has distribution μ , we say X is a solution of the martingale problem for (A, μ) in $D_E[0, \infty)$.

Remark 2.4. Since X has sample paths in $D_E[0, \infty)$, the fact that it is $\{\mathcal{F}_t\}$ -adapted implies that $\int_0^t g(X(s))ds$ is $\{\mathcal{F}_t\}$ -adapted, for every $g \in B(E)$.

Remark 2.5. The requirement that X have sample paths in $D_E[0, \infty)$ is usually fulfilled provided the state space E is selected appropriately. Moreover, if $A \subset C_b(E) \times C_b(E)$, $\mathcal{D}(A)$ is dense in $C_b(E)$ in the topology of uniform convergence on compact sets and for each compact $K \subset E$, $\varepsilon > 0$, and $T > 0$, there exists a compact $K' \subset E$ such that

$$P\{X(t) \in K', t \leq T, X(0) \in K\} \geq (1 - \varepsilon)P\{X(0) \in K\},$$

for every progressive process such that (2.1) is a martingale, then every such process has a modification with sample paths in $D_E[0, \infty)$ (See Theorem 4.3.6 of [10].)

Remark 2.6. The martingale property is equivalent to the requirement that

$$E[(f(X(t+r)) - f(X(t)) - \int_t^{t+r} g(X(s))ds) \prod_i h_i(X(t_i))] = 0$$

for all choices of $(f, g) \in A$, $h_i \in B(E)$, $t, r \geq 0$, and $0 \leq t_i \leq t$. Consequently, the property of being a solution of a martingale problem is a property of the finite-dimensional distributions of X .

In particular, for the martingale problem in $D_E[0, \infty)$, the property of being a solution is a property of the distribution of X on $D_E[0, \infty)$. Much of what follows in the next sections will be formulated in terms of the collection $\Pi \subset \mathcal{P}(D_E[0, \infty))$ of distributions of solutions of the martingale problem. For some purposes, it will be convenient to assume that X is the canonical process defined on $(\Omega, \mathcal{F}, \mathbb{P}) = (D_E[0, \infty), \mathcal{B}(D_E[0, \infty))), P$, for some $P \in \Pi$.

In view of Remark 2.6 it is clear that uniqueness of the solution to the martingale problem for an operator A is to be meant as uniqueness of the finite-dimensional distributions.

Definition 2.7. We say that uniqueness holds for the martingale problem for A if, for every μ , any two solutions of the martingale problem for (A, μ) have the same finite-dimensional distributions. If we restrict our attention to solutions in $D_E[0, \infty)$, then uniqueness holds if any two solutions for (A, μ) have the same distribution on $D_E[0, \infty)$.

One of the most important consequences of the martingale approach to Markov processes is that uniqueness of one-dimensional distributions implies uniqueness of finite-dimensional distributions.

Theorem 2.8. Suppose that for each $\mu \in \mathcal{P}(E)$, any two solutions of the martingale problem for (A, μ) have the same one-dimensional distributions. Then any two solutions have the same finite-dimensional distributions. If any two solutions of the martingale problem for (A, μ) in $D_E[0, \infty)$ have the same one-dimensional distributions, then they have the same distribution on $D_E[0, \infty)$.

Proof. This is a classical result. See for instance Theorem 4.4.2 and Corollary 4.4.3 of [10]. □

For $\mu \in \mathcal{P}(E)$ and $f \in B(E)$ we will use the notation

$$\mu f = \int_E f(x)\mu(dx). \tag{2.3}$$

Lemma 2.9. Let X be a $\{\mathcal{F}_t\}$ -adapted stochastic process with sample paths in $D_E[0, \infty)$, with initial distribution μ , $f, g \in B(E)$ and $\lambda > 0$. Then (2.2) is a $\{\mathcal{F}_t\}$ -martingale if and only if

$$M_f^\lambda(t) = e^{-\lambda t} f(X(t)) - f(X(0)) + \int_0^t e^{-\lambda s} (\lambda f(X(s)) - g(X(s))) ds \tag{2.4}$$

is a $\{\mathcal{F}_t\}$ -martingale. In particular, if (2.2) is a $\{\mathcal{F}_t\}$ -martingale

$$\mu f = E[\int_0^\infty e^{-\lambda s} (\lambda f(X(s)) - g(X(s))) ds]. \tag{2.5}$$

Proof. The general statement is a special case of Lemma 4.3.2 of [10]. If f is continuous, as will typically be the case in the next sections, M_f will be cadlag and we can apply Itô's formula to obtain

$$e^{-\lambda t} f(X(t)) - f(X(0)) = \int_0^t (-f(X(s))\lambda e^{-\lambda s} + e^{-\lambda s} g(X(s))) ds + \int_0^t e^{-\lambda s} dM_f(s),$$

where the last term on right is a $\{\mathcal{F}_t\}$ -martingale. (Note that, since all the processes involved are cadlag, we do not need to require the filtration $\{\mathcal{F}_t\}$ to satisfy the ‘usual conditions’.) Conversely, if (2.4) is a $\{\mathcal{F}_t\}$ -martingale, the assertion follows by applying Itô’s formula to $f(X(t)) = e^{\lambda t} \left(e^{-\lambda t} f(X(t)) \right)$.

In particular, if (2.2) is a $\{\mathcal{F}_t\}$ -martingale

$$E[f(X(0)) - E[e^{-\lambda t} f(X(t))]] = E\left[\int_0^t e^{-\lambda s} (\lambda f(X(s)) - g(X(s))) ds\right],$$

and the second statement follows by taking $t \rightarrow \infty$. □

Lemma 2.10. *Let X be a solution of the martingale problem for $A \subset C_b(E) \times C_b(E)$ in $D_E[0, \infty)$ with respect to a filtration $\{\mathcal{F}_t\}$. Let $\tau \geq 0$ be a finite $\{\mathcal{F}_t\}$ -stopping time and $H \geq 0$ be a \mathcal{F}_τ -measurable random variable such that $0 < E[H] < \infty$. Define $P^{\tau, H}$ by*

$$P^{\tau, H}(C) = \frac{E[H \mathbf{1}_C(X(\tau + \cdot))]}{E[H]}, \quad C \in \mathcal{B}(D_E[0, \infty)). \tag{2.6}$$

Then $P^{\tau, H}$ is the distribution of a solution of the martingale problem for A in $D_E[0, \infty)$.

Proof. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space on which X is defined, and define \mathbb{P}^H on (Ω, \mathcal{F}) by

$$\mathbb{P}^H(C) = \frac{E^{\mathbb{P}}[H \mathbf{1}_C]}{E^{\mathbb{P}}[H]}, \quad C \in \mathcal{F}.$$

Define X^τ by $X^\tau(t) = X(\tau + t)$. X^τ is adapted to the filtration $\{\mathcal{F}_{\tau+t}\}$ and for $0 \leq t_1 < \dots < t_n < t_{n+1}$ and $f_1, \dots, f_n \in B(E)$,

$$\begin{aligned} & E^{\mathbb{P}^H} \left[\left\{ f(X^\tau(t_{n+1})) - f(X^\tau(t_n)) - \int_{t_n}^{t_{n+1}} Af(X^\tau(s)) ds \right\} \prod_{i=1}^n f_i(X^\tau(t_i)) \right] \\ &= \frac{1}{E^{\mathbb{P}}[H]} E^{\mathbb{P}} \left[H \left\{ f(X(\tau + t_{n+1})) - f(X(\tau + t_n)) \right. \right. \\ &\quad \left. \left. - \int_{\tau+t_n}^{\tau+t_{n+1}} Af(X(s)) ds \right\} \prod_{i=1}^n f_i(X(\tau + t_i)) \right] \\ &= 0 \end{aligned}$$

by the optional sampling theorem. Therefore, under \mathbb{P}^H , X^τ is a solution of the martingale problem. $P^{\tau, H}$, given by (2.6), is the distribution of X^τ on $D_E[0, \infty)$. □

Lemma 2.11. *Let $\lambda > 0$. Suppose $u, h \in B(E)$ satisfy*

$$\mu u = E\left[\int_0^\infty e^{-\lambda t} h(X(t)) dt\right], \tag{2.7}$$

for every solution of the martingale problem for A in $D_E[0, \infty)$ with initial distribution μ and for every $\mu \in \mathcal{P}(E)$. Then

$$u(X(t)) - \int_0^t (\lambda u(X(s)) - h(X(s))) ds \tag{2.8}$$

is a $\{\mathcal{F}_t^X\}$ -martingale for every solution X of the martingale problem for A in $D_E[0, \infty)$.

Proof. The lemma is Lemma 4.5.18 of [10], but we repeat the proof here for the convenience of the reader.

Let X be a solution of the martingale problem for A on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For $t \geq 0$ and $B \in \mathcal{F}_t$ with $\mathbb{P}(B) > 0$, define \mathbb{Q} on (Ω, \mathcal{F}) by

$$\mathbb{Q}(C) = \frac{E^{\mathbb{P}}[\mathbf{1}_B \mathbf{1}_C]}{\mathbb{P}(B)}, \quad C \in \mathcal{F}.$$

Then

$$\begin{aligned} E^{\mathbb{P}}[\mathbf{1}_B e^{\lambda t} \int_t^\infty e^{-\lambda s} h(X(s)) ds] &= E^{\mathbb{P}}[\mathbf{1}_B \int_0^\infty e^{-\lambda s} h(X(t+s)) ds] \\ &= \mathbb{P}(B) E^{\mathbb{Q}}[\int_0^\infty e^{-\lambda s} h(X(t+s)) ds]. \end{aligned}$$

By the same arguments as in the proof of Lemma 2.10, $X(t + \cdot)$ is a solution of the martingale problem for A on the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ with respect to the filtration $\{\mathcal{F}_{t+\cdot}\}$, with initial distribution $\nu(C) = \mathbb{Q}(X(t) \in C) = \mathbb{P}(X(t) \in C | B)$. Hence, by the assumptions of the lemma,

$$E^{\mathbb{P}}[\mathbf{1}_B e^{\lambda t} \int_t^\infty e^{-\lambda s} h(X(s)) ds] = \mathbb{P}(B) \nu u = E^{\mathbb{P}}[\mathbf{1}_B u(X(t))].$$

Therefore

$$E[\int_0^\infty e^{-\lambda s} h(X(s)) ds | \mathcal{F}_t] = e^{-\lambda t} u(X(t)) + \int_0^t e^{-\lambda s} h(X(s)) ds,$$

and the assertion follows from Lemma 2.9 with $f = u$ and $g = \lambda u - h$. □

Our approach to proving uniqueness of the solution of the martingale problem for (A, μ) relies on the following theorem.

Definition 2.12. A class of functions $M \subset C_b(E)$ is separating if, for $\mu, \nu \in \mathcal{P}(E)$, $\mu f = \nu f$ for all $f \in M$ implies $\mu = \nu$.

Theorem 2.13. For each $\lambda > 0$, suppose that

$$E[\int_0^\infty e^{-\lambda s} h(X(s)) ds] = E[\int_0^\infty e^{-\lambda s} h(Y(s)) ds], \tag{2.9}$$

for any two solutions of the martingale problem for A in $D_E[0, \infty)$ with the same initial distribution and for all h in a separating class of functions M_λ . Then any two solutions of the martingale problem for A in $D_E[0, \infty)$ with the same initial distribution have the same distribution on $D_E[0, \infty)$.

Proof. The proof is actually implicit in the proof of Corollary 4.4.4 of [10], but we give it here for clarity. For any two solutions of the martingale problem for A in $D_E[0, \infty)$ with the same initial distribution, we have, for all $h \in M_\lambda$,

$$\int_0^\infty e^{-\lambda s} E[h(X(s))] ds = \int_0^\infty e^{-\lambda s} E[h(Y(s))] ds. \tag{2.10}$$

Consider the measures on $(E, \mathcal{B}(E))$ defined by

$$m^X(C) = \int_0^\infty e^{-\lambda s} E[1_C(X(s))] ds, \quad m^Y(C) = \int_0^\infty e^{-\lambda s} E[1_C(Y(s))] ds.$$

Then (2.10) reads

$$m^X h = m^Y h,$$

and, as M_λ is separating, (2.10) holds for all $h \in B(E)$. Since the identity holds for each $\lambda > 0$, by uniqueness of the Laplace transform,

$$E[h(X(s))] = E[h(Y(s))], \quad \text{for almost every } s \geq 0,$$

and by the right continuity of X and Y , for all $s \geq 0$. Consequently, X and Y have the same one-dimensional distributions, and hence by Theorem 2.8, the same finite-dimensional distributions and the same distribution on $D_E[0, \infty)$. \square

Corollary 2.14. *Suppose that for each $\lambda > 0$, $\mathcal{R}(\lambda - A)$ is separating. Then for each initial distribution $\mu \in \mathcal{P}(E)$, any two cadlag solutions of the martingale problem for (A, μ) have the same distribution on $D_E[0, \infty)$.*

Proof. The assertion follows immediately from Lemma 2.11 and Theorem 2.13. \square

Martingale problems and dissipative operators are closely related.

Definition 2.15. *A linear operator $A \subset B(E) \times B(E)$ is dissipative provided*

$$\|\lambda f - g\| \geq \lambda \|f\|,$$

for each $(f, g) \in A$ and each $\lambda > 0$.

Lemma 2.16. *Suppose that for each $x \in E$, there exists a solution of the martingale problem for (A, δ_x) . Then A is dissipative.*

Proof. By Lemma 2.9,

$$|f(x)| \leq E\left[\int_0^\infty e^{-\lambda s} |\lambda f(X(s)) - g(X(s))| ds\right] \leq \frac{1}{\lambda} \|\lambda f - g\|.$$

\square

2.2 Viscosity solutions

Let $A \subset C_b(E) \times C_b(E)$. Theorem 2.14 implies that if for each $\lambda > 0$, the equation

$$\lambda u - g = h \tag{2.11}$$

has a solution $(u, g) \in A$ for every h in a class of functions $M_\lambda \subseteq C_b(E)$ that is separating, then for each initial distribution μ , the martingale problem for (A, μ) has at most one solution. Unfortunately in many situations it is hard to verify that (2.11) has a solution in A . Thus one is lead to consider a weaker notion of solution, namely the notion of *viscosity solution*.

Definition 2.17. (viscosity (semi)solution)

Let A be as above, $\lambda > 0$, and $h \in C_b(E)$.

- a) $u \in B(E)$ is a viscosity subsolution of (2.11) if and only if u is upper semicontinuous and if $(f, g) \in A$ and $x_0 \in E$ satisfy

$$\sup_x (u - f)(x) = (u - f)(x_0), \tag{2.12}$$

then

$$\lambda u(x_0) - g(x_0) \leq h(x_0). \tag{2.13}$$

b) $u \in B(E)$ is a viscosity supersolution of (2.11) if and only if u is lower semicontinuous and if $(f, g) \in A$ and $x_0 \in E$ satisfy

$$\inf_x (u - f)(x) = (u - f)(x_0), \tag{2.14}$$

then

$$\lambda u(x_0) - g(x_0) \geq h(x_0). \tag{2.15}$$

A function $u \in C_b(E)$ is a viscosity solution of (2.11) if it is both a subsolution and a supersolution.

In the theory of viscosity solutions, usually existence of a viscosity solution follows by existence of a viscosity subsolution and a viscosity supersolution, together with the following *comparison principle*.

Definition 2.18. The comparison principle holds for (2.11) when every subsolution is pointwise less than or equal to every supersolution.

Remark 2.19. To better motivate the notion of viscosity solution in the context of martingale problems, assume that there exists a solution of the martingale problem for (A, δ_x) for each $x \in E$. Suppose that there exists $v \in C_b(E)$ such that

$$e^{-\lambda t} v(X(t)) + \int_0^t e^{-\lambda s} h(X(s)) ds \tag{2.16}$$

is a $\{\mathcal{F}_t^X\}$ -martingale for every solution X of the martingale problem for A . Let $(f, g) \in A$ and x_0 satisfy

$$\sup_x (v - f)(x) = (v - f)(x_0).$$

Let X be a solution of the martingale problem for (A, δ_{x_0}) . Then

$$e^{-\lambda t} (v(X(t)) - f(X(t))) + \int_0^t e^{-\lambda s} (h(X(s)) - \lambda f(X(s)) + g(X(s))) ds$$

is a $\{\mathcal{F}_t^X\}$ -martingale by Lemma 2.9, and

$$\begin{aligned} & E\left[\int_0^t e^{-\lambda s} (\lambda v(X(s)) - g(X(s)) - h(X(s))) ds\right] \\ &= E\left[\int_0^t e^{-\lambda s} \lambda (v(X(s)) - f(X(s))) ds\right] \\ &\quad + E[e^{-\lambda t} (v(X(t)) - f(X(t))) - (v(x_0) - f(x_0))] \\ &\leq 0. \end{aligned}$$

Dividing by t and letting $t \rightarrow 0$, we see that

$$\lambda v(x_0) - g(x_0) \leq h(x_0),$$

so v is a subsolution for (2.11). A similar argument shows that it is also a supersolution and hence a viscosity solution. We will give conditions such that if the comparison principle holds for some h , then a viscosity solution v exists and (2.16) is a martingale for every solution of the martingale problem for A .

In the case of a domain with boundary, in order to uniquely determine the solution of the martingale problem for A one usually must specify some boundary conditions, by means of boundary operators $B_1 \dots, B_m$.

Let $E_0 \subseteq E$ be an open set and let

$$\partial E_0 = \cup_{k=1}^m E_k,$$

for disjoint, nonempty Borel sets E_1, \dots, E_m .

Let $A \subseteq C_b(\bar{E}_0) \times C_b(\bar{E}_0)$, $B_k \subseteq C_b(\bar{E}_0) \times C(\bar{E}_0)$, $k = 1, \dots, m$, be linear operators with a common domain \mathcal{D} . For simplicity we will assume that E is compact (hence the subscript b will be dropped) and that A, B_1, \dots, B_m are single valued.

Definition 2.20. Let A, B_1, \dots, B_m be as above, and let $\lambda > 0$. For $h \in C_b(\bar{E}_0)$, consider the equation

$$\begin{aligned} \lambda u - Au &= h, & \text{on } E_0 \\ -B_k u &= 0, & \text{on } E_k, \quad k = 1, \dots, m. \end{aligned} \tag{2.17}$$

a) $u \in B(\bar{E}_0)$ is a viscosity subsolution of (2.17) if and only if u is upper semicontinuous and if $f \in \mathcal{D}$ and $x_0 \in \bar{E}_0$ satisfy

$$\sup_x (u - f)(x) = (u - f)(x_0), \tag{2.18}$$

then

$$\lambda u(x_0) - Af(x_0) \leq h(x_0), \quad \text{if } x_0 \in E_0, \tag{2.19}$$

$$(\lambda u(x_0) - Af(x_0) - h(x_0)) \wedge \min_{k: x_0 \in \bar{E}_k} (-B_k f(x_0)) \leq 0, \quad \text{if } x_0 \in \partial E_0 \tag{2.20}$$

b) $u \in B(\bar{E}_0)$ is a viscosity supersolution of (2.17) if and only if u is lower semicontinuous and if $f \in \mathcal{D}$ and $x_0 \in \bar{E}_0$ satisfy

$$\inf_x (u - f)(x) = (u - f)(x_0), \tag{2.21}$$

then

$$\lambda u(x_0) - Af(x_0) \geq h(x_0), \quad \text{if } x_0 \in E_0, \tag{2.22}$$

$$(\lambda u(x_0) - Af(x_0) - h(x_0)) \vee \max_{k: x_0 \in \bar{E}_k} (-B_k f(x_0)) \geq 0, \quad \text{if } x_0 \in \partial E_0.$$

A function $u \in C(\bar{E}_0)$ is a viscosity solution of (2.17) if it is both a subsolution and a supersolution.

Remark 2.21. The above definition, with the ‘relaxed’ requirement that on the boundary either the interior inequality or the boundary inequality be satisfied by at least one among $-B_1 f, \dots, -B_m f$ is the standard one in the theory of viscosity solutions where it is used in particular because it is stable under limit operations and because it can be localized. As will be clear in Section 5, it suits perfectly our approach to martingale problems with boundary conditions.

3 Comparison principle and uniqueness for martingale problems

In this section, we restrict our attention to

$$A \subset C_b(E) \times C_b(E)$$

and consider the martingale problem for A in $D_E[0, \infty)$. Let $\Pi \subset \mathcal{P}(D_E[0, \infty))$ denote the collection of distributions of solutions of the martingale problem for A in $D_E[0, \infty)$, and, for $\mu \in \mathcal{P}(E)$, let $\Pi_\mu \subset \Pi$ denote the subcollection with initial distribution μ . If $\mu = \delta_x$, we will write Π_x for Π_{δ_x} . In this section X will be the canonical process on $D_E[0, \infty)$. Assume the following condition.

Condition 3.1.

- a) $\mathcal{D}(A)$ is dense in $C_b(E)$ in the topology of uniform convergence on compact sets.
- b) For each $\mu \in \mathcal{P}(E)$, $\Pi_\mu \neq \emptyset$.
- c) If $\mathcal{K} \subset \mathcal{P}(E)$ is compact, then $\cup_{\mu \in \mathcal{K}} \Pi_\mu$ is compact. (See Proposition 3.3 below.)

Remark 3.2. In working with these conditions, it is simplest to take the usual Skorohod topology on $D_E[0, \infty)$. (See, for example, Sections 3.5-3.9 of [10].) The results of this paper also hold if we take the Jakubowski topology ([17]). The σ -algebra of Borel sets $\mathcal{B}(D_E[0, \infty))$ is the same for both topologies and, in fact, is simply the smallest σ -algebra under which all mappings of the form $x \in D_E[0, \infty) \rightarrow x(t), t \geq 0$, are measurable.

It is also relevant to note that mappings of the form

$$x \in D_E[0, \infty) \rightarrow \int_0^\infty e^{-\lambda t} h(x(t)) dt, \quad h \in C_b(E), \lambda > 0,$$

are continuous under both topologies. The Jakubowski topology could be particularly useful for extensions of the results of Section 5 to constrained martingale problems in which the boundary terms are not local-time integrals.

Proposition 3.3. *In addition to Condition 3.1(a), assume that for each compact $K \subset E$, $\varepsilon > 0$, and $T > 0$, there exists a compact $K' \subset E$ such that*

$$P\{X(t) \in K', t \leq T, X(0) \in K\} \geq (1 - \varepsilon)P\{X(0) \in K\}, \quad \forall P \in \Pi.$$

Then Condition 3.1(c) holds.

Proof. The assertion is part of the thesis of Theorem 4.5.11 (b) of [10]. □

Let $\lambda > 0$, and for $h \in C_b(E)$, define

$$u_+(x) = u_+(x, h) = \sup_{P \in \Pi_x} E^P \left[\int_0^\infty e^{-\lambda t} h(X(t)) dt \right], \tag{3.1}$$

$$u_-(x) = u_-(x, h) = \inf_{P \in \Pi_x} E^P \left[\int_0^\infty e^{-\lambda t} h(X(t)) dt \right]. \tag{3.2}$$

$$\pi_+(\Pi_\mu, h) = \sup_{P \in \Pi_\mu} E^P \left[\int_0^\infty e^{-\lambda t} h(X(t)) dt \right], \tag{3.3}$$

$$\pi_-(\Pi_\mu, h) = \inf_{P \in \Pi_\mu} E^P \left[\int_0^\infty e^{-\lambda t} h(X(t)) dt \right]. \tag{3.4}$$

Lemma 3.4. *Under Condition 3.1, for $h \in C_b(E)$, $u_+(x, h)$ is upper semicontinuous (hence measurable), and*

$$\pi_+(\Pi_\mu, h) = \int_E u_+(x, h) \mu(dx) \quad \forall \mu \in \mathcal{P}(E). \tag{3.5}$$

The analogous result holds for u_- and π_- .

Proof. For π_+ , the lemma is a combination of Theorem 4.5.11(a), Lemma 4.5.8, Lemma 4.5.9 and Lemma 4.5.10 of [10], but we recall here the main steps of the proof for the convenience of the reader. Throughout the proof, h will be fixed and will be omitted. In addition we will use the notation $\pi_+(\Pi_\mu, h) = \pi_+(\mu)$.

First of all let us show that, for $\mu_n \rightarrow \mu$,

$$\limsup_{n \rightarrow \infty} \pi_+(\mu_n) \leq \pi_+(\mu).$$

In fact, by the compactness of Π_μ there is $P \in \Pi_\mu$ that achieves the supremum. Moreover, by the compactness of $\Pi_\mu \cup \cup_n \Pi_{\mu_{n_i}}$, for every convergent subsequence $\{\pi_+(\mu_{n_i})\} = \{E^{P_{n_i}}[\int_0^\infty e^{-\lambda t} h(X(t)) dt]\}$, we can extract a subsequence $\{P_{n_{i_j}}\}$ that converges to some $P \in \Pi_\mu$. Since $\int_0^\infty e^{-\lambda t} h(x(t)) dt$ is continuous on $D_E[0, \infty)$, we then have

$$\lim_{i \rightarrow \infty} \pi_+(\mu_{n_i}) = \lim_{j \rightarrow \infty} E^{P_{n_{i_j}}}[\int_0^\infty e^{-\lambda t} h(X(t)) dt] = E^P[\int_0^\infty e^{-\lambda t} h(X(t)) dt] \leq \pi_+(\mu).$$

This yields, in particular, the upper semicontinuity (and hence the measurability) of $u_+(x) = \pi_+(\delta_x)$.

Next, Condition 3.1 (b) and (c) implies that, for $\mu_1, \mu_2 \in \mathcal{P}(E)$, $0 \leq \alpha \leq 1$,

$$\pi_+(\alpha\mu_1 + (1 - \alpha)\mu_2) = \alpha\pi_+(\mu_1) + (1 - \alpha)\pi_+(\mu_2),$$

(Theorem 4.5.11(a), Lemma 4.5.8 and Lemma 4.5.10 of [10]. We will not recall this part of the proof). This yields, for $\{\mu_i\} \subset \mathcal{P}(E)$, $\alpha_i \geq 0$, $\sum_i \alpha_i = 1$:

$$\pi_+(\sum_i \alpha_i \mu_i) = \sum_1^N \alpha_i \pi_+(\frac{1}{\sum_1^N \alpha_i} \sum_1^N \alpha_i \mu_i) + \sum_{N+1}^\infty \alpha_i \pi_+(\frac{1}{\sum_{N+1}^\infty \alpha_i} \sum_{N+1}^\infty \alpha_i \mu_i),$$

and hence

$$\left| \pi_+(\sum_i \alpha_i \mu_i) - \sum_i \alpha_i \pi_+(\mu_i) \right| \leq 2 \frac{\|h\|}{\lambda} \sum_{N+1}^\infty \alpha_i, \quad \forall N.$$

Finally, for each n , let $\{E_i^n\}$ be a countable collection of disjoint subsets of E with diameter less than $\frac{1}{n}$ and such that $E = \bigcup_i E_i^n$. In addition let $x_i^n \in E_i^n$ satisfy $u_+(x_i^n) \geq \sup_{E_i^n} u_+(x) - \frac{1}{n}$. Define $u_n(x) = \sum_i u_+(x_i^n) 1_{E_i^n}(x)$ and $\mu_n = \sum_i \mu(E_i^n) \delta_{x_i^n}$. Then $\{u_n\}$ converges to u_+ pointwise and boundedly, and $\{\mu_n\}$ converges to μ . Therefore

$$\int_E u_+(x) \mu(dx) = \lim_n \int_E u_n(x) \mu(dx) = \lim_n \sum_i \pi_+(\delta_{x_i^n}) \mu(E_i^n) = \lim_n \pi_+(\mu_n) \leq \pi_+(\mu).$$

To prove the opposite inequality, let $\mu_i^n(B) = \mu(B \cap E_i^n) / \mu(E_i^n)$, for $\mu(E_i^n) > 0$, and $u_n(x) = \sum_i \pi_+(\mu_i^n) 1_{E_i^n}(x)$. For each $x \in E$, for every n there exists a (unique) $i(n)$ such that $x \in E_{i(n)}^n$. Then $u_n(x) = \pi_+(\mu_{i(n)}^n)$ and $\mu_{i(n)}^n \rightarrow \delta_x$, hence $\limsup_n u_n(x) \leq u_+(x)$. Therefore

$$\pi_+(\mu) = \pi_+(\sum_i \mu(E_i^n) \mu_i^n) = \int_E u_n(x) \mu(dx) \leq \int_E \limsup_n u_n(x) \mu(dx) \leq \int_E u_+(x) \mu(dx),$$

where the last but one inequality follows from the fact that the u_n are uniformly bounded.

To prove the assertion for π_- use the fact that $\pi_-(\Pi_\mu, h) = -\pi_+(\Pi_\mu, -h)$. □

Lemma 3.5. Assume that Condition 3.1 holds. Then u_+ is a viscosity subsolution of (2.11) and u_- is a viscosity supersolution of the same equation.

Proof. Since $u_-(x, h) = -u_+(x, -h)$ it is enough to consider u_+ . Let $(f, g) \in A$. Suppose x_0 is a point such that $u_+(x_0) - f(x_0) = \sup_x (u_+(x) - f(x))$. Since we can always add a constant to f , we can assume $u_+(x_0) - f(x_0) = 0$. By compactness (Condition 3.1(c)), we have

$$u_+(x_0) = E^P \left[\int_0^\infty e^{-\lambda t} h(X(t)) dt \right]$$

for some $P \in \Pi_{x_0}$.

For $\epsilon > 0$, define

$$\tau_\epsilon = \epsilon \wedge \inf\{t > 0 : r(X(t), x_0) \geq \epsilon \text{ or } r(X(t-), x_0) \geq \epsilon\} \tag{3.6}$$

and let $H_\epsilon = e^{-\lambda\tau_\epsilon}$. Then, by Lemma 2.9,

$$\begin{aligned} 0 &= u_+(x_0) - f(x_0) \\ &= E^P \left[\int_0^\infty e^{-\lambda t} (h(X(t)) - \lambda f(X(t)) + g(X(t))) dt \right] \\ &= E^P \left[\int_0^{\tau_\epsilon} e^{-\lambda t} (h(X(t)) - \lambda f(X(t)) + g(X(t))) dt \right] \\ &\quad + E^P \left[e^{-\lambda\tau_\epsilon} \int_0^\infty e^{-\lambda t} (h(X(t+\tau_\epsilon)) - \lambda f(X(t+\tau_\epsilon)) + g(X(t+\tau_\epsilon))) dt \right] \\ &= E^P \left[\int_0^{\tau_\epsilon} e^{-\lambda t} (h(X(t)) - \lambda f(X(t)) + g(X(t))) dt \right] \\ &\quad + E^P[H_\epsilon] E^{P^{\tau_\epsilon, H_\epsilon}} \left[\int_0^\infty e^{-\lambda t} (h(X(t)) - \lambda f(X(t)) + g(X(t))) dt \right]. \end{aligned}$$

Setting $\mu_\epsilon(\cdot) = P^{\tau_\epsilon, H_\epsilon}(X(0) \in \cdot)$, by Lemma 2.10 and Lemma 2.9, the above chain of equalities can be continued as (with the notation (2.3))

$$\leq E^P \left[\int_0^{\tau_\epsilon} e^{-\lambda t} (h(X(t)) - \lambda f(X(t)) + g(X(t))) dt \right] + E^P[H_\epsilon](\pi_+(\Pi_{\mu_\epsilon}, h) - \mu_\epsilon f) ,$$

and, by Lemma 3.4,

$$\begin{aligned} &= E^P \left[\int_0^{\tau_\epsilon} e^{-\lambda t} (h(X(t)) - \lambda f(X(t)) + g(X(t))) dt \right] + E^P[H_\epsilon](\mu_\epsilon u_+ - \mu_\epsilon f) \\ &= E^P \left[\int_0^{\tau_\epsilon} e^{-\lambda t} (h(X(t)) - \lambda f(X(t)) + g(X(t))) dt \right] + E^P [e^{-\lambda\tau_\epsilon} (u_+(X(\tau_\epsilon)) - f(X(\tau_\epsilon)))] \\ &\leq E^P \left[\int_0^{\tau_\epsilon} e^{-\lambda t} (h(X(t)) - \lambda f(X(t)) + g(X(t))) dt \right], \end{aligned}$$

where the last inequality uses the fact that $u_+ - f \leq 0$. Therefore

$$\begin{aligned} 0 &\leq \lim_{\epsilon \rightarrow 0} \frac{E^P \left[\int_0^{\tau_\epsilon} e^{-\lambda t} (h(X(t)) - \lambda f(X(t)) + g(X(t))) dt \right]}{E^P[\tau_\epsilon]} \\ &= h(x_0) - \lambda f(x_0) + g(x_0) \\ &= h(x_0) - \lambda u_+(x_0) + g(x_0). \end{aligned}$$

□

Corollary 3.6. *Let $h \in C_b(E)$. If in addition to Condition 3.1, the comparison principle holds for equation (2.11), then $u = u_+ = u_-$ is the unique viscosity solution of equation (2.11).*

Theorem 3.7. *Assume that Condition 3.1 holds. For $\lambda > 0$, let M_λ be the set of $h \in C_b(E)$, such that the comparison principle holds for (2.11). If for each $\lambda > 0$, M_λ is separating, then uniqueness holds for the martingale problem for A in $D_E[0, \infty)$.*

Proof. If the comparison principle for (2.11) holds for some $h \in C_b(E)$, then by Lemma 3.5, $u_+ = u_-$. Then, by the definition of u_+ and u_- and Lemma 3.4, for any two solutions $P_1, P_2 \in \Pi_\mu$, we must have

$$E^{P_1} \left[\int_0^\infty e^{-\lambda t} h(X(t)) dt \right] = E^{P_2} \left[\int_0^\infty e^{-\lambda t} h(X(t)) dt \right].$$

Consequently, if M_λ is separating, Theorem 2.13 implies $P_1 = P_2$. □

Remark 3.8. Another way of viewing the role of the comparison principle in the issue of uniqueness for the martingale problem for A is the following.

Suppose the comparison principle holds for some h and let $u_+ = u_- = u$. By Lemmas 3.4 and 2.11,

$$u(X(t)) - \int_0^t (\lambda u(X(s)) - h(X(s))) ds,$$

is a $\{\mathcal{F}_t^X\}$ -martingale for every $P \in \Pi$. Then, even though u_+ and u_- are defined in nonlinear ways, the linearity of the martingale property ensures that A can be extended to the linear span A^u of $A \cup \{(u, \lambda u - h)\}$ and every solution of the martingale problem for A will be a solution of the martingale problem for A^u . By applying this procedure to all functions $h \in M_\lambda$, we obtain an extension \hat{A} of A such that every solution of the martingale problem for A will be a solution of the martingale problem for \hat{A} and such that $\mathcal{R}(\lambda - \hat{A}) \supset M_\lambda$ and hence is separating. Therefore uniqueness follows from Corollary 2.14.

Notice that, even if the comparison principle does not hold, under Condition 3.1, by Lemma 4.5.18 of [10], for each $\mu \in \mathcal{P}(E)$, there exists $P \in \Pi_\mu$ such that under P

$$u_+(X(t)) - \int_0^t (\lambda u_+(X(s)) - h(X(s))) ds$$

is a $\{\mathcal{F}_t^X\}$ -martingale.

Remark 3.9. If, for some h , there exists $(u, g) \in \hat{A}$ such that $\lambda u - g = h$ (essentially u is the analog of a stochastic solution as defined in [27]), then, by Lemma 2.9 and Remark 2.19, u is a viscosity solution of (2.11).

4 Alternative definitions of viscosity solution

Different definitions of viscosity solution may be useful, depending on the setting. Here we discuss two other possibilities. As mentioned in the Introduction, the first, which is stated in terms of solutions of the martingale problem, is a modification of a definition used in [9], while the second is a stronger version of definitions in [11]. We show that Lemma 3.5 still holds under these alternative definitions and hence all the results of Section 3 carry over. Both definitions are stronger in the sense that the inequalities (2.13) and (2.15) required by the previous definition are required by these definitions. Consequently, in both cases, it should be easier to prove the comparison principle.

\mathcal{T} will denote the set of $\{\mathcal{F}_t^X\}$ -stopping times.

Definition 4.1. (Stopped viscosity (semi)solution)

Let $A \subset C_b(E) \times C_b(E)$, $\lambda > 0$, and $h \in C_b(E)$.

- a) $u \in B(E)$ is a stopped viscosity subsolution of (2.11) if and only if u is upper semicontinuous and if $(f, g) \in A$, $x_0 \in E$, and there exists a strictly positive $\tau_0 \in \mathcal{T}$ such that

$$\sup_{P \in \Pi_{x_0}, \tau \in \mathcal{T}} \frac{E^P[e^{-\lambda\tau \wedge \tau_0}(u - f)(X(\tau \wedge \tau_0))]}{E^P[e^{-\lambda\tau \wedge \tau_0}]} = (u - f)(x_0), \tag{4.1}$$

then

$$\lambda u(x_0) - g(x_0) \leq h(x_0). \tag{4.2}$$

b) $u \in B(E)$ is a stopped viscosity supersolution of (2.11) if and only if u is lower semicontinuous and if $(f, g) \in A$, $x_0 \in E$, and there exists a strictly positive $\tau_0 \in \mathcal{T}$ such that

$$\inf_{P \in \Pi_{x_0}, \tau \in \mathcal{T}} \frac{E^P[e^{-\lambda\tau \wedge \tau_0}(u - f)(X(\tau \wedge \tau_0))]}{E^P[e^{-\lambda\tau \wedge \tau_0}]} = (u - f)(x_0), \tag{4.3}$$

then

$$\lambda u(x_0) - g(x_0) \geq h(x_0). \tag{4.4}$$

A function $u \in C_b(E)$ is a stopped viscosity solution of (2.11) if it is both a subsolution and a supersolution.

Remark 4.2. If $(u - f)(x_0) = \sup_x (u - f)(x)$, then (4.1) is satisfied. Consequently, every stopped sub/supersolution in the sense of Definition 4.1 is a sub/supersolution in the sense of Definition 2.17.

Remark 4.3. Definition 4.1 requires (4.2) ((4.4)) to hold only at points x_0 for which (4.1) ((4.3)) is verified for some τ_0 . Note that, as in Definition 2.17, such an x_0 might not exist.

Definition 4.1 essentially requires a local maximum principle and is related to the notion of characteristic operator as given in [8].

For Definition 4.1, we have the following analog of Lemma 3.5.

Lemma 4.4. Assume that Condition 3.1 holds. Then u_+ given by (3.1) is a stopped viscosity subsolution of (2.11) and u_- given by (3.2) is a stopped viscosity supersolution of the same equation.

Proof. Let $(f, g) \in A$. Suppose x_0 is a point such that (4.1) holds for u_+ for some $\tau_0 \in \mathcal{T}$, $\tau_0 > 0$. Since we can always add a constant to f , we can assume $u_+(x_0) - f(x_0) = 0$. By the same arguments used in the proof of Lemma 3.5, defining τ_ϵ and H_ϵ in the same way, we obtain, for some $P \in \Pi_{x_0}$ (independent of ϵ),

$$\begin{aligned} 0 &= u_+(x_0) - f(x_0) \\ &\leq E^P \left[\int_0^{\tau_\epsilon \wedge \tau_0} e^{-\lambda t} (h(X(t)) - \lambda f(X(t)) + g(X(t))) dt \right] \\ &\quad + E^P [e^{-\lambda\tau_\epsilon \wedge \tau_0} (u_+(X(\tau_\epsilon \wedge \tau_0)) - f(X(\tau_\epsilon \wedge \tau_0)))] \\ &\leq E^P \left[\int_0^{\tau_\epsilon \wedge \tau_0} e^{-\lambda t} (h(X(t)) - \lambda f(X(t)) + g(X(t))) dt \right], \end{aligned}$$

where the last inequality uses (4.1) and the fact that $u_+(x_0) - f(x_0) = 0$. Then the result follows as in Lemma 3.5. □

The following is essentially Definition 7.1 of [11].

Definition 4.5. (Sequential viscosity (semi)solution)

Let $A \subset C_b(E) \times C_b(E)$, $\lambda > 0$, and $h \in C_b(E)$.

a) $u \in B(E)$ is a sequential viscosity subsolution of (2.11) if and only if u is upper semicontinuous and for each $(f, g) \in A$ and each sequence $y_n \in E$ satisfying

$$\lim_{n \rightarrow \infty} (u - f)(y_n) = \sup_x (u - f)(x), \tag{4.5}$$

we have

$$\limsup_{n \rightarrow \infty} (\lambda u(y_n) - g(y_n) - h(y_n)) \leq 0. \tag{4.6}$$

b) $u \in B(E)$ is a sequential viscosity supersolution of (2.11)

if and only if u is lower semicontinuous and for each $(f, g) \in A$ and each sequence $y_n \in E$ satisfying

$$\lim_{n \rightarrow \infty} (u - f)(y_n) = \inf_x (u - f)(x), \tag{4.7}$$

we have

$$\liminf_{n \rightarrow \infty} (\lambda u(y_n) - g(y_n) - h(y_n)) \geq 0. \tag{4.8}$$

A function $u \in C_b(E)$ is a sequential viscosity solution of (2.11) if it is both a subsolution and a supersolution.

Remark 4.6. For E compact, every viscosity sub/supersolution is a sequential viscosity sub/supersolution.

For sequential viscosity semisolutions, we have the following analog of Lemma 3.5. $C_{b,u}(E)$ denotes the space of bounded, uniformly continuous functions on E .

Lemma 4.7. For $\epsilon > 0$, define

$$\tau_\epsilon = \epsilon \wedge \inf\{t > 0 : r(X(t), X(0)) \geq \epsilon \text{ or } r(X(t-), X(0)) \geq \epsilon\}.$$

Assume $A \subset C_{b,u}(E) \times C_{b,u}(E)$, for each $\epsilon > 0$, $\inf_{P \in \Pi} E^P[\tau_\epsilon] > 0$, and that Condition 3.1 holds. Then, for $h \in C_{b,u}(E)$, u_+ given by (3.1) is a sequential viscosity subsolution of (2.11) and u_- given by (3.2) is a sequential viscosity supersolution of the same equation.

Proof. Let $(f, g) \in A$. Suppose $\{y_n\}$ is a sequence such that (4.5) holds for u_+ . Since we can always add a constant to f , we can assume $\sup_x (u_+ - f)(x) = 0$. Let $H_\epsilon = e^{-\lambda\tau_\epsilon}$. Then, by the same arguments as in Lemma 3.5, we have, for some $P_n \in \Pi_{y_n}$ (independent of ϵ),

$$(u_+ - f)(y_n) \leq E^{P_n} \left[\int_0^{\tau_\epsilon} e^{-\lambda t} (h(X(t)) - \lambda f(X(t)) + g(X(t))) dt \right],$$

where we have used the fact that $\sup_x (u_+ - f)(x) = 0$. Therefore

$$\frac{(u_+ - f)(y_n)}{E^{P_n}[\tau_\epsilon]} \leq \frac{E^{P_n} \left[\int_0^{\tau_\epsilon} e^{-\lambda t} (h(X(t)) - \lambda f(X(t)) + g(X(t))) dt \right]}{E^{P_n}[\tau_\epsilon]}.$$

Replacing ϵ by ϵ_n going to zero sufficiently slowly so that the left side converges to zero, the uniform continuity of f , g , and h implies the right side is asymptotic to $h(y_n) - \lambda f(y_n) + g(y_n)$ giving

$$\begin{aligned} 0 &\leq \liminf_{n \rightarrow \infty} (h(y_n) - \lambda f(y_n) + g(y_n)) \\ &= \liminf (h(y_n) - \lambda u_+(y_n) + g(y_n)). \end{aligned}$$

□

The following theorem is essentially Lemma 7.4 of [11]. It gives the intuitively natural result that if $h \in \overline{\mathcal{R}(\lambda - A)}$ (where the closure is taken under uniform convergence), then the comparison principle holds for sequential viscosity semisolutions of $\lambda u - Au = h$.

If E is compact, the same results hold for viscosity semisolutions, by Remark 4.6.

Theorem 4.8. Suppose $h \in C_b(E)$ and there exist $(f_n, g_n) \in A$ satisfying $\sup_x |\lambda f_n(x) - g_n(x) - h| \rightarrow 0$. Then the comparison principle holds for sequential viscosity semisolutions of (2.11).

Proof. Suppose \underline{u} is a sequential viscosity subsolution. Set $h_n = \lambda f_n - g_n$. For $\epsilon_n > 0$, $\epsilon_n \rightarrow 0$, there exist $y_n \in E$ satisfying $\underline{u}(y_n) - f_n(y_n) \geq \sup_x(\underline{u}(x) - f_n(x)) - \epsilon_n/\lambda$ and $\lambda \underline{u}(y_n) - g_n(y_n) - h(y_n) \leq \epsilon_n$. Then

$$\begin{aligned} \sup_x(\lambda \underline{u}(x) - \lambda f_n(x)) &\leq \lambda \underline{u}(y_n) - \lambda f_n(y_n) + \epsilon_n \\ &\leq h(y_n) + g_n(y_n) - \lambda f_n(y_n) + 2\epsilon_n \\ &= h(y_n) - h_n(y_n) + 2\epsilon_n \\ &\rightarrow 0. \end{aligned}$$

Similarly, if \bar{u} is a supersolution of $\lambda u - Au = h$,

$$\liminf_{n \rightarrow \infty} \inf_x(\bar{u}(x) - f_n(x)) \geq 0,$$

and it follows that $\underline{u} \leq \bar{u}$. □

5 Martingale problems with boundary conditions

The study of stochastic processes that are constrained to some set \bar{E}_0 and must satisfy some boundary condition on ∂E_0 , described by one or more boundary operators B_1, \dots, B_m , is typically carried out by incorporating the boundary condition in the definition of the domain $\mathcal{D}(A)$ (see Remark 5.12 below). However, this approach restricts the problems that can be dealt with to fairly regular ones, so we follow the formulation of a constrained martingale problem given in [19]. (See also [20, 22]).

Let $E_0 \subseteq E$ be an open set and let

$$\partial E_0 = \cup_{k=1}^m E_k,$$

for disjoint, nonempty Borel sets E_1, \dots, E_m . Let $A \subseteq C_b(\bar{E}_0) \times C_b(\bar{E}_0)$, $B_k \subseteq C_b(\bar{E}_0) \times C_b(\bar{E}_0)$, $k = 1, \dots, m$, be linear operators with a common domain \mathcal{D} such that $(1, 0) \in A$, $(1, 0) \in B_k$, $k = 1, \dots, m$. For simplicity we will assume that E is compact (hence the subscript b will be dropped) and that A, B_1, \dots, B_m are single-valued.

Definition 5.1. A stochastic process X with sample paths in $D_{\bar{E}_0}[0, \infty)$ is a solution of the constrained martingale problem for $(A, E_0; B_1, E_1; \dots; B_m, E_m)$ provided there exist a filtration $\{\mathcal{F}_t\}$ and continuous, nondecreasing processes $\gamma_1, \dots, \gamma_m$ such that $X, \gamma_1, \dots, \gamma_m$ are $\{\mathcal{F}_t\}$ -adapted,

$$\gamma_k(t) = \int_0^t \mathbf{1}_{\bar{E}_k}(X(s-)) d\gamma_k(s),$$

and for each $f \in \mathcal{D}$,

$$M_f(t) = f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds - \sum_{k=1}^m \int_0^t B_k f(X(s-))d\gamma_k(s) \tag{5.1}$$

is a $\{\mathcal{F}_t\}$ -martingale.

Remark 5.2. $\gamma_1, \dots, \gamma_m$ will be called *local times* since γ_k increases only when X is in \bar{E}_k . Without loss of generality, we can assume that the γ_k are $\{\mathcal{F}_t^X\}$ -adapted. (Replace γ_k by its dual, predictable projection on $\{\mathcal{F}_t^X\}$.) Definition 5.1 does not require that the γ_k be uniquely determined by the distribution of X , but if γ_k^1 and γ_k^2 , $k = 1, \dots, m$, are continuous and satisfy the martingale requirement with the same filtration, we must have

$$\sum_{k=1}^m \int_0^t B_k f(X(s-))d\gamma_k^1(s) - \sum_{k=1}^m \int_0^t B_k f(X(s-))d\gamma_k^2(s) = 0,$$

since this expression will be a continuous martingale with finite variation paths.

Remark 5.3. The main example of a constrained martingale problem in the sense of the above definition is the constrained martingale problem that describes a reflected diffusion process. In this case, A is a second order elliptic operator and the B_k are first order differential operators. Although there is a vast literature on this topic, there are still relevant cases of reflected diffusions that have not been uniquely characterized as solutions of martingale problems or stochastic differential equations. In Section 6.1, the results of this section are used in one of these cases. More general constrained diffusions where the B_k are second order elliptic operators, for instance diffusions with sticky reflection, also satisfy Definition 5.1.

Definition 5.1 is a special case of a more general definition of constrained martingale problem given in [22]. This broader definition allows for more general boundary behavior, such as models considered in [4].

Many results for solutions of martingale problems carry over to solutions of constrained martingale problems. In particular Lemma 2.9 still holds. In addition the following lemma holds.

Lemma 5.4. *Let X be a stochastic process with sample paths in $D_{\bar{E}_0}[0, \infty)$, $\gamma_1, \dots, \gamma_m$ be continuous, nondecreasing processes such that $X, \gamma_1, \dots, \gamma_m$ are $\{\mathcal{F}_t\}$ -adapted. Then for $f \in \mathcal{D}$ such that (5.1) is a $\{\mathcal{F}_t\}$ -martingale and $\lambda > 0$,*

$$M_f^\lambda(t) = e^{-\lambda t} f(X(t)) - f(X(0)) + \int_0^t e^{-\lambda s} (\lambda f(X(s)) - Af(X(s))) ds - \sum_{k=1}^m \int_0^t e^{-\lambda s} B_k f(X(s-)) d\gamma_k(s)$$

is a $\{\mathcal{F}_t\}$ -martingale.

Proof. Since $\mathcal{D} \subset C(E)$, M_f is cadlag, we can apply Itô's formula to $e^{-\lambda t} f(X(t))$ and obtain

$$e^{-\lambda t} f(X(t)) - f(X(0)) = \int_0^t (-f(X(s))\lambda e^{-\lambda s} + e^{-\lambda s} Af(X(s))) ds + \sum_{k=1}^m \int_0^t e^{-\lambda s} B_k f(X(s-)) d\gamma_k(s) + \int_0^t e^{-\lambda s} dM_f(s).$$

□

Lemma 2.10 is replaced by Lemma 5.5 below.

Lemma 5.5. *a) The set of distributions of solutions of the constrained martingale problem for $(A, E_0; B_1, E_1; \dots; B_m, E_m)$ is convex.*

b) Let $X, \gamma_1, \dots, \gamma_m$ satisfy Definition 5.1. Let $\tau \geq 0$ be a bounded $\{\mathcal{F}_t\}$ -stopping time and $H \geq 0$ be a \mathcal{F}_τ -measurable random variable such that $0 < E[H] < \infty$. Then the measure $P^{\tau, H} \in \mathcal{P}(D_E[0, \infty))$ defined by

$$P^{\tau, H}(C) = \frac{E[H \mathbf{1}_C(X(\tau + \cdot))]}{E[H]}, \quad C \in \mathcal{B}(D_{\bar{E}_0}[0, \infty)), \quad (5.2)$$

is the distribution of a solution of the constrained martingale problem for $(A, E_0; B_1, E_1; \dots; B_m, E_m)$.

Proof. Part (a) is immediate. For Part (b), let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space on which $X, \gamma_1, \dots, \gamma_m$ are defined, and define \mathbb{P}^H by

$$\mathbb{P}^H(C) = \frac{E^\mathbb{P}[H \mathbf{1}_C]}{E^\mathbb{P}[H]}, \quad C \in \mathcal{F}.$$

Define X^τ and γ_k^τ by $X^\tau(t) = X(\tau + t)$ and $\gamma_k^\tau(t) = \gamma_k(\tau + t) - \gamma_k(\tau)$. X^τ and the γ_k^τ are adapted to the filtration $\{\mathcal{F}_{\tau+t}\}$ and for $0 \leq t_1 < \dots < t_n < t_{n+1}$ and $f_1, \dots, f_n \in B(E)$,

$$\begin{aligned} & E^{\mathbb{P}^H} \left[\left\{ f(X^\tau(t_{n+1})) - f(X^\tau(t_n)) - \int_{t_n}^{t_{n+1}} Af(X^\tau(s))ds - \sum_{k=1}^m \int_{t_n}^{t_{n+1}} B_k f(X^\tau(s-))d\gamma_k^\tau(s) \right\} \right. \\ & \quad \left. \Pi_{i=1}^n f_i(X^\tau(t_i)) \right] \\ &= \frac{1}{E^{\mathbb{P}}[H]} E^{\mathbb{P}} \left[H \left\{ f(X(\tau + t_{n+1})) - f(X(\tau + t_n)) \right. \right. \\ & \quad \left. \left. - \int_{\tau+t_n}^{\tau+t_{n+1}} Af(X(s))ds - \sum_{k=1}^m \int_{\tau+t_n}^{\tau+t_{n+1}} B_k f(X(s-))d\gamma_k(s) \right\} \right. \\ & \quad \left. \Pi_{i=1}^n f_i(X(\tau + t_i)) \right] \\ &= 0 \end{aligned}$$

by the optional sampling theorem. Therefore, under \mathbb{P}^H , X^τ is a solution of the constrained martingale problem with local times $\gamma_1^\tau, \dots, \gamma_m^\tau$. $P^{\tau,H}$, given by (5.2), is the distribution of X^τ on $D_{\overline{E}_0}[0, \infty)$. \square

As in Section 3, let Π denote the set of distributions of solutions of the constrained martingale problem and Π_μ denote the set of distributions of solutions with initial condition μ . In the rest of this section X is the canonical process on $D_E[0, \infty)$ and $\gamma_1, \dots, \gamma_m$ are a set of $\{\mathcal{F}_t^X\}$ -adapted local times (see Remark 5.2). We assume that the following conditions hold. See Section 5.1 below for settings in which these conditions are valid. Recall that we are assuming E is compact.

Condition 5.6.

- a) \mathcal{D} is dense in $C(\overline{E}_0)$ in the topology of uniform convergence.
- b) For each $\mu \in \mathcal{P}(\overline{E}_0)$, $\Pi_\mu \neq \emptyset$ (see Proposition 5.13).
- c) Π is compact (see Proposition 5.13).
- d) For each $P \in \Pi$ and $\lambda > 0$, there exist $\gamma_1, \dots, \gamma_m$ satisfying the requirements of Definition 5.1 such that $E^P \left[\int_0^\infty e^{-\lambda t} d\gamma_k(t) \right] < \infty$, $k = 1, \dots, m$ (see Proposition 5.13).

Remark 5.7. For $P \in \Pi$, Condition 5.6(d) and Lemma 5.4 give

$$\mu f = E \left[\int_0^\infty e^{-\lambda s} (\lambda f(X(s)) - Af(X(s)))ds - \sum_{k=1}^m \int_0^\infty e^{-\lambda s} B_k f(X(s-))d\gamma_k(s) \right]. \quad (5.3)$$

Remark 5.8. We can take the topology on $D_E[0, \infty)$ to be either the Skorohod topology or the Jakubowski topology (see Remark 3.2).

The definitions of u_+ , u_- , π_+ and π_- are still given by (3.1), (3.2), (3.3) and (3.4). With Condition 5.6 replacing Condition 3.1, the proof of Lemma 3.4 carries over (Lemma 5.5 above guarantees that Lemmas 4.5.8 and 4.5.10 in [10] can be applied).

Lemma 5.9. Assume Condition 5.6 holds. Then u_+ is a viscosity subsolution of (2.17) and u_- is a viscosity supersolution of the same equation.

Proof. The proof is similar to the proof of Lemma 3.5, so we will only sketch the argument. For $f \in \mathcal{D}$, let x_0 satisfy $\sup_{x \in \overline{E}_0} (u_+ - f)(x) = u_+(x_0) - f(x_0)$. By adding a constant to f if necessary, we can assume that $u_+(x_0) - f(x_0) = 0$.

With τ_ϵ as in the proof of Lemma 3.5, by Lemmas 5.4, 5.5 and 3.4, and the compactness of Π_{x_0} (Condition 5.6(c)), for some $P \in \Pi_{x_0}$ (independent of ϵ) we have

$$0 \leq E^P \left[\int_0^{\tau_\epsilon} e^{-\lambda t} (h(X(t)) - \lambda f(X(t)) + Af(X(t))) dt + \sum_{k=1}^m \int_0^{\tau_\epsilon} e^{-\lambda t} B_k f(X(t-)) d\gamma_k(t) \right],$$

Dividing the expectations by $E^P[\lambda^{-1}(1 - e^{-\lambda\tau_\epsilon}) + \sum_{k=1}^m \int_0^{\tau_\epsilon} e^{-\lambda t} d\gamma_k(t)]$ and letting $\epsilon \rightarrow 0$, we must have

$$0 \leq (h(x_0) - \lambda f(x_0) + Af(x_0)) \vee \max_{k: x_0 \in \bar{E}_k} B_k f(x_0),$$

which, since $f(x_0) = u_+(x_0)$, implies (2.20), if $x_0 \in \partial E_0$, and (2.19), if $x_0 \in E_0$. □

Corollary 5.10. *Let $h \in C(\bar{E}_0)$. If, in addition to Condition 5.6, the comparison principle holds for equation (2.17), then $u = u_+ = u_-$ is the unique viscosity solution of equation (2.17).*

The following theorem is the analog of Theorem 3.7.

Theorem 5.11. *Assume Condition 5.6 holds. For $\lambda > 0$, let M_λ be the set of $h \in C(\bar{E}_0)$ such that the comparison principle holds for (2.17). If for every $\lambda > 0$, M_λ is separating, then the distribution of the solution X of the constrained martingale problem for $(A, E_0; B_1, E_1; \dots; B_m, E_m)$ is uniquely determined.*

Proof. The proof of this result is in the spirit of Remark 3.8. Let \hat{A} be the collection of $(f, g) \in B(\bar{E}_0) \times B(\bar{E}_0)$ such that $f(X(t)) - \int_0^t g(X(s)) ds$ is a $\{\mathcal{F}_t^X\}$ -martingale for all $P \in \Pi$. Denote by $\hat{\Pi}$ the set of the distributions of solutions of the martingale problem for \hat{A} , and by $\hat{\Pi}_\mu$ the set of solutions with initial distribution μ . Then, by construction, for each $\mu \in \mathcal{P}(\bar{E}_0)$, $\Pi_\mu \subseteq \hat{\Pi}_\mu$. By the comparison principle, Lemmas 5.9, 3.4 and 2.11, for each $h \in M_\lambda$ and $u = u_+ = u_-$ given by (3.1) (or equivalently, (3.2)), $(u, \lambda u - h)$ belongs to \hat{A} , or equivalently the pair (u, h) belongs to $\lambda - \hat{A}$. Consequently $\mathcal{R}(\lambda - \hat{A}) \supseteq M_\lambda$ is separating and the thesis follows from Corollary 2.14. □

Remark 5.12. Differently from Remark 3.8, the operator \hat{A} is not an extension of A as an operator on the domain \mathcal{D} , but it is an extension of A restricted to the domain $\mathcal{D}_0 = \{f \in \mathcal{D} : B_k f(x) = 0 \ \forall x \in \bar{E}_k, k = 1, \dots, m\}$. The distribution of the solution X of the constrained martingale problem for $(A, E_0; B_1, E_1; \dots; B_m, E_m, \mu)$ is uniquely determined even though the same might not hold for the solution of the martingale problem for $(A|_{\mathcal{D}_0}, \mu)$.

5.1 Sufficient conditions for the validity of Condition 5.6

In what follows, we assume that $E - E_0 = \cup_{k=1}^m \tilde{E}_k$, where the \tilde{E}_k are disjoint Borel sets satisfying $\tilde{E}_k \supset E_k, k = 1, \dots, m$.

Proposition 5.13. *Assume Condition 5.6(a) and that the following hold:*

- i) *There exist linear operators $\tilde{A}, \tilde{B}_1, \dots, \tilde{B}_m : \tilde{\mathcal{D}} \subseteq C(E) \rightarrow C(E)$ with $\tilde{\mathcal{D}}$ dense in $C(E)$, $(1, 0) \in \tilde{A}$, $(1, 0) \in \tilde{B}_k, k = 1, \dots, m$, that are extensions of A, B_1, \dots, B_m in the sense that for every $f \in \mathcal{D}$ there exists $\tilde{f} \in \tilde{\mathcal{D}}$ such that $f = \tilde{f}|_{\bar{E}_0}$, $Af = \tilde{A}\tilde{f}|_{\bar{E}_0}$, and $B_k f = \tilde{B}_k \tilde{f}|_{\bar{E}_0}, k = 1, \dots, m$, and such that the martingale problem for each of $\tilde{A}, \tilde{B}_1, \dots, \tilde{B}_m$ with initial condition δ_x has a solution for every $x \in E$.*

- ii) If $E \neq \bar{E}_0$, there exists $\varphi \in \tilde{\mathcal{D}}$ such that $\varphi = 0$ on \bar{E}_0 , $\varphi > 0$ on $E - \bar{E}_0$ and $\tilde{A}\varphi = 0$ on \bar{E}_0 , $\tilde{B}_k\varphi \leq 0$ on \bar{E}_k , $k = 1, \dots, m$.
- iii) There exists $\{\varphi_n\}$, $\varphi_n \in \mathcal{D}$, such that $\sup_{n,x} |\varphi_n(x)| < \infty$ and $B_k\varphi_n(x) \geq n$ on \bar{E}_k for all $k = 1, \dots, m$.

Then b) c) and d) in Condition 5.6 are verified.

Proof. Condition 5.6(b). We will obtain a solution of the constrained martingale problem for $(A, E_0; B_1, E_1; \dots; B_m, E_m)$ by constructing a solution of the constrained martingale problem for $(\tilde{A}, E_0; \tilde{B}_1, \tilde{E}_1; \dots; \tilde{B}_m, \tilde{E}_m)$ and showing that any such solution that starts in \bar{E}_0 stays in \bar{E}_0 for all times. Following [19], we will construct a solution of the constrained martingale problem for $(\tilde{A}, E_0; \tilde{B}_1, \tilde{E}_1; \dots; \tilde{B}_m, \tilde{E}_m)$ from a solution of the corresponding patchwork martingale problem.

$\tilde{A}, \tilde{B}_1, \dots, \tilde{B}_m$ are dissipative operators by i) and Lemma 2.16. Then, by Lemma 1.1 in [19], for each initial distribution on E , there exists a solution of the patchwork martingale problem for $(\tilde{A}, E_0; \tilde{B}_1, \tilde{E}_1; \dots; \tilde{B}_m, \tilde{E}_m)$. In addition, if $E \neq \bar{E}_0$, by ii) and the same argument used in the proof of Lemma 1.4 in [19], for every solution Y of the patchwork martingale problem for $(\tilde{A}, E_0; \tilde{B}_1, \tilde{E}_1; \dots; \tilde{B}_m, \tilde{E}_m)$ with initial distribution concentrated on \bar{E}_0 , $Y(t) \in \bar{E}_0$ for all $t \geq 0$. Therefore Y is also a solution of the patchwork martingale problem for $(A, E_0; B_1, E_1; \dots; B_m, E_m)$. By iii) and Lemma 1.8, Lemma 1.9, Proposition 2.2, and Proposition 2.3 in [19], from Y , a solution X of the constrained martingale problem for $(A, E_0; B_1, E_1; \dots; B_m, E_m)$ can be constructed.

Condition 5.6(c). If X is a solution of the constrained martingale problem for $(A, E_0; B_1, E_1; \dots; B_m, E_m)$ and $\gamma_1, \dots, \gamma_m$ are associated local times, then $\eta_0(t) = \inf\{s : s + \gamma_1(s) + \dots + \gamma_m(s) > t\}$ is strictly increasing and diverging to infinity as t goes to infinity, with probability one, and $Y = X \circ \eta_0$ is a solution of the patchwork martingale problem for $(A, E_0; B_1, E_1; \dots; B_m, E_m)$, $\eta_0, \eta_1 = \gamma_1 \circ \eta_0, \dots, \eta_m = \gamma_m \circ \eta_0$ are associated increasing processes (see the proof of Corollary 2.5 of [19]). Let $\{(X^n, \gamma_1^n, \dots, \gamma_m^n)\}$ be a sequence of solutions of the constrained martingale problem for $(A, E_0; B_1, E_1; \dots; B_m, E_m)$ with initial conditions $\{\mu^n\}$, $\mu^n \in \mathcal{P}(E)$, with associated local times. Since $\mathcal{P}(E)$ is compact, we may assume, without loss of generality, that $\{\mu^n\}$ converges to μ . Let $\{(Y^n, \eta_0^n, \eta_1^n, \dots, \eta_m^n)\}$ be the sequence of the corresponding solutions of the patchwork martingale problem and associated increasing processes. Then by the density of \mathcal{D} and Theorems 3.9.1 and 3.9.4 of [10], $\{(Y^n, \eta_0^n, \eta_1^n, \dots, \eta_m^n)\}$ is relatively compact under the Skorohod topology on $D_{E \times \mathbf{R}^{m+1}}[0, \infty)$.

Let $\{(Y^{n_k}, \eta_0^{n_k}, \eta_1^{n_k}, \dots, \eta_m^{n_k})\}$ be a subsequence converging to a limit $(Y, \eta_0, \eta_1, \dots, \eta_m)$. Then Y is a solution of the patchwork martingale problem for $(A, E_0; B_1, E_1; \dots; B_m, E_m)$ with initial condition μ and $\eta_0, \eta_1, \dots, \eta_m$ are associated increasing processes. By iii) and Lemma 1.8 and Lemma 1.9 in [19], η_0 is strictly increasing and diverging to infinity as t goes to infinity, with probability one. It follows that $\{(\eta_0^{n_k})^{-1}\}$ converges to $(\eta_0)^{-1}$ and hence $\{(X^{n_k}, \gamma_1^{n_k}, \dots, \gamma_m^{n_k})\} = \{(Y^{n_k} \circ (\eta_0^{n_k})^{-1}, \eta_1^{n_k} \circ (\eta_0^{n_k})^{-1}, \dots, \eta_m^{n_k} \circ (\eta_0^{n_k})^{-1})\}$ converges to $(Y \circ (\eta_0)^{-1}, \eta_1 \circ (\eta_0)^{-1}, \dots, \eta_m \circ (\eta_0)^{-1})$ and $Y \circ (\eta_0)^{-1}$ with associated local times $\eta_1 \circ (\eta_0)^{-1}, \dots, \eta_m \circ (\eta_0)^{-1}$ is a solution of the constrained martingale problem for $(A, E_0; B_1, E_1; \dots; B_m, E_m)$ with initial condition μ .

Condition 5.6(d). Let φ_1 be the function of iii) for $n = 1$. By Lemma 5.4 and iii) we have

$$E \left[\sum_{k=1}^m \int_0^t e^{-\lambda s} d\gamma_k(s) \right] \leq e^{-\lambda t} E[\varphi_1(X(t))] - E[\varphi_1(X(0))] + \int_0^t e^{-\lambda s} E[\lambda \varphi_1(X(s)) - A\varphi_1(X(s))] ds.$$

□

6 Examples

Several examples of application of the results of the previous sections can be given by exploiting comparison principles proved in the literature. Here we will discuss in detail two examples.

The first example is a class of diffusion processes reflecting in a domain $D \subseteq \mathbf{R}^d$ according to an oblique direction of reflection which may become tangential. This case is not covered by the existing literature on reflecting diffusions, which assumes that the direction of reflection is uniformly bounded away from the tangent hyperplane.

The second example is a large class of jump diffusion processes with jump component of unbounded variation and possibly degenerate diffusion matrix. In this case uniqueness results are already available in the literature (see e.g. [15], [13], [21]) but we believe it is still a good benchmark to show how our method works.

6.1 Diffusions with degenerate oblique direction of reflection

Let $D \subseteq \mathbf{R}^d$, $d \geq 2$, be a bounded domain with C^3 boundary, i.e.

$$D = \{x \in \mathbf{R}^d : \psi(x) > 0\}, \quad \partial D = \{x \in \mathbf{R}^d : \psi(x) = 0\}, \\ |\nabla\psi(x)| > 0, \quad \text{for } x \in \partial D,$$

for some function $\psi \in C^3(\mathbf{R}^d)$, where ∇ denotes the gradient, viewed as a row vector. Let $l : \bar{D} \rightarrow \mathbf{R}^d$ be a vector field in $C^2(\bar{D})$ such that

$$|l(x)| > 0 \text{ and } \langle l(x), \nu(x) \rangle \geq 0, \quad \forall x \in \partial D, \tag{6.1}$$

ν being the unit inward normal vector field, and let

$$\partial_0 D = \{x \in \partial D : \langle l(x), \nu(x) \rangle = 0\}. \tag{6.2}$$

We assume that $\partial_0 D$ has dimension $d - 2$. More precisely, for $d \geq 3$, we assume that $\partial_0 D$ has a finite number of connected components, each of the form

$$\{x \in \partial D : \psi(x) = 0, \tilde{\psi}(x) = 0\}, \tag{6.3}$$

where ψ is the function above and $\tilde{\psi}$ is another function in $C^2(\mathbf{R}^d)$ such that the level set $\{x \in \partial D : \tilde{\psi}(x) = 0\}$ is bounded and $|\nabla\tilde{\psi}(x)| > 0$ on it. For $d = 2$, we assume that $\partial_0 D$ consists of a finite number of points. In addition, we assume that $l(x)$ is never tangential to $\partial_0 D$.

Our goal is to prove uniqueness of the reflecting diffusion process with generator of the form

$$Af(x) = \frac{1}{2} \text{tr} (D^2 f(x) \sigma(x) \sigma(x)^T) + \nabla f(x) b(x), \tag{6.4}$$

where σ and b are Lipschitz continuous functions on \bar{D} , and direction of reflection l . We will characterize this reflecting diffusion process as the unique solution of the constrained martingale problem for $(A, D; B, \partial D)$, where A is given by (6.4),

$$Bf(x) = \nabla f(x) l(x), \tag{6.5}$$

and the common domain of A and B is $\mathcal{D} = C^2(\bar{D})$. Our tools will be the results of Section 5 and the comparison principle proved by [25].

Proposition 6.1. *Condition 5.6 is verified.*

Proof. Condition 5.6a) is obviously verified. Therefore we only need to prove that the assumptions of Proposition 5.13 are satisfied. Let $0 < r < 1$ be small enough that for

$d(x, \partial D) < \frac{4}{3}r$ the normal projection of x on ∂D , $\pi_\nu(x)$, is well defined and $|\nabla\psi(x)| > 0$. Set $U(\overline{D}) = \{x : d(x, \overline{D}) < r\}$. Let $\chi(c)$ be a nondecreasing function in $C^\infty(\mathbf{R})$ such that $0 \leq \chi(c) \leq 1$, $\chi(c) = 1$ for $c \geq \frac{2r}{3}$, $\chi(c) = 0$ for $c \leq \frac{r}{3}$. We can extend l to a Lipschitz continuous vector field on $\overline{U(\overline{D})}$ by setting, for $x \in \overline{U(\overline{D})} - \overline{D}$,

$$l(x) = (1 - \chi(d(x, \partial D)))l(\pi_\nu(x)).$$

We can also extend σ and b to Lipschitz continuous functions on $\overline{U(\overline{D})}$ by setting, for $x \in \overline{U(\overline{D})} - \overline{D}$,

$$\begin{aligned} \sigma(x) &= (1 - \chi(d(x, \partial D)))\sigma(\pi_\nu(x)), \\ b(x) &= (1 - \chi(d(x, \partial D)))b(\pi_\nu(x)). \end{aligned}$$

Clearly, both the martingale problem for A , with domain $C^2(\overline{U(\overline{D})})$, and the martingale problem for B , with the same domain, have a solution for every initial condition δ_x , $x \in \overline{U(\overline{D})}$. Since every $f \in C^2(\overline{D})$ can be extended to a function $\tilde{f} \in C^2(\overline{U(\overline{D})})$ and

$$Af = (A\tilde{f})\Big|_{\overline{D}}, \quad Bf = (B\tilde{f})\Big|_{\overline{D}},$$

Condition (i) in Proposition 5.13 is verified.

Next, consider the function φ defined as

$$\varphi(x) \begin{cases} = 0, & \text{for } x \in \overline{D}, \\ = \exp\{\frac{-1}{d(x, \partial D)}\}, & \text{for } x \in \overline{U(\overline{D})} - \overline{D}, \end{cases}$$

where $U(\overline{D})$ is as above. Since ∂D is of class C^3 , $\varphi \in C^2(\overline{U(\overline{D})})$. Moreover

$$\nabla\varphi(x) = -|\nabla\varphi(x)|\nu(\pi_\nu(x)), \quad \text{for } x \in \overline{U(\overline{D})} - \overline{D}.$$

Therefore φ satisfies Condition (ii) in Proposition 5.13.

Finally, in order to verify iii) of Proposition 5.13, we just need to modify slightly the proof of Lemma 3.1 in [25]. Suppose first that $\partial_0 D$ is connected. Let $\tilde{\psi}$ be the function in (6.3). Since $l(x)$ is never tangent to $\partial_0 D$, it must hold $\nabla\tilde{\psi}(x)l(x) \neq 0$ for each $x \in \partial_0 D$, and hence, possibly replacing ψ by $-\psi$, we can assume that

$$\tilde{\psi}(x) = 0, \quad \nabla\tilde{\psi}(x)l(x) > 0, \quad \forall x \in \partial_0 D. \tag{6.6}$$

Let $U(\partial_0 D)$ be a neighborhood of $\partial_0 D$ such that $\inf_{\overline{U(\partial_0 D)}} \nabla\tilde{\psi}(x)l(x) > 0$, and for each $n \in \mathbf{N}$, set

$$\begin{aligned} \partial_0^n D &= \left\{x \in \partial D \cap \overline{U(\partial_0 D)} : |\tilde{\psi}(x)| < \frac{1}{2n}\right\}, \\ \tilde{C}_n &= \frac{1}{\inf_{\partial_0^n D} \nabla\tilde{\psi}(x)l(x)}, \\ C_n &= \frac{\tilde{C}_n \sup_{\partial D} |\nabla\tilde{\psi}(x)l(x)| + 1}{\inf_{\partial D - \partial_0^n D} \nabla\tilde{\psi}(x)l(x)}. \end{aligned}$$

Let χ_n be a function in $C^\infty(\mathbf{R})$ such that $\chi_n(c) = nc$ for $|c| \leq \frac{1}{2n}$, $\chi_n(c) = -1$ for $c \leq -\frac{1}{n}$, $\chi_n(c) = 1$ for $c \geq \frac{1}{n}$, $0 \leq \chi'_n(c) \leq n$ for every $c \in \mathbf{R}$, and define

$$\varphi_n(x) = \chi_n(C_n\psi(x)) + \tilde{C}_n\chi_n(\tilde{\psi}(x)).$$

Then $|\varphi_n(x)|$ is bounded by $1 + \frac{1}{\inf_{\partial_0^n D} \nabla\tilde{\psi}(x)l(x)}$ and we have, for $x \in \partial_0^n D$,

$$\nabla\varphi_n(x)l(x) = n \left[C_n \nabla\psi(x)l(x) + \tilde{C}_n \nabla\tilde{\psi}(x)l(x) \right] \geq n,$$

and for $x \in \partial D - \partial_0^n D$,

$$\nabla \varphi_n(x)l(x) = nC_n \nabla \psi(x)l(x) + \tilde{C}_n \chi'_n(\tilde{\psi}(x)) \nabla \tilde{\psi}(x)l(x) \geq n.$$

If $\partial_0 D$ is not connected, there is a function $\tilde{\psi}^k$ satisfying (6.6) for each connected component $\partial_0^k D$. Let $U^k(\partial_0^k D)$ be neighborhoods such that $\inf_{\overline{U^k(\partial_0^k D)}} \nabla \tilde{\psi}^k(x)l(x) > 0$. We can assume, without loss of generality, that $\overline{U^k(\partial_0^k D)} \subseteq V^k(\partial_0^k D)$, where $V^k(\partial_0^k D)$ are pairwise disjoint and $\tilde{\psi}^k$ vanishes outside $V^k(\partial_0^k D)$. Then, defining $\partial_0^{k,n} D$ and \tilde{C}_n^k as above,

$$C_n^k = \frac{\tilde{C}_n^k \sup_{\partial D} |\nabla \tilde{\psi}^k(x)l(x)| + 1}{\inf_{\partial D - \cup_k \partial_0^{k,n} D} \nabla \psi(x)l(x)},$$

and φ_n^k as above, $\varphi_n(x) = \sum_k \varphi_n^k(x)$ verifies iii) of Proposition 5.13. □

Theorem 2.6 of [25] gives the comparison principle for a class of linear and nonlinear equations that includes, in particular, the partial differential equation with boundary conditions

$$\begin{aligned} \lambda u(x) - Au(x) &= h(x), & \text{in } D, \\ -Bu(x) &= 0, & \text{on } \partial D, \end{aligned} \tag{6.7}$$

where h is a Lipschitz continuous function, and A, B are given by (6.4), (6.5) and verify, in addition to the the assumptions formulated at the beginning of this section, the following local condition on $\partial_0 D$.

Condition 6.2.

For every $x_0 \in \partial_0 D$, let ϕ be a C^2 diffeomorphism from the closure of a suitable neighborhood V of the origin into the closure of a suitable neighborhood of x_0 , $U(x_0)$, such that $\phi(0) = x_0$ and the d th column of $J\phi(z)$, $J_d\phi(z)$, satisfies

$$J_d\phi(z) = -l(\phi(z)), \quad \forall z \in \phi^{-1}(\overline{\partial D \cap U(x_0)}). \tag{6.8}$$

Let \tilde{A} ,

$$\tilde{A}f(z) = \frac{1}{2} \text{tr} (D^2 f(z) \tilde{\sigma}(z) \tilde{\sigma}^T(z)) + \nabla f(z) \tilde{b}(z),$$

be the operator such that

$$\tilde{A}(f \circ \phi)(z) = Af(\phi(z)), \quad \forall z \in \phi^{-1}(\overline{D \cap U(x_0)}).$$

Assume

- a) $\tilde{b}^i, i = 1, \dots, d - 1$, is a function of the first $d - 1$ coordinates (z^1, \dots, z^{d-1}) only, and \tilde{b}_d is a function of z_d only.
- b) $\tilde{\sigma}^{ij}, i = 1, \dots, d - 1, j = 1, \dots, d$ is a function of the first $d - 1$ coordinates (z^1, \dots, z^{d-1}) only.

Remark 6.3. For every $x_0 \in \partial_0 D$, some coordinate of $l(x_0)$, say the d th coordinate, must be nonzero. Then in (6.8) we can choose $U(x_0)$ such that in $\overline{U(x_0)}$ $l^d(x) \neq 0$ and we can replace $l(x)$ by $l(x)/|l^d(x)|$, since this normalization does not change the boundary condition of (6.7) in $D \cap U(x_0)$ (i.e. any viscosity sub/supersolution of (6.7) in $\overline{D \cap U(x_0)}$ is a viscosity sub/supersolution of (6.7) in $\overline{D \cap U(x_0)}$ with the normalized vector field and conversely).

Moreover, since (6.8) must be verified only in $\phi^{-1}(\overline{\partial D \cap U(x_0)})$, in the construction of ϕ we can use any C^2 vector field \tilde{l} that agrees with l , or the above normalization of l , on $\partial D \cap U(x_0)$.

Therefore, whenever Condition 6.2 is satisfied Theorem 5.11 applies and there exists one and only one diffusion process reflecting in D according to the degenerate oblique direction of reflection l .

The following is a concrete example where Condition 6.2 is satisfied.

Example 6.4. Let

$$D = B_1(0) \subseteq \mathbf{R}^2.$$

and suppose the direction of reflection l satisfies (6.1) with the strict inequality at every $x \in \partial D$ except at $x_0 = (1, 0)$, where

$$l(1, 0) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Of course, in a neighborhood of $x_0 = (1, 0)$ we can always assume that l depends only on the second coordinate x_2 . In addition, by Remark 6.3, we can suppose

$$l_2(x) = -1.$$

Consider

$$\sigma(x) = \sigma = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

and a drift b that, in a neighborhood of $x_0 = (1, 0)$, depends only on the second coordinate, i.e.

$$b(x) = b(x_2).$$

Assume that, in a neighborhood of $x_0 = (1, 0)$, the direction of reflection l is parallel to b . Then we can find a nonlinear change of coordinates ϕ such that Condition 6.2 is verified, namely

$$\begin{aligned} \phi_2(z) &= z_2 \\ \phi_1(z) &= -\int_0^{z_2} l_1(\zeta_2) d\zeta_2 + z_1 + 1, \end{aligned}$$

which yields

$$\tilde{\sigma}(z) = \sigma, \quad \tilde{b}_1(z) = 0, \quad \tilde{b}_2(z) = b_2(z_2).$$

6.2 Jump diffusions with degenerate diffusion matrix

Consider the operator

$$Af = Lf + Jf$$

$$Lf(x) = \frac{1}{2} \text{tr} (a(x) D^2 f(x)) + \nabla f(x) b(x) \tag{6.9}$$

$$Jf(x) = \int_{\mathbf{R}^{d'} - \{0\}} [f(x + \eta(x, z)) - f(x) - \nabla f(x) \eta(x, z) I_{|z| < 1}] m(dz),$$

where ∇ is viewed as a row vector. Assume:

Condition 6.5.

- a) $a = \sigma \sigma^T$, σ and b are continuous.
- b) $\eta(\cdot, z)$ is continuous for every z , $\eta(x, \cdot)$ is Borel measurable for every x , $\sup_{|z| < 1} |\eta(x, z)| < +\infty$ for every x and

$$|\eta(x, z) I_{|z| < 1}| \leq \rho(z) (1 + |x|) I_{|z| < 1},$$

for some positive, measurable function ρ such that $\lim_{|z| \rightarrow 0} \rho(z) = 0$.

c) m is a Borel measure such that

$$\int_{\mathbb{R}^{d'} - \{0\}} [\rho(z)^2 I_{|z| < 1} + I_{|z| \geq 1}] m(dz) < +\infty.$$

Then, with $\mathcal{D}(A) = \{f + c : f \in C_c^2(\mathbb{R}^d), c \in \mathbb{R}\}$, $A \subset C_b(E) \times C_b(E)$ and A satisfies Condition 3.1.

A comparison principle for bounded subsolutions and supersolutions of the equation (2.11) when A is given by (6.9) is proven in [16], as a special case of a more general result, under the following assumptions:

Condition 6.6.

a) σ and b are Lipschitz continuous.

b)

$$|\eta(x, z) - \eta(y, z)| I_{|z| < 1} \leq \rho(z) |x - y| I_{|z| < 1}.$$

c) h is uniformly continuous.

Then, under the above assumptions, our result of Theorem 3.7 applies and uniqueness of the solution of the martingale problem for A is granted.

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