

Existence and uniqueness of reflecting diffusions in cusps

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Abstract

We consider stochastic differential equations with (oblique) reflection in a 2-dimensional domain that has a cusp at the origin, i.e. in a neighborhood of the origin has the form $\{(x_1, x_2) : 0 < x_1 \leq \delta_0, \psi_1(x_1) < x_2 < \psi_2(x_1)\}$, with $\psi_1(0) = \psi_2(0) = 0$, $\psi_1'(0) = \psi_2'(0) = 0$.

Given a vector field g of directions of reflection at the boundary points other than the origin, defining directions of reflection at the origin $g^i(0) := \lim_{x_1 \rightarrow 0^+} g(x_1, \psi_i(x_1))$, $i = 1, 2$, and assuming there exists a vector e^* such that $\langle e^*, g^i(0) \rangle > 0$, $i = 1, 2$, and $e_1^* > 0$, we prove weak existence and uniqueness of the solution starting at the origin and strong existence and uniqueness starting away from the origin.

Our proof uses a new scaling result and a coupling argument.

Key words: Oblique reflection, stochastic differential equation, diffusion process, cusp, boundary singularity.

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1 Introduction

In this work we prove existence and uniqueness of reflecting diffusions in 2-dimensional domains with cusps. By saying that the domain has a cusp, we mean that in a neighborhood of some point, which we take to be the origin, the domain, \mathcal{O} , has the form

$$\{(x_1, x_2) \in \mathcal{O} : 0 < x_1 \leq \delta_0\} = \{(x_1, x_2) : 0 < x_1 \leq \delta_0, \psi_1(x_1) < x_2 < \psi_2(x_1)\},$$

where we assume ψ_1 and ψ_2 are C^1 with

$$\psi_1(0) = \psi_2(0) = 0, \quad \psi_1'(0) = \psi_2'(0) = 0,$$

and, in general, the boundary is C^1 away from the origin.

The direction of reflection, g , is assumed Lipschitz continuous on the smooth part of the boundary, with a uniformly positive scalar product with the inward normal. At the tip,

$$g^i(0) := \lim_{x_1 \rightarrow 0^+} g(x_1, \psi_i(x_1)), \quad i = 1, 2,$$

is assumed to exist, and for some $e^* \in \mathbb{R}^2$

$$\langle e^*, g \rangle > 0, \quad \forall g \in \left\{ \left[\begin{array}{c} 1 \\ 0 \end{array} \right], g^1(0), g^2(0) \right\}.$$

See Section 2 for the complete formulation of our assumptions.

In the case of a domain \mathcal{O} of the form

$$\mathcal{O} := \{(x_1, x_2) : 0 < x_1, \psi_1(x_1) := -x_1^{\beta_1} < x_2 < \psi_2(x_1) := x_1^{\beta_2}\},$$

(where either $\beta_1 = \beta_2 > 1$ or $\beta_1 > 2\beta_2 - 1$, $\beta_2 > 1$), under the assumption that on each of $\{(x_1, x_2) : 0 < x_1, x_2 = -x_1^{\beta_1}\}$ and $\{(x_1, x_2) : 0 < x_1, x_2 = x_1^{\beta_2}\}$ the direction of reflection forms a constant angle with the inward normal, weak existence and uniqueness of reflecting Brownian motions have been exhaustively studied by DeBlassie and Toby (1993a). DeBlassie and Toby (1993b) gives a complete characterization of the cases in which, in the above setup, the reflecting Brownian motion is a semimartingale. For general continuous ψ_1, ψ_2 ($\psi_1(0) = \psi_2(0) = 0$, $\psi_2(x_1) > \psi_1(x_1)$ for every $x_1 > 0$), the case when the direction of reflection on $\{(x_1, x_2) : 0 < x_1, x_2 = \psi_1(x_1)\}$ and $\{(x_1, x_2) : 0 < x_1, x_2 = \psi_2(x_1)\}$ is given by constant, opposite vertical vectors - a case when the process is not a semimartingale - has been studied by Burdzy and Toby (1995) and Burdzy et al. (2009). In higher dimension, normally reflecting diffusions in domains with Hölder cusps have been studied by Fukushima and Tomisaki (1996) by analytical techniques.

Here we characterize the reflecting diffusion as the solution of a stochastic differential equation with reflection (SDER), which will always be a semimartingale. In particular, we recover the results by DeBlassie and Toby (1993a) and DeBlassie and Toby (1993b) for the cases when the process is a semimartingale, except for the case when $g^1(0)$ and $g^2(0)$ point at each other and $\beta_2 < 2$.

First, we show that our conditions imply that, starting away from the origin, the origin is never reached. Therefore we easily obtain strong existence and uniqueness of the reflecting diffusion starting away from the origin from known results on existence and uniqueness in smooth domains (Section 3).

Moreover, the fact that, starting away from the origin, the reflecting diffusion is well defined for all times allows us to obtain a weak solution of the SDER starting at the origin as the limit of solutions starting away from the origin (Section 4.1). To this end, we employ a random time change of the SDER (the same that is used in Kurtz (1990) to obtain a solution of a patchwork martingale problem from a solution of the corresponding constrained

martingale problem) that makes it particularly simple to prove relative compactness of the processes.

The main result of this paper, however, is weak uniqueness of the solution to the SDER starting at the origin (Section 4.3). Our assumptions on the direction of reflection guarantee that any solution starting at the origin immediately leaves it. Since the distribution of a solution starting away from the origin is uniquely determined, the distribution of a solution starting at the origin is determined by its exit distribution from an arbitrarily small neighborhood of the origin. The crucial ingredient that allows us to understand the behavior of the process near the origin is a scaling result (Section 4.2). Combined with an adaptation of Lindvall and Rogers (1986), this scaling result allows us to use a coupling argument to show that indeed all solutions starting at the origin must have the same exit distribution from every neighborhood of the origin. For a more detailed discussion of our approach, see the beginning of Section 4. Some technical lemmas that are needed in our argument are proved in Section 5. The Feller property is proved in Section 4.4.

The most general uniqueness result for SDER in piecewise \mathcal{C}^1 domains can be found in Dupuis and Ishii (1993). Reflecting diffusions in piecewise smooth domains are characterized as solutions of constrained martingale problems in Kurtz (1990), and Costantini and Kurtz (2015) reduces the problem of proving uniqueness for the solution of a constrained martingale problem (as well as of a martingale problem in a general Polish space) to that of proving a comparison principle for viscosity semisolutions of the corresponding resolvent equation. None of these results applies to the situation we are considering here. In particular, Dupuis and Ishii (1993) makes the assumption that the convex cone generated by the normal vectors at each point does not contain any straight line, which is violated at the tip of the cusp.

Finally, we wish to mention that our work was partly motivated by diffusion approximations for some queueing models where domains with cusplike singularities appear (see e.g. Kang et al. (2009) and Kang and Williams (2012)). These models are in higher dimension, but this paper is intended as a first contribution in the direction of understanding reflecting diffusions in such domains.

1.1 Notation

$\langle \cdot, \cdot \rangle$ denotes the scalar product of two vectors.

For any matrix M (or vector v), M^T (v^T) denotes its transpose.

$\text{tr}M$ denotes the trace of a matrix.

$\mathbf{1}_E$ is the indicator function of a set E .

$B_r(x) \subseteq \mathbb{R}^d$ is the ball of radius r centered at x and $S_r(x) \subseteq \mathbb{R}^d$ is the sphere of radius r centered at x .

$|\cdot|$ denotes indifferently the absolute value of a number, the norm of a vector or of a matrix, while $\|\cdot\|$ denotes the supremum norm of a bounded, real valued function.

For $f : \mathbb{R}^{d_1} \rightarrow \mathbb{R}^{d_2}$ with first order partial derivatives, Df denotes the Jacobian matrix of f .

For $f : \mathbb{R}^d \rightarrow \mathbb{R}$ with second order partial derivatives D^2f denotes the Hessian matrix.

For an open set $E \subseteq \mathbb{R}^d$, $\mathcal{C}^i(E)$ denotes the set of real valued functions defined on E with continuous partial derivatives up to the order i . For E closed, $\mathcal{C}^i(E)$ denotes the set of

real valued functions defined on an open neighborhood of E that admit continuous partial derivatives up to the order i .

For a complete, separable, metric space E , $\mathcal{C}_E[0, \infty)$ is the space of E valued, continuous functions on $[0, \infty)$.

$\mathcal{L}(\cdot)$ denotes the law (distribution) of a random variable or a stochastic process.

The total variation distance between two probability measure $\mu_1, \mu_2 \in \mathcal{P}(E)$ on (E, \mathcal{B}) is denoted $\|\mu_1 - \mu_2\|_{TV}$, that is

$$\|\mu_1 - \mu_2\|_{TV} = \sup_{A \in \mathcal{B}} |\mu_1(A) - \mu_2(A)|.$$

Throughout the paper c and C denote positive constants depending only on the data of the problem. When necessary, they are indexed $c_0, c_1, \dots, C_0, C_1, \dots$ and the dependence on the data or other parameters is explicitly pointed out.

2 Formulation of the problem and assumptions

We are interested in studying diffusion processes with oblique reflection in the closure of a simply connected 2-dimensional domain $\mathcal{O} \subset [0, \infty) \times \mathbb{R}$ with a boundary $\partial\mathcal{O}$ that is C^1 except at a single point (which we will take to be the origin 0), where the domain has a cusp. More precisely \mathcal{O} satisfies the following.

Condition 2.1

- (a) \mathcal{O} is a bounded, simply connected domain in $[0, \infty) \times \mathbb{R}$ with $0 \in \partial\mathcal{O}$.
- (b) $\partial\mathcal{O}$ is C^1 except at 0.
- (c) There exists a $\delta_0 > 0$ and $\psi_1, \psi_2 \in C^1(\mathbb{R})$ with $\psi_1 \leq \psi_2$ and

$$\psi_1(0) = \psi_2(0) = 0, \quad \psi_1'(0) = \psi_2'(0) = 0$$

such that

$$\{(x_1, x_2) \in \mathcal{O} : x_1 \leq \delta_0\} = \{(x_1, x_2) : 0 < x_1 \leq \delta_0, \psi_1(x_1) < x_2 < \psi_2(x_1)\},$$

and

$$\lim_{x_1 \rightarrow 0^+} \frac{\psi_1(x_1)}{\psi_2(x_1) - \psi_1(x_1)} = L \in (-\infty, \infty).$$

Remark 2.2 *The last requirement in Condition 2.1(c) is used in the scaling of Section 4.2, specifically in (4.13) and (4.14). Heuristically, it ensures that the cusp is not "too narrow".*

For $x \in \partial\mathcal{O} - \{0\}$, let $n(x)$ denote the unit inward normal vector at x . The direction of reflection is assigned at all points of the boundary except the origin and is given by a unit vector field g verifying the following condition.

Condition 2.3

a) $g : \partial\mathcal{O} - \{0\} \rightarrow \mathbb{R}^2$ is locally Lipschitz continuous and satisfies

$$\inf_{x \in \partial\mathcal{O}-0} \langle g(x), n(x) \rangle > 0.$$

The mappings

$$x_1 \in (0, \delta_0] \rightarrow g^i(x_1) := g(x_1, \psi_i(x_1)), \quad i = 1, 2,$$

are Lipschitz continuous and hence the limits

$$g^i(0) := \lim_{x_1 \rightarrow 0^+} g(x_1, \psi_i(x_1)), \quad i = 1, 2,$$

exist.

b) Let $G(0)$ be the convex cone generated by

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, g^1(0), g^2(0) \right\}.$$

There exists $e^* \in \mathbb{R}^2$ such that

$$\langle e^*, g \rangle > 0, \quad \forall g \in G(0).$$

Of course, without loss of generality, we can suppose that $|e^*| = 1$.

Remark 2.4 Condition 2.3(b) can be reformulated as follows. In a neighborhood of the origin, we can view \mathcal{O} as being the intersection of three \mathcal{C}^1 domains,

$$\{x : x_2 > \psi_1(x_1)\}, \quad \{x : x_2 < \psi_2(x_1)\}, \quad \{x : x_1 > 0\},$$

with unit inward normal vector at the origin, respectively,

$$n^1(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad n^2(0) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad n^0(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then, letting the normal cone at the origin, $N(0)$, be the closed, convex cone generated by $\{n^1(0), n^2(0), n^3(0)\}$, Condition 2.3(b) is equivalent to requiring that there exists $e^* \in N(0)$ such that

$$\langle e^*, g \rangle > 0, \quad \forall g \in G(0),$$

where we can think of $G(0)$ as the closed, convex cone generated by the directions of reflection at the origin for each of the three domains. In other terms, Condition 2.3(b) is the analog of the condition usually assumed in the literature for polyhedral domains (see e.g. Varadhan and Williams (1985), Taylor and Williams (1993) or Dai and Williams (1996)). Note that, in contrast, the condition that there exists $e^{**} \in G(0)$ such that

$$\langle e^{**}, n \rangle > 0, \quad \forall n \in N(0),$$

can never be satisfied at a cusp, because $n^2(0) = -n^1(0)$.

As in the literature on polyhedral domains, Condition 2.3(b) ensures existence of a semi-martingale reflecting diffusion starting at the origin (see the proof of Theorem 4.1).

Remark 2.5 Note that, under Condition 2.3, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ can be expressed as a positive linear combination of $g^1(0)$ and $g^2(0)$, so that $G(0)$ coincides with the closed, convex cone generated by

$$\{g^1(0), g^2(0)\}.$$

We seek to characterize the diffusion process with directions of reflection g as the solution of a stochastic differential equation driven by a standard Brownian motion W :

$$\begin{aligned} X(t) &= X(0) + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dW(s) + \int_0^t \gamma(s)d\Lambda(s), & t \geq 0, \\ \gamma(t) &\in G_1(X(t)), & d\Lambda - a.e., \quad t \geq 0, \\ X(t) &\in \bar{\mathcal{O}}, & \int_0^t \mathbf{1}_{\partial\mathcal{O}}(X(s))d\Lambda(s) = \Lambda(t), \quad t \geq 0, \end{aligned} \quad (2.1)$$

where Λ is nondecreasing, $G_1(0)$ is the convex hull of $g^1(0)$ and $g^2(0)$ and for $x \in \partial\mathcal{O} - \{0\}$ $G_1(x) := \{g(x)\}$, and γ is almost surely measurable.

We make the following assumptions on the coefficients.

Condition 2.6

- a) σ and b are Lipschitz continuous on $\bar{\mathcal{O}}$.
- b) $(\sigma\sigma^T)(0)$ is nonsingular.

We will denote

$$Af(x) := Df(x)b(x) + \frac{1}{2}\text{tr}((\sigma\sigma^T)(x)D^2f(x)). \quad (2.2)$$

Definition 2.7 A stochastic process X is compatible with a Brownian motion W if for each $t \geq 0$, $W(t + \cdot) - W(t)$ is independent of $\mathcal{F}_t^{W, X}$, where $\{\mathcal{F}_t^{W, X}\}$ is the filtration generated by W and X .

Definition 2.8 Given a standard Brownian motion W and $X(0) \in \bar{\mathcal{O}}$ independent of W , X is a strong solution of (2.1) if X is adapted to the filtration generated by $X(0)$ and W and the equation is satisfied for some γ and Λ . (Note that (2.1) immediately implies that $\int_0^t \gamma(s)d\Lambda(s)$ is adapted to $\{\mathcal{F}_t^{W, X}\}$ although γ and Λ need not be.)

(X, W) , defined on some probability space, is a weak solution of (2.1) if W is a standard Brownian motion, X is compatible with W , and the equation is satisfied for some γ and Λ .

Given an initial distribution $\mu \in \mathcal{P}(\bar{\mathcal{O}})$, weak uniqueness or uniqueness in distribution holds if for all weak solutions with $P\{X(0) \in \cdot\} = \mu$, X has the same distribution on $\mathcal{C}_{\bar{\mathcal{O}}}[0, \infty)$.

Strong uniqueness holds if for any standard Brownian motion W and weak solutions (X, W) , (\tilde{X}, W) such that $X(0) = \tilde{X}(0)$ a.s. and (X, \tilde{X}) is compatible with W , $X = \tilde{X}$ a.s.

Remark 2.9 *Of course, any strong solution is a weak solution. Existence of a weak solution and strong uniqueness imply that the weak solution is a strong solution (c.f., Yamada and Watanabe (1971) and Kurtz (2014)).*

For processes starting away from the tip, existence and uniqueness follows from results of Dupuis and Ishii (1993) and the fact that under our conditions, the solution never hits the tip. For processes starting at the tip, we only prove weak existence and uniqueness.

3 Strong existence and uniqueness starting at $x^0 \neq 0$

Our first result is that, for every $x^0 \in \overline{\mathcal{O}} - \{0\}$, (2.1) has a unique strong solution with $X(0) = x^0$, well-defined for all times. In fact, by Dupuis and Ishii (1993), for each $n > 0$, the solution, X , is well-defined up to

$$\tau_n := \inf\{t \geq 0 : X_1(t) \leq \frac{1}{n}\}, \quad (3.1)$$

so the proof consists in showing that, almost surely,

$$\lim_{n \rightarrow +\infty} \tau_n = +\infty. \quad (3.2)$$

We will do this by means of a modification of the Lyapunov function used in Section 2.2 of Varadhan and Williams (1985).

Theorem 3.1 *Let W be a standard Brownian motion. Then, for every $x^0 \in \overline{\mathcal{O}} - \{0\}$, there is a unique strong solution to (2.1) with $X(0) = x^0$.*

Proof. As anticipated above, by Dupuis and Ishii (1993) there is one and only one stochastic process X that satisfies (2.1) for $t < \lim_{n \rightarrow +\infty} \tau_n$, where τ_n is defined by (3.1). Therefore, we only have to prove (3.2). Define

$$V(x) := |(\sigma\sigma^T)(0)^{-1/2}x|^{-p} \cos(\vartheta((\sigma\sigma^T)(0)^{-1/2}x) + \xi),$$

where $\vartheta(z) \in (-\pi, \pi]$ is the angular polar coordinate of z , $p \in (0, 1)$ and

$$\xi := \vartheta((\sigma\sigma^T)(0)^{1/2}e^*) - 2\vartheta_0, \quad \vartheta_0 := \lim_{x \in \overline{\mathcal{O}} - \{0\}, x \rightarrow 0} \vartheta((\sigma\sigma^T)(0)^{-1/2}x),$$

(notice that $-\frac{\pi}{2} < \vartheta_0 < \frac{\pi}{2}$). Then one can check that, if p is taken sufficiently close to 1,

$$\begin{aligned} \lim_{x \in \overline{\mathcal{O}} - \{0\}, x \rightarrow 0} V(x) &= +\infty \\ \lim_{x \in \overline{\mathcal{O}} - \{0\}, x \rightarrow 0} AV(x) &= -\infty, \\ \lim_{x \in \partial\mathcal{O} - \{0\}, x \rightarrow 0} DV(x)g(x) &= -\infty. \end{aligned}$$

Therefore there exists $0 < \delta \leq \delta_0$ such that

$$\inf_{x \in \overline{\mathcal{O}} - 0, x_1 \leq \delta} V(x) > 0,$$

$$\sup_{x \in \overline{\mathcal{O}} - 0, x_1 \leq \delta} AV(x) < 0$$

and

$$\sup_{x \in \partial\mathcal{O} - 0, x_1 \leq \delta} DV(x)g(x) < 0.$$

Let

$$\alpha_\delta = \inf\{t \geq 0 : X_1(t) \leq \delta/2\}$$

and

$$\beta_\delta = \inf\{t \geq \alpha_\delta : X_1(t) \geq \delta\}.$$

Without loss of generality, we can assume that $x_1^0 > \delta$.

By Itô's formula, for $n^{-1} < \delta/2$,

$$\begin{aligned} V(X(t \wedge \tau_n \wedge \beta_\delta)) &= V(X(t \wedge \alpha_\delta)) + \int_{t \wedge \alpha_\delta}^{t \wedge \tau_n \wedge \beta_\delta} AV(X(s))ds \\ &\quad + \int_{t \wedge \alpha_\delta}^{t \wedge \tau_n \wedge \beta_\delta} DV(X(s))\sigma(X(s))dW(s) + \int_{t \wedge \alpha_\delta}^{t \wedge \tau_n \wedge \beta_\delta} DV(X(s))g(s)d\Lambda(s). \\ &\leq V(X(t \wedge \alpha_\delta)) + \int_{t \wedge \alpha_\delta}^{t \wedge \tau_n \wedge \beta_\delta} DV((X(s))\sigma(X(s))dW(s), \end{aligned}$$

Hence, by multiplying by $\mathbf{1}_{\{\alpha_\delta < \infty\}}$, taking expectations and letting t go to infinity,

$$\mathbb{E}[V(X(\tau_n \wedge \beta_\delta))\mathbf{1}_{\{\alpha_\delta < \infty\}}] \leq \mathbb{E}[V(X(\alpha_\delta))\mathbf{1}_{\{\alpha_\delta < \infty\}}].$$

Thus,

$$\inf_{x \in \overline{\mathcal{O}}, x_1 = 1/n} V(x)P\{\tau_n < \beta_\delta | \alpha_\delta < \infty\} + \inf_{x \in \overline{\mathcal{O}}, x_1 = \delta} V(x)P\{\beta_\delta < \tau_n | \alpha_\delta < \infty\} \leq \sup_{x \in \overline{\mathcal{O}}, x_1 = \delta/2} V(x).$$

Consequently, if X_1 hits $\delta/2$, then, with probability one, it hits δ before it hits 0. In particular, with probability one, X_1 never hits 0. \square

Remark 3.2 *Theorem 3.1 implies existence and uniqueness of a strong solution to (2.1) for every initial condition such that $\mathbb{P}(X(0) \in \overline{\mathcal{O}} - \{0\}) = 1$ which in turn implies existence and uniqueness in distribution of a weak solution to (2.1) for every initial distribution μ such that $\mu(\overline{\mathcal{O}} - \{0\}) = 1$.*

4 Weak existence and uniqueness starting at $x^0 = 0$

In this section we prove weak existence and uniqueness for the solution of (2.1) starting at the origin.

In order to prove existence (Theorem 4.1), we start with a sequence of solutions to (2.1) starting at $x^n \in \overline{\mathcal{O}} - \{0\}$, where $\{x^n\}$ converges to the origin. For every n , we consider a random time change of the solution, the same time change that is used in Kurtz (1990) to construct a solution to a patchwork martingale problem from a solution to the corresponding

constrained martingale problem. The time changed processes and the time changes are relatively compact, and any limit point satisfies the time changed version of (2.1) with $X(0) = 0$. The key point of the proof is to show that the limit time change is invertible. The process obtained is a weak solution to (2.1) defined for all times.

Weak uniqueness of the solution of (2.1) starting at the origin (Theorem 4.7 below) is the main result of this paper. Our proof takes inspiration from the one used in Taylor and Williams (1993) for reflecting Brownian motion in the nonnegative orthant. The argument of that paper, in the case when, starting away from the origin, the origin is not reached, can essentially be reformulated as follows: First, it is shown that, for any solution of the SDER starting at the origin, the exit time from $B_\delta(0)$, $\delta > 0$, is finite and tends to zero as $\delta \rightarrow 0$, almost surely, and that any two solutions of the SDER, starting at the origin, that have the same exit distributions from $B_\delta(0)$ for all $\delta > 0$ sufficiently small, have the same distribution; next it is proved that, for any $\xi \in \partial B_1(0)$, ξ in the nonnegative orthant, letting $X^{\delta\xi}$ be the solution of the SDER starting at $\delta\xi$ and $\tau_{2\delta}$ be its exit time from $B_{2\delta}(0)$, $\mathbb{P}(X^{\delta\xi}(\tau_{2\delta})/(2\delta) \in \cdot)$ is independent of δ and hence defines the transition kernel of a Markov chain on $\partial B_1(0)$. This Markov chain is shown to be ergodic and that in turn ensures that, for any initial distribution on $\overline{\mathcal{O}} \cap (\partial B_{\delta/2^n}(0))$, the exit distribution from $B_\delta(0)$ converges, as n goes to infinity, to a uniquely determined distribution. Consequently, any two solutions of the SDER starting at the origin have the same exit distributions from $B_\delta(0)$.

The first part of our argument is the same as in Taylor and Williams (1993), except that we find it more convenient to use the exit distribution from $\{x : x_1 < \delta\}$ rather than from $B_\delta(0)$. We prove that, for any solution of (2.1) starting at the origin, the exit time from $\{x : x_1 < \delta\}$, $\delta > 0$, is finite and tends to zero as $\delta \rightarrow 0$, almost surely (Lemma 4.2), and we construct a sequence $\{\delta_n\}$, of positive numbers decreasing to zero such that any two solutions of (2.1) starting at the origin have the same exit distributions from $\{x : x_1 < \delta_n\}$ which in turn implies the solutions have the same distribution (Lemma 4.6). This construction cannot be obtained by the arguments used in Taylor and Williams (1993). Instead, it is achieved by a coupling argument based on the rescaling result of Section 4.2, together with an adaptation of Lindvall and Rogers (1986) (Lemma 5.3).

4.1 Existence

Theorem 4.1 *There exists a weak solution to (2.1) starting at $x^0 = 0$.*

Proof. Consider a sequence $\{x^n\} \subseteq \overline{\mathcal{O}} - \{0\}$ that converges to the origin. Let $(X^n, \gamma^n, \Lambda^n)$ satisfy (2.1) starting at x^n . Define

$$K_0^n(t) := \inf\{s \geq 0 : s + \Lambda^n(s) > t\},$$

and set

$$Y^n(t) := X^n(K_0^n(t)), \quad M^n(t) := W(K_0^n(t)), \quad K_1^n(t) := \Lambda^n(K_0^n(t)), \quad \eta^n(t) := \gamma^n(K_0^n(t)).$$

Then K_0^n, K_1^n are nonnegative and nondecreasing,

$$K_0^n(t) + K_1^n(t) = t, \quad t \geq 0,$$

and, by (2.1),

$$Y^n(t) = x^n + \int_0^t \sigma(Y^n(s))dM^n(s) + \int_0^t b(Y^n(s))dK_0^n(s) + \int_0^t \eta^n(s)dK_1^n(s), \quad (4.1)$$

$$Y^n(t) \in \overline{\mathcal{O}}, \quad \eta^n(t) \in G_1(Y^n(t)), \quad dK_1^n - a.e., \quad \int_0^t \mathbf{1}_{\partial\mathcal{O}}(Y^n(s))dK_1^n(s) = K_1^n(t),$$

where M^n is a continuous, square integrable martingale with quadratic variation

$$[M^n](t) = K_0^n(t)I, \quad t \geq 0.$$

With reference to Theorem 5.4 in Kurtz and Protter (1991), let

$$U^n(t) = \int_0^t \eta^n(s)dK_1^n(s).$$

Since U^n , K_0^n , and K_1^n are all Lipschitz continuous with Lipschitz constants bounded by 1, $\{(Y^n, U^n, M^n, K_0^n, K_1^n)\}$ is relatively compact in distribution in the appropriate space of continuous functions. Taking a convergent subsequence with limit (Y, U, M, K_0, K_1) , Y satisfies

$$Y(t) = Y(0) + \int_0^t \sigma(Y(s))dM(s) + \int_0^t b(Y(s))dK_0(s) + U(t),$$

where $M(t) = W(K_0(t))$ for a standard Brownian motion W . Since $|U^n(s) - U^n(t)| \leq |K_1^n(s) - K_1^n(t)|$, the same inequality holds for U and K_1 and hence

$$U(t) = \int_0^t \eta(s)dK_1(s).$$

It remains only to characterize η .

Invoking the Skorohod representation theorem, we assume that $(Y^n, U^n, M^n, K_0^n, K_1^n) \rightarrow (Y, U, M, K_0, K_1)$ uniformly over compact time intervals, almost surely. Then the argument of Theorem 3.1 of Costantini (1992) yields that

$$\eta(t) \in G_1(Y(t)), \quad dK_1 - a.e., \quad \int_0^t \mathbf{1}_{\partial\mathcal{O}}(Y(s))dK_1(s) = K_1(t). \quad (4.2)$$

Finally, let us show that K_0 is invertible and K_0^{-1} is defined on all $[0, \infty)$. Suppose, by contradiction, that K_0 is constant on some time interval $[t_1, t_2]$, $0 \leq t_1 < t_2$. Then, since

$$K_0(t) + K_1(t) = t, \quad (4.3)$$

we have

$$K_1(s) - K_1(t) = s - t, \quad \text{for } t_1 \leq t < s \leq t_2. \quad (4.4)$$

In particular, by (4.2), $Y(t) \in \partial\mathcal{O}$ for all $t \in [t_1, t_2]$. If, for some $t \in [t_1, t_2]$, $Y(t) \in \partial\mathcal{O} - \{0\}$, then for $s > t$ close enough to t , we have $Y(r) \in \partial\mathcal{O} - \{0\}$, $r \in [t, s]$ and, by Condition 2.3(a),

$$\inf_{r \in [t, s]} \langle g(Y(r)), \nu(Y(t)) \rangle > 0.$$

Hence

$$\langle Y(s) - Y(t), n(Y(t)) \rangle = \int_t^s \langle g(Y(r)), n(Y(t)) \rangle dK_1(r) > 0,$$

which implies, for s close enough to t ,

$$Y(s) \in \mathcal{O},$$

and this contradicts (4.4). On the other hand, if $Y(t) = 0$ for all $t \in [t_1, t_2]$, then,

$$\int_{t_1}^t \eta(r) dK_1(r) = 0,$$

while Condition 2.3(b) gives

$$\left\langle \int_{t_1}^t \eta(r) dK_1(r), e^* \right\rangle \geq \inf_{G_1(0)} \langle g, e^* \rangle (t - t_1) > 0.$$

Therefore K_0 is strictly increasing.

In order to see that K_0 diverges as t goes to infinity, we can use the argument of Lemma 1.9 of Kurtz (1990), provided there is a \mathcal{C}^2 function φ such that

$$\inf_{x \in \partial \mathcal{O}} \inf_{g \in G_1(x)} D\varphi(x)g > 0. \quad (4.5)$$

Let e^* be the vector in Condition 2.3(b), and let $r^* > 0$ be such that

$$\inf_{x \in \partial \mathcal{O}, \langle e^*, x \rangle \leq 2r^*} \inf_{g \in G_1(x)} \langle e^*, g \rangle > 0.$$

By Condition 2.1(b),

$$\{x \in \mathcal{O} : \langle e^*, x \rangle \geq r^*\} = \{x : \Psi(x) > 0, \langle e^*, x \rangle \geq r^*\},$$

for some \mathcal{C}^1 function Ψ such that $\inf_{x: \Psi(x)=0} |D\Psi(x)| > 0$. Then (see, e.g., Crandall et al. (1992), Lemma 7.6) there exists a \mathcal{C}^2 function Φ such that

$$\inf_{x \in \partial \mathcal{O}, \langle e^*, x \rangle \geq r^*} D\Phi(x)g(x) > 0.$$

Of course we can always assume

$$\inf_{x \in \partial \mathcal{O}, \langle e^*, x \rangle \geq r^*} \Phi(x) \geq 2r^*.$$

Therefore the function

$$\varphi(x) := \langle e^*, x \rangle \chi\left(\frac{\langle e^*, x \rangle - r^*}{r^*}\right) + [1 - \chi\left(\frac{\langle e^*, x \rangle - r^*}{r^*}\right)]\Phi(x),$$

where $\chi : \mathbb{R} \rightarrow [0, 1]$ is a smooth, nonincreasing function such that $\chi(r) = 1$ for $r \leq 0$, $\chi(r) = 0$ for $r \geq 1$, satisfies (4.5).

We conclude our proof by setting

$$X(t) := Y(K_0^{-1}(t)), \quad \Lambda(t) := K_1(K_0^{-1}(t)), \quad \gamma(t) := \eta(K_0^{-1}(t)).$$

It can be easily checked that X , Λ and γ verify (2.1) with $x^0 = 0$. \square

The above argument shows that the solution leaves the tip immediately and the results of the previous section show that the solution never returns. We can say more about the initial behavior of a solution.

For every solution X of (2.1), let

$$\tau_\delta^X = \inf\{t \geq 0 : X_1(t) \geq \delta\}, \quad \delta > 0. \quad (4.6)$$

Lemma 4.2 *There exists $C > 0$ such that for all δ sufficiently small and all solutions, X , of (2.1) starting at the origin,*

$$\mathbb{E}[\tau_\delta^X] \leq C\delta^2.$$

Proof. Here X is fixed, so we will omit the superscript X . Let e^* be the vector in Condition 2.3(b) and

$$f(x) = \frac{1}{2}\langle e^*, x \rangle^2.$$

Then, by Ito's formula,

$$\mathbb{E}[f(X(t \wedge \tau_\delta))] = \mathbb{E}\left[\int_0^{t \wedge \tau_\delta} Af(X(s))ds + \int_0^{t \wedge \tau_\delta} \langle e^*, X(s) \rangle \langle e^*, \gamma(s) \rangle d\Lambda(s)\right].$$

Observe that

$$\lim_{x \in \overline{\mathcal{O}} - \{0\}, x \rightarrow 0} \frac{|x|}{x_1} = 1, \quad \lim_{x \in \overline{\mathcal{O}} - \{0\}, x \rightarrow 0} \frac{\langle e^*, x \rangle}{e_1^* x_1} = 1,$$

and hence, for δ sufficiently small,

$$|X(t \wedge \tau_\delta)|^2 \leq 4\delta^2,$$

$$\int_0^{t \wedge \tau_\delta} \langle e^*, X(s) \rangle \langle e^*, \gamma(s) \rangle d\Lambda(s) \geq \frac{1}{2} \int_0^{t \wedge \tau_\delta} e_1^* X_1(s) \langle e^*, \gamma(s) \rangle d\Lambda(s) \geq 0,$$

where the last inequality follows from Condition 2.3(b). In addition

$$\lim_{x \in \overline{\mathcal{O}} - \{0\}, x \rightarrow 0} Af(x) = \lim_{x \in \overline{\mathcal{O}} - \{0\}, x \rightarrow 0} (\langle e^*, x \rangle \langle e^*, b(x) \rangle + \frac{1}{2}(e^*)^T(\sigma\sigma^T)(x)e^*) = \frac{1}{2}(e^*)^T(\sigma\sigma^T)(0)e^*.$$

Therefore, for δ sufficiently small,

$$2\delta^2 \geq \mathbb{E}[f(X(t \wedge \tau_\delta))] \geq \frac{1}{4}(e^*)^T(\sigma\sigma^T)(0)e^*\mathbb{E}[t \wedge \tau_\delta],$$

which yields the assertion by taking the limit as t goes to infinity. \square

Remark 4.3 *By looking at the proof of Lemma 4.2, we see that we have proved, more generally, that, for every $x^0 \in \overline{\mathcal{O}}$ with $x_1^0 < \delta$, for every solution X of (2.1) starting at x^0 ,*

$$\mathbb{E}[\tau_\delta^X] \leq C (4\delta^2 - \langle e^*, x^0 \rangle^2).$$

With reference to Condition 2.1(c) and Lemma 4.2, without loss of generality, in what follows, we will assume that δ_0 satisfies the following:

Condition 4.4

(a)

$$\sup_{0 \leq x_1 \leq \delta_0} |\psi'_1(x_1)| + |\psi'_2(x_1)| < \frac{1}{2}, \quad (4.7)$$

(b) *For $x_1 \leq \delta_0$, $(\sigma\sigma^T)(x)$ is strictly positive definite and*

$$\sup_{x \in \overline{\mathcal{O}}, x_1 \leq \delta_0} |\sigma(x)^{-1}| < 2|\sigma(0)^{-1}|, \quad \sup_{x \in \overline{\mathcal{O}}, x_1 \leq \delta_0} |\sigma(x) - \sigma(0)| < \frac{1}{2}|\sigma(0)^{-1}|^{-1}. \quad (4.8)$$

(c) *There exists $C > 0$ such that for $0 < \delta \leq \delta_0$ and all solutions of (2.1) starting at 0,*

$$\mathbb{E}[\tau_\delta^X] \leq C\delta^2.$$

4.2 Scaling near the tip

The following scaling result is central to our argument.

Let δ_0 satisfy Condition 4.4, and define $\{\delta_n\}$ recursively by

$$q_n := (\psi_2 - \psi_1)(\delta_n), \quad \delta_{n+1} := \delta_n - q_n, \quad n \geq 0. \quad (4.9)$$

Then $\delta_n > 0$ for all n , and $\{\delta_n\}$ is decreasing and converges to zero. In addition,

$$\lim_{n \rightarrow \infty} \frac{q_n}{\delta_n} = 0. \quad (4.10)$$

Lemma 4.5 *Let $\{x^n\} \subseteq \overline{\mathcal{O}} - \{0\}$, be such that $\{\bar{x}^n\} := \{q_n^{-1}(x_1^n - \delta_{n+1}, x_2^n)\}$ converges to a point \bar{x}^0 , and, for each n , let X^n be a solution of (2.1) starting at x^n .*

Then the sequence of processes

$$\{\bar{X}^n\} := \{q_n^{-1}(X_1^n(q_n^2 \cdot) - \delta_{n+1}, X_2^n(q_n^2 \cdot))\} \quad (4.11)$$

converges in distribution to a reflecting Brownian motion in $(-\infty, \infty) \times [L, L+1]$ with directions of reflection $g^1(0)$ on $(-\infty, \infty) \times \{L\}$ and $g^2(0)$ on $(-\infty, \infty) \times \{L+1\}$, respectively, covariance matrix $(\sigma\sigma^T)(0)$ and initial condition \bar{x}^0 .

Proof. \bar{X}^n is a solution of the rescaled SDER

$$\begin{aligned}\bar{X}^n(t) &= \bar{x}^n + q_n \int_0^t b((q_n \bar{X}_1^n(s) + \delta_{n+1}, q_n \bar{X}_2^n(s))) ds \\ &\quad + \int_0^t \sigma((q_n \bar{X}_1^n(s) + \delta_{n+1}, q_n \bar{X}_2^n(s))) dW(s) \\ &\quad + \int_0^t \bar{\gamma}^n(s) d\bar{\Lambda}^n(s), \quad t \geq 0,\end{aligned}\tag{4.12}$$

and

$$\bar{X}^n(t) \in \bar{\mathcal{O}}^n, \quad \bar{\gamma}^n(t) \in G_1(\bar{X}^n(t)), \quad d\bar{\Lambda}^n - a.e., \quad \bar{\Lambda}^n(t) = \int_0^t \mathbf{1}_{\partial \bar{\mathcal{O}}^n}(\bar{X}^n(s)) d\bar{\Lambda}^n(s), \quad t \geq 0,$$

where

$$\bar{\mathcal{O}}^n := \{x : (q_n x_1 + \delta_{n+1}, q_n x_2) \in \mathcal{O}\}, \quad \partial \bar{\mathcal{O}}^n = \{x : (q_n x_1 + \delta_{n+1}, q_n x_2) \in \partial \mathcal{O}\}.$$

Observe that, by the last requirement in Condition 2.1(c),

$$\bar{\mathcal{O}}^n \rightarrow \Delta := (-\infty, \infty) \times [L, L+1]$$

in the sense that the boundaries converge uniformly on compact subsets of $(-\infty, \infty)$, that is,

$$\lim_{n \rightarrow \infty} \frac{\psi_1(q_n x_1 + \delta_{n+1})}{q_n} = \lim_{n \rightarrow \infty} \frac{\psi_1(q_n(x_1 - 1) + \delta_n)}{q_n} = \lim_{n \rightarrow \infty} \frac{\psi_1(\delta_n)}{q_n} = L,\tag{4.13}$$

(notice that for n sufficiently large, $0 < q_n(x_1 - 1) + \delta_n < \delta_0$), and analogously

$$\lim_{n \rightarrow \infty} \frac{\psi_2(q_n x_1 + \delta_{n+1})}{q_n} = \lim_{n \rightarrow \infty} \frac{\psi_2(\delta_n)}{q_n} = L + 1.\tag{4.14}$$

The second term on the right of (4.12) converges to zero. By applying the same time-change argument as in Theorem 4.1, we see that $\{\bar{X}^n\}$ is relatively compact and

$$(q_n \bar{X}_1^n(s) + \delta_{n+1}, q_n \bar{X}_2^n(s)) \rightarrow 0.$$

Consequently, $\{\bar{X}^n\}$ converges in distribution to \bar{X} satisfying

$$\bar{X}(t) = \bar{x}^0 + \sigma(0)\bar{W}(t) + g^1(0)\Lambda_L(t) + g^2(0)\Lambda_{L+1}(t),\tag{4.15}$$

where \bar{W} is a standard Brownian motion, $\bar{X}(t) \in \Delta$, Λ_L is nondecreasing and increases only when $\bar{X}_2 = L$, and Λ_{L+1} is nondecreasing and increases only when $\bar{X}_2 = L+1$, that is, \bar{X} is a reflecting Brownian motion in $(-\infty, \infty) \times [L, L+1]$ with directions of reflection $g^1(0)$ on $(-\infty, \infty) \times \{L\}$ and $g^2(0)$ on $(-\infty, \infty) \times \{L+1\}$, respectively, covariance matrix $(\sigma\sigma^T)(0)$ and initial condition \bar{x}^0 . Then the thesis follows from the fact that the distribution of \bar{X} is uniquely determined. \square

4.3 Uniqueness

Lemma 4.6 *Suppose any two weak solutions, X, \tilde{X} , of (2.1) starting at the origin satisfy*

$$\mathcal{L}(X(\tau_{\delta_n}^X)) = \mathcal{L}(\tilde{X}(\tau_{\delta_n}^{\tilde{X}})), \quad (4.16)$$

for all n sufficiently large. Then the solution of (2.1) starting at the origin is unique in distribution.

Proof. For each n , consider the stochastic processes defined pathwise as

$$X(\tau_{\delta_n}^X + \cdot), \quad \tilde{X}(\tau_{\delta_n}^{\tilde{X}} + \cdot).$$

Since, starting away from 0, we have strong and weak uniqueness, if (4.16) holds, we have

$$\mathcal{L}(X(\tau_{\delta_n}^X + \cdot)) = \mathcal{L}(\tilde{X}(\tau_{\delta_n}^{\tilde{X}} + \cdot)).$$

On the other hand, $\tau_{\delta_n}^X$ and $\tau_{\delta_n}^{\tilde{X}}$ converge to zero, as $n \rightarrow \infty$, almost surely. Therefore we have $X(\tau_{\delta_n}^X + \cdot) \rightarrow X$ and $\tilde{X}(\tau_{\delta_n}^{\tilde{X}} + \cdot) \rightarrow \tilde{X}$, uniformly over compact time intervals, almost surely. Consequently, $\mathcal{L}(X) = \mathcal{L}(\tilde{X})$. \square

Theorem 4.7 *The solution of (2.1) starting at $x^0 = 0$ is unique in distribution.*

Proof. In the sequel, X and \tilde{X} will be weak solutions of (2.1) starting at the origin. Let $\{\delta_n\} = \{\delta_n\}_{n \geq 0}$ be given by (4.9). We want to show that for n sufficiently large

$$\|\mathcal{L}(X_2(\tau_{\delta_n}^X)) - \mathcal{L}(\tilde{X}_2(\tau_{\delta_n}^{\tilde{X}}))\|_{TV} \leq (1 - \kappa) \|\mathcal{L}(X_2(\tau_{\delta_{n+2}}^X)) - \mathcal{L}(\tilde{X}_2(\tau_{\delta_{n+2}}^{\tilde{X}}))\|_{TV}, \quad (4.17)$$

where $\kappa \in (0, 1)$ is defined below. Then by iterating (4.17), we obtain

$$\|\mathcal{L}(X_2(\tau_{\delta_n}^X)) - \mathcal{L}(\tilde{X}_2(\tau_{\delta_n}^{\tilde{X}}))\|_{TV} = 0,$$

for n sufficiently large (and hence for all n). The theorem then follows from Lemma 4.6.

To obtain κ , we construct below two solutions of (2.1), χ , starting at $(\delta_{n+2}, \chi_2(0)) \in \overline{\mathcal{O}}$, and $\tilde{\chi}$, starting at $(\delta_{n+2}, \tilde{\chi}_2(0)) \in \overline{\mathcal{O}}$, that are coupled in such a way that, letting

$$\tau_2 := \inf\{t \geq 0 : \chi_1(t) \geq \delta_n\}, \quad \tilde{\tau}_2 := \inf\{t \geq 0 : \tilde{\chi}_1(t) \geq \delta_n\}, \quad (4.18)$$

we have

$$\mathbb{P}(\chi(\tau_2) = \tilde{\chi}(\tilde{\tau}_2)) \geq \kappa > 0, \quad (4.19)$$

where κ is independent of $\chi_2(0)$ and $\tilde{\chi}_2(0)$ and of n sufficiently large.

Then (4.19) implies

$$\|\mathcal{L}(\chi(\tau_2)) - \mathcal{L}(\tilde{\chi}(\tilde{\tau}_2))\|_{TV} \leq (1 - \kappa),$$

Consequently, with the notation of Lemma 5.4, denoting by P the transition function from $[\psi_1(\delta_{n+2}), \psi_2(\delta_{n+2})]$ to $[\psi_1(\delta_n), \psi_2(\delta_n)]$ defined by

$$\begin{aligned} P(x_2, \cdot) &:= \mathbb{P}(\chi_2(\tau_2) \in \cdot | \chi_2(0) = x_2) = \mathbb{P}(X_2(\tau_{\delta_n}^X) \in \cdot | X_2(\tau_{\delta_{n+2}}^X) = x_2), \\ &= \mathbb{P}(\tilde{\chi}_2(\tilde{\tau}_2) \in \cdot | \tilde{\chi}_2(0) = x_2) = \mathbb{P}(\tilde{X}_2(\tau_{\delta_n}^{\tilde{X}}) \in \cdot | \tilde{X}_2(\tau_{\delta_{n+2}}^{\tilde{X}}) = x_2), \end{aligned}$$

we have

$$\|P\nu - P\tilde{\nu}\|_{TV} \leq (1 - \kappa),$$

for any two probability distributions ν and $\tilde{\nu}$ on $[\psi_1(\delta_{n+2}), \psi_2(\delta_{n+2})]$. Therefore (4.17) follows from Lemma 5.4.

We conclude the proof with the construction of the coupled solutions χ and $\tilde{\chi}$. We start χ and $\tilde{\chi}$ as two independent solutions of (2.1), with initial condition $(\delta_{n+2}, \chi_2(0))$ and $(\delta_{n+2}, \tilde{\chi}_2(0))$, respectively, and we run them until the times

$$\tau_1 := \inf\{t \geq 0 : \chi_1(t) \geq \delta_{n+1}\}, \quad \tilde{\tau}_1 := \inf\{t \geq 0 : \tilde{\chi}_1(t) \geq \delta_{n+1}\}.$$

We then consider the solution, (Z, \tilde{Z}) , with initial distribution $(\chi(\tau_1), \tilde{\chi}(\tilde{\tau}_1))$, of the coupled SDE (5.6) with $\beta = b$, $\varsigma = \sigma$ and B independent of $(\chi(\tau_1), \tilde{\chi}(\tilde{\tau}_1))$, until the times

$$\Theta := \inf\{t \geq 0 : Z(t) \notin Q\}, \quad \tilde{\Theta} := \inf\{t \geq 0 : \tilde{Z}(t) \notin Q\},$$

where Q is the rectangle

$$Q := (\delta_{n+1} - \frac{1}{4}q_{n+1}, \delta_{n+1} + \frac{1}{4}q_{n+1}) \times I_{n+1}^{\epsilon_0+1/2},$$

and $I_{n+1}^{\epsilon_0+1/2}$ is the interval in Lemma 5.2 for a value of ϵ_0 to be chosen later. We set

$$\chi(\tau_1 + t) := Z(t), \quad \tilde{\chi}(\tilde{\tau}_1 + t) := \tilde{Z}(t), \quad \text{for } t \leq \Theta \wedge \tilde{\Theta}.$$

For $\epsilon_0 \leq 1/4$, by (4.7),

$$\bar{Q} \subseteq \mathcal{O},$$

therefore χ and $\tilde{\chi}$ are solutions of (2.1) up to $\tau_1 + \Theta \wedge \tilde{\Theta}$ and $\tilde{\tau}_1 + \Theta \wedge \tilde{\Theta}$ respectively. Moreover, by (4.8), the assumptions of Lemma 5.3 are satisfied. Fix $p_0 \in (0, \frac{1}{4})$, let C_0 be the constant in Lemma 5.3, and choose $\epsilon_0 \leq C_0/4$. Then, for $x_2, \tilde{x}_2 \in I_{n+1}^{\epsilon_0}$, it holds

$$|x_2 - \tilde{x}_2| \leq C_0 \frac{q_{n+1}}{4}, \quad \overline{B_{q_{n+1}/4}(\delta_{n+1}, x_2)}, \overline{B_{q_{n+1}/4}(\delta_{n+1}, \tilde{x}_2)} \subseteq \bar{Q}.$$

Therefore Lemma 5.3 yields that

$$\mathbb{P}(Z(\Theta \wedge \tilde{\Theta}) = \tilde{Z}(\Theta \wedge \tilde{\Theta}) | Z_2(0) \in I_{n+1}^{\epsilon_0}, \tilde{Z}_2(0) \in I_{n+1}^{\epsilon_0}) \geq p_0.$$

Combining this with Lemma 5.2, we get,

$$\begin{aligned} & \mathbb{P}(\chi(\tau_1 + \Theta \wedge \tilde{\Theta}) = \tilde{\chi}(\tilde{\tau}_1 + \Theta \wedge \tilde{\Theta})) \\ & \geq \mathbb{P}(\chi(\tau_1 + \Theta \wedge \tilde{\Theta}) = \tilde{\chi}(\tilde{\tau}_1 + \Theta \wedge \tilde{\Theta}) | \chi_2(\tau_1) \in I_{n+1}^{\epsilon_0}, \tilde{\chi}_2(\tilde{\tau}_1) \in I_{n+1}^{\epsilon_0}) \mathbb{P}(\chi_2(\tau_1) \in I_{n+1}^{\epsilon_0}, \tilde{\chi}_2(\tilde{\tau}_1) \in I_{n+1}^{\epsilon_0}) \\ & \geq \mathbb{P}(Z(\Theta \wedge \tilde{\Theta}) = \tilde{Z}(\Theta \wedge \tilde{\Theta}) | Z_2(0) \in I_{n+1}^{\epsilon_0}, \tilde{Z}_2(0) \in I_{n+1}^{\epsilon_0}) \eta_{\epsilon_0}^2 \\ & \geq p_0 \eta_{\epsilon_0}^2 := \kappa. \end{aligned}$$

Finally, we define $\chi(\tau_1 + \Theta \wedge \tilde{\Theta} + \cdot)$ as the solution of (2.1) starting at $\chi(\tau_1 + \Theta \wedge \tilde{\Theta})$ and $\tilde{\chi}(\tilde{\tau}_1 + \Theta \wedge \tilde{\Theta} + \cdot) = \chi(\tau_1 + \Theta \wedge \tilde{\Theta} + \cdot)$, if $\chi(\tau_1 + \Theta \wedge \tilde{\Theta}) = \tilde{\chi}(\tilde{\tau}_1 + \Theta \wedge \tilde{\Theta})$, and as the solution of (2.1) starting at $\tilde{\chi}(\tilde{\tau}_1 + \Theta \wedge \tilde{\Theta})$ otherwise. Since, with Q as above, by (4.7),

$$\tau_1 + \Theta \wedge \tilde{\Theta} < \tau_2, \quad \tilde{\tau}_1 + \Theta \wedge \tilde{\Theta} < \tilde{\tau}_2,$$

χ and $\tilde{\chi}$ have the desired property. \square

4.4 The Feller property

We conclude with the observation that the family of distributions $\{P^x\}_{x \in \bar{\mathcal{O}}}$, where P^x is the distribution of the unique weak solution of (2.1) starting at x , enjoys the Feller property.

Proposition 4.8 *Let X^x be the unique weak solution of (2.1) starting at x . Then the mapping $x \in \bar{\mathcal{O}} \rightarrow X^x$ is continuous in distribution.*

Proof. The proof is exactly the same as that of Theorem 4.1. In fact, once it is known that the weak solution of (2.1) starting at the origin is unique, the proof of Proposition 4.8 amounts to showing that X^x is continuous in distribution at the origin. \square

5 Technical lemmas

Lemma 5.1 *Let $\bar{X}^{(0,x_2)}$ satisfy (4.15) with $\bar{X}^{(0,x_2)}(0) = (0, x_2)$, $x_2 \in [L, L+1]$, and let*

$$\bar{\tau}_1 := \inf\{t \geq 0 : \bar{X}_1^{(0,x_2)}(t) \geq 1\}.$$

Then

(i) $\bar{\tau}_1$ is a.s. finite.

(ii) For every $0 < \epsilon < 1/2$,

$$\inf_{L \leq x_2 \leq L+1} \mathbb{P}(\bar{X}_2^{(0,x_2)}(\bar{\tau}_1) \in (L + \frac{1}{2} - \epsilon, L + \frac{1}{2} + \epsilon)) > 0. \quad (5.1)$$

Proof. To simplify notation, whenever possible without loss of clarity, we will omit the superscript on \bar{X} .

(i) Let e^* be the vector in Condition 2.3(b). Then, for all $N > 0$,

$$\inf\{t \geq 0 : \langle \bar{X}(t), e^* \rangle \geq N + x_2 e_2^*\} \leq \inf\{t \geq 0 : \langle \sigma(0)W(t), e^* \rangle \geq N\} < +\infty \quad a.s..$$

On the other hand

$$\bar{X}_1(t) = \frac{1}{e_1^*} (\langle \bar{X}(t), e^* \rangle - \bar{X}_2(t)e_2^*) \geq \frac{1}{e_1^*} (\langle \bar{X}(t), e^* \rangle - (|L| + 1)).$$

Therefore, for N large enough,

$$\bar{\tau}_1 \leq \inf\{t \geq 0 : \langle \bar{X}(t), e^* \rangle \geq N + x_2 e_2^*\}.$$

(ii) We can suppose, without loss of generality, $\epsilon < 1/2$. Let $f : \mathbb{R}^2 \rightarrow [0, 1]$ be a smooth function such that

$$f(1, L + \frac{1}{2}) = 1, \quad f(x) = 0 \text{ for } x \notin B_\epsilon((1, L + \frac{1}{2})).$$

We can estimate the probability in the left hand side of (5.1) by

$$\mathbb{P}(\bar{X}_2(\bar{\tau}_1) \in (L + \frac{1}{2} - \epsilon, L + \frac{1}{2} + \epsilon)) \geq \mathbb{E}[f(\bar{X}(\bar{\tau}_1))].$$

For $x \in (-\infty, 1] \times [L, L+1]$, let \bar{X}^x satisfy (4.15) with $\bar{x}^0 = x$, $\bar{\tau}_1(\bar{X}^x) := \inf\{t \geq 0 : \bar{X}_1^x(t) \geq 1\}$ and set

$$u(x) := \mathbb{E}[f(\bar{X}^x(\bar{\tau}_1(\bar{X}^x)))].$$

u is continuous on $(-\infty, 1] \times [L, L+1]$ by the continuous mapping theorem, because \bar{X} is a Feller process and $\bar{\tau}_1$ is a functional on $\mathcal{C}_{(-\infty, 1] \times [L, L+1]}[0, \infty)$ almost surely continuous under the law of \bar{X}^x , for every x . For any bounded, smooth domain Q such that $\bar{Q} \subseteq (-\infty, 1) \times (L, L+1)$, u is the classical solution of the Dirichlet problem with itself as boundary datum. Therefore $u \in \mathcal{C}^2((-\infty, 1) \times (L, L+1))$ and

$$\text{tr}((\sigma\sigma^T)(0)D^2u(x)) = 0, \quad \forall x \in (-\infty, 1) \times (L, L+1).$$

For $0 < \eta < 1/2$, let

$$Q_\eta := (-1 + \eta, 1 - \eta) \times (L + \eta, L + 1 - \eta).$$

By the Harnack inequality,

$$\inf_{x \in Q_\eta} u(x) \geq c_\eta \sup_{x \in Q_\eta} u(x), \quad (5.2)$$

for some $c_\eta > 0$. For η small enough the right hand side of (5.2) is strictly positive and hence

$$u(0, x_2) > 0, \quad \forall x_2 \in (L, L+1).$$

Now let

$$\bar{\tau}_{Q_\eta}^{(0, L+1)} := \inf\{t \geq 0 : \bar{X}^{(0, L+1)}(t) \in \bar{Q}_\eta\},$$

and fix η small enough that the right hand side of (5.2) is strictly positive and that

$$\mathbb{P}(\bar{\tau}_{Q_\eta}^{(0, L+1)} < \bar{\tau}_1^{(0, L+1)}) > 0.$$

Then

$$\begin{aligned} u(0, L+1) &= \mathbb{E}[f(\bar{X}^{(0, L+1)}(\bar{\tau}_1^{(0, L+1)}))] \\ &\geq \mathbb{E}[\mathbf{1}_{\{\bar{\tau}_{Q_\eta}^{(0, L+1)} < \bar{\tau}_1^{(0, L+1)}\}} f(\bar{X}^{(0, L+1)}(\bar{\tau}_1^{(0, L+1)}))] \\ &= \mathbb{E}[\mathbf{1}_{\{\bar{\tau}_{Q_\eta}^{(0, L+1)} < \bar{\tau}_1^{(0, L+1)}\}} u(\bar{X}^{(0, L+1)}(\bar{\tau}_{Q_\eta}^{(0, L+1)}))] \\ &\geq \inf_{x \in Q_\eta} u(x) \mathbb{P}(\bar{\tau}_{Q_\eta}^{(0, L+1)} < \bar{\tau}_1^{(0, L+1)}) \\ &> 0. \end{aligned}$$

Analogously

$$u(0, L) > 0,$$

and the assertion follows by the continuity of u on $\{0\} \times [L, L+1]$. \square

Lemma 5.2 *Let $\{\delta_n\}$ be defined by (4.9), X^{x^0} be the solution of (2.1) starting at $x^0 \in \overline{\mathcal{O}} - \{0\}$, $\tau_{\delta_n}^{x^0} := \inf\{t \geq 0 : X_1^{x^0}(t) \geq \delta_n\}$, $n \geq 0$, and, for $0 < \epsilon < 1$, I_n^ϵ be the open interval of length $\epsilon(\psi_2 - \psi_1)(\delta_n)$ centered at $\frac{(\psi_1 + \psi_2)(\delta_n)}{2}$. Then there exists $n_\epsilon \geq 0$ and $\eta_\epsilon > 0$ such that*

$$\inf_{n \geq n_\epsilon} \inf_{x_2: (\delta_{n+1}, x_2) \in \overline{\mathcal{O}}} \mathbb{P}(X_2^{(\delta_{n+1}, x_2)}(\tau_{\delta_n}^{(\delta_{n+1}, x_2)}) \in I_n^\epsilon) = \eta_\epsilon. \quad (5.3)$$

Proof. Let $\{x_2^n\}$, $\psi_1(\delta_{n+1}) \leq x_2^n \leq \psi_2(\delta_{n+1})$ be such that $q_n^{-1}x_2^n$ converges to $\bar{x}_2^0 \in [L, L+1]$, and let X^n denote the solution of (2.1) starting at (δ_{n+1}, x_2^n) . Let \bar{X}^n denote the scaled process (4.11) with initial condition $(0, q_n^{-1}x_2^n)$, and let \bar{X} denote the limiting reflecting Brownian motion satisfying (4.15) starting at $(0, \bar{x}_2^0)$. Define

$$\begin{aligned} \tau^n = \tau_{\delta_n}^{(\delta_{n+1}, x_2^n)} &:= \inf\{t \geq 0 : X_1^n(t) \geq \delta_n\} \\ \bar{\tau}_1^n &:= \inf\{t \geq 0 : \bar{X}_1^n(t) \geq 1\} \\ \bar{\tau}_1 &:= \inf\{t \geq 0 : \bar{X}_1(t) \geq 1\} \end{aligned}$$

Notice that τ^n is a.s. finite by Remark 4.3 and that

$$\tau^n = q_n^{-2} \bar{\tau}_1^n.$$

Since the first exit time from $(-\infty, 1) \times \mathbb{R}$ is a continuous functional on a set of paths that has probability one under the distribution of \bar{X} , by the continuous mapping theorem we may assume that $\bar{X}_2^n(\bar{\tau}_1^n)$ converges in distribution to $\bar{X}_2(\bar{\tau}_1)$. Then

$$\begin{aligned} \liminf_n \mathbb{P}(X_2^n(\tau^n) \in I_n^\epsilon) &\geq \liminf_n \mathbb{P}\left(X_2^n(\tau^n) \in (q_n(L + \frac{1}{2} - \frac{\epsilon}{4}), q_n(L + \frac{1}{2} + \frac{\epsilon}{4}))\right) \\ &= \liminf_n \mathbb{P}\left(\bar{X}_2^n(\bar{\tau}_1^n) \in (L + \frac{1}{2} - \frac{\epsilon}{4}, L + \frac{1}{2} + \frac{\epsilon}{4})\right) \\ &\geq \mathbb{P}\left(\bar{X}_2(\bar{\tau}_1) \in (L + \frac{1}{2} - \frac{\epsilon}{4}, L + \frac{1}{2} + \frac{\epsilon}{4})\right) \\ &\geq \inf_{L \leq x_2 \leq L+1} \mathbb{P}(\bar{X}_2^{(0, x_2)}(\bar{\tau}_1^{(0, x_2)}) \in (L + \frac{1}{2} - \frac{\epsilon}{4}, L + \frac{1}{2} + \frac{\epsilon}{4})), \end{aligned}$$

and the assertion follows by (5.1) and by the arbitrariness of $\{x_2^n\}$. \square

The following lemma, which uses the coupling of Lindvall and Rogers (1986), may be of independent interest.

Lemma 5.3 *Let $\beta : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\varsigma : \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d$ be Lipschitz continuous and bounded and let $\varsigma \varsigma^T$ be uniformly positive definite. Suppose that*

$$\sup_{x, \tilde{x}} |\varsigma(x) - \varsigma(\tilde{x})| < 2(\sup_x |\varsigma(x)^{-1}|)^{-1}. \quad (5.4)$$

Define

$$K(x, \tilde{x}) := I - 2 \frac{\varsigma(\tilde{x})^{-1}(x - \tilde{x})(x - \tilde{x})^T(\varsigma(\tilde{x})^{-1})^T}{|\varsigma(\tilde{x})^{-1}(x - \tilde{x})|^2}, \quad (5.5)$$

Let B be a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and (Z, \tilde{Z}) be the solution of the system of stochastic differential equations

$$\begin{aligned} dZ(t) &= \beta(Z(t))dt + \varsigma(Z(t))dB(t), & Z(0) &= x^0, \\ d\tilde{Z}(t) &= \beta(\tilde{Z}(t))dt + \varsigma(\tilde{Z}(t))K(Z(t), \tilde{Z}(t))dB(t), & \tilde{Z}(0) &= \tilde{x}^0 \neq x^0 \end{aligned} \quad (5.6)$$

for $t < \zeta$, where

$$\zeta := \lim_{\epsilon \rightarrow 0} \zeta_\epsilon, \quad \zeta_\epsilon := \inf\{t \geq 0 : |Z(t) - \tilde{Z}(t)| < \epsilon\}, \quad (5.7)$$

and $\tilde{Z}(t) = Z(t)$ for $t \geq \zeta$, on the set $\{\zeta < \infty\}$. (Notice that K is locally Lipschitz continuous on $\mathbb{R}^d \times \mathbb{R}^d - \{(x, x), x \in \mathbb{R}^d\}$.)

Then \tilde{Z} is a diffusion process with generator $Df(x)\beta(x) + \frac{1}{2}\text{tr}(\varsigma(x)\varsigma(x)^T D^2 f(x))$ and, for every p_0 , $0 < p_0 < \frac{1}{4}$, there exists a positive constant $C_0 < 1$, depending only on p_0 , β and ς , such that, setting

$$\vartheta_\rho := \inf\{t \geq 0 : |Z(t) - x^0| \geq \rho\}, \quad \tilde{\vartheta}_\rho := \inf\{t \geq 0 : |\tilde{Z}(t) - \tilde{x}^0| \geq \rho\},$$

for $\rho \leq 1$,

$$|x^0 - \tilde{x}^0| \leq C_0 \rho \quad \text{implies} \quad \mathbb{P}(\zeta \leq \vartheta_\rho \wedge \tilde{\vartheta}_\rho) \geq p_0.$$

Proof. The fact that \tilde{Z} has generator $Df(x)\beta(x) + \frac{1}{2}\text{tr}(\varsigma(x)\varsigma(x)^T D^2 f(x))$ follows from the fact that $K(x, \tilde{x})$ is an orthogonal matrix.

As in Lindvall and Rogers (1986), consider

$$U(t) := |Z(t) - \tilde{Z}(t)|.$$

For $t < \zeta$, U satisfies

$$dU(t) = a(t)dt + \alpha(t)dW(t),$$

where

$$\begin{aligned} a(t) &:= \frac{\langle Z(t) - \tilde{Z}(t), \beta(Z(t)) - \beta(\tilde{Z}(t)) \rangle}{|Z(t) - \tilde{Z}(t)|} + \frac{\text{tr}((\varsigma(Z(t)) - \varsigma(\tilde{Z}(t))) (\varsigma(Z(t)) - \varsigma(\tilde{Z}(t)))^T)}{2|Z(t) - \tilde{Z}(t)|} \\ &\quad - \frac{|(\varsigma(Z(t)) - \varsigma(\tilde{Z}(t)))^T \frac{Z(t) - \tilde{Z}(t)}{|Z(t) - \tilde{Z}(t)|}|^2}{2|Z(t) - \tilde{Z}(t)|} \\ \alpha(t) &:= \left| (\varsigma(Z(t)) - \varsigma(\tilde{Z}(t)))^T K(Z(t), \tilde{Z}(t)) \frac{Z(t) - \tilde{Z}(t)}{|Z(t) - \tilde{Z}(t)|} \right|, \end{aligned}$$

and W is a standard Brownian motion. Then, as in Lindvall and Rogers (1986), setting, for $u > 0$,

$$\begin{aligned} h(u) &:= \sup_{|x - \tilde{x}| = u} \left\{ \left| (\varsigma(x) - \varsigma(\tilde{x})) K(x, \tilde{x}) \frac{x - \tilde{x}}{|x - \tilde{x}|} \right|^{-2} \right. \\ &\quad \left. \times \left[\frac{\langle x - \tilde{x}, \beta(x) - \beta(\tilde{x}) \rangle}{|x - \tilde{x}|} + \frac{\text{tr}((\varsigma(x) - \varsigma(\tilde{x})) (\varsigma(x) - \varsigma(\tilde{x}))^T) - |(\varsigma(x) - \varsigma(\tilde{x}))^T \frac{x - \tilde{x}}{|x - \tilde{x}|}|^2)}{2|x - \tilde{x}|} \right] \right\}, \end{aligned}$$

we have

$$a(t) \leq \alpha(t)^2 h(U(t)).$$

In addition, by (5.4), the Lipschitz property of β and ς and the boundedness of ς ,

$$\begin{aligned} \underline{\alpha} &:= \inf_t \alpha(t) > 0 \\ \bar{\alpha} &:= \sup_t \alpha(t) \leq 2\|\varsigma\| \\ |a(t)| &\leq \bar{\alpha}U(t), \quad \text{a.s.} \\ |h(u)| &\leq C u \end{aligned} \tag{5.8}$$

(See the computations on page 866 of Lindvall and Rogers (1986).) In particular, setting $h(0) := 0$, h is continuous at 0 and locally square integrable on $[0, \infty)$. Therefore, applying Itô's formula to the process U and the function

$$H(u) := \int_1^u dv \exp\left(-2 \int_1^v h(z) dz\right), \quad u \geq 0, \tag{5.9}$$

we see that, for $u := |x^0 - \tilde{x}^0|$, $\theta_{2u} := \inf\{t \geq 0 : U(t) > 2u\}$, $\epsilon < 2u$,

$$\mathbb{P}(\zeta_\epsilon < \theta_{2u}) \geq \frac{H(2u) - H(u)}{H(2u) - H(\epsilon)}.$$

On the other hand, we have, for any $0 < t \leq 1$,

$$\begin{aligned} \mathbb{P}(\zeta_\epsilon < \vartheta_\rho \wedge \tilde{\vartheta}_\rho) &\geq \mathbb{P}(\zeta_\epsilon \leq t, t < \vartheta_\rho \wedge \tilde{\vartheta}_\rho) \\ &\geq \mathbb{P}(\zeta_\epsilon \leq t) - \mathbb{P}(\vartheta_\rho \wedge \tilde{\vartheta}_\rho \leq t) \\ &\geq \mathbb{P}(\zeta_\epsilon < \theta_{2u}, \zeta_\epsilon \wedge \theta_{2u} \leq t) - \mathbb{P}(\vartheta_\rho \wedge \tilde{\vartheta}_\rho \leq t) \\ &\geq \mathbb{P}(\zeta_\epsilon < \theta_{2u}) - \mathbb{P}(\zeta_\epsilon \wedge \theta_{2u} > t) - \mathbb{P}(\vartheta_\rho \wedge \tilde{\vartheta}_\rho \leq t) \\ &\geq \frac{H(2u) - H(u)}{H(2u) - H(\epsilon)} - \mathbb{P}\left(\sup_{s \leq t} |U(s) - u| < u\right) - \mathbb{P}(\vartheta_\rho \wedge \tilde{\vartheta}_\rho \leq t). \end{aligned}$$

Then, noting that $\mathbb{P}(\vartheta_\rho \wedge \tilde{\vartheta}_\rho < \infty) = 1$, taking the limit as ϵ goes to zero, we obtain

$$\begin{aligned} \mathbb{P}(\zeta \leq \vartheta_\rho \wedge \tilde{\vartheta}_\rho) &\geq \frac{H(2u) - H(u)}{H(2u) - H(0)} - \mathbb{P}\left(\sup_{s \leq t} |U(s) - u| < u\right) \\ &\quad - \mathbb{P}(\vartheta_\rho \wedge \tilde{\vartheta}_\rho \leq t). \end{aligned}$$

Now, we can easily see (for instance applying Itô's formula to the function $f(x) = |x - x^0|^2$) that

$$\mathbb{P}(\vartheta_\rho \wedge \tilde{\vartheta}_\rho \leq t) \leq \frac{C_1 t}{\rho^2},$$

where C_1 depends only on b and ς . Of course we can suppose, without loss of generality, that $C_1 \geq 1$. We will take, for $\rho \leq 1$,

$$t = \frac{1}{C_1} \left(\frac{1}{4} - p_0\right) \rho^2 \leq 1. \tag{5.10}$$

We have, for $u \leq C_0\rho$, where $C_0 \leq 1$ is a constant to be chosen later,

$$\begin{aligned} & \mathbb{P}\left(\sup_{s \leq t} |U(s) - u| < u\right) \\ &= \mathbb{P}\left(\sup_{s \leq t} \left| \int_0^s a(r) dr + \int_0^s \alpha(r) dW(r) \right| < u, \sup_{s \leq t} U(s) < 2u\right) \\ &\leq \mathbb{P}\left(\sup_{s \leq t} \left| \int_0^s \alpha(r) dW(r) \right| < C_0(1 + 2\bar{a}) \sqrt{\frac{4C_1}{1 - 4p_0}} \sqrt{t}\right), \end{aligned}$$

where in the last inequality we have used the fact that $|a(t)| \leq \bar{a}U(t)$ and (5.10). We take C_0 small enough that

$$C := C_0(1 + 2\bar{a}) \sqrt{\frac{4C_1}{1 - 4p_0}} < \underline{\alpha}. \quad (5.11)$$

Then setting $\theta_{C\sqrt{t}}^\alpha := \inf\{s \geq 0 : |\int_0^s \alpha(r) dW(r)| \geq C\sqrt{t}\}$,

$$\begin{aligned} & \mathbb{P}\left(\sup_{s \leq t} \left| \int_0^s \alpha(r) dW(r) \right| < C\sqrt{t}\right) \\ &= \mathbb{P}\left(\theta_{C\sqrt{t}}^\alpha > t, \left(\int_0^t \alpha(s) dW(s)\right)^2 < C^2 t\right) \\ &= \mathbb{P}\left(\theta_{C\sqrt{t}}^\alpha > t, 2 \int_0^t \left(\int_0^s \alpha(r) dW(r)\right) \alpha(s) dW(s) + \int_0^t \alpha(s)^2 ds < C^2 t\right) \\ &\leq \mathbb{P}\left(\int_0^{t \wedge \theta_{C\sqrt{t}}^\alpha} \left(\int_0^s \alpha(r) dW(r)\right) \alpha(s) dW(s) < \frac{1}{2}(C^2 - \underline{\alpha}^2)t\right), \end{aligned}$$

where the last inequality uses (5.8). Since $C < \underline{\alpha}$, the above chain of inequalities can be continued as

$$\begin{aligned} & \leq \mathbb{P}\left(\left| \int_0^{t \wedge \theta_{C\sqrt{t}}^\alpha} \left(\int_0^s \alpha(r) dW(r)\right) \alpha(s) dW(s) \right| > \frac{1}{2}(\underline{\alpha}^2 - C^2)t\right) \\ & \leq \frac{4}{(\underline{\alpha}^2 - C^2)^2 t^2} \mathbb{E}\left[\int_0^{t \wedge \theta_{C\sqrt{t}}^\alpha} \left(\int_0^s \alpha(r) dW(r)\right)^2 \alpha(s)^2 ds\right] \\ & \leq \frac{4C^2 t}{(\underline{\alpha}^2 - C^2)^2 t^2} \mathbb{E}\left[\int_0^t \alpha(s)^2 ds\right] \\ & \leq \frac{4C^2 \bar{\alpha}^2 t^2}{(\underline{\alpha}^2 - C^2)^2 t^2}. \end{aligned}$$

Finally, observe that, by (5.9) and the fact that h is locally integrable on $[0, \infty)$, H is differentiable on $[0, \infty)$ so that

$$\lim_{u \rightarrow 0^+} \frac{H(2u) - H(u)}{H(2u) - H(0)} = \lim_{u \rightarrow 0^+} \frac{2H'(2u) - H'(u)}{2H'(2u)} = \frac{1}{2}.$$

Then, by choosing C_0 in (5.11) small enough that C satisfies

$$\frac{4C^2 \bar{\alpha}^2}{(\underline{\alpha}^2 - C^2)^2} \leq \frac{1}{4} - p_0,$$

and that, for $u \leq C_0$,

$$\frac{H(2u) - H(u)}{H(2u) - H(0)} \geq \frac{1}{2} - p_0,$$

the assertion is proved. \square

Lemma 5.4 *Let E be a complete, separable metric space, and for $\mu_1, \mu_2 \in \mathcal{P}(E)$, define*

$$\|\mu_1 - \mu_2\|_{TV} = \sup_{A \in \mathcal{B}(E)} |\mu_1(A) - \mu_2(A)|.$$

(i) *For $\mu_1, \mu_2 \in \mathcal{P}(E)$, there exist $\nu_0, \nu_1, \nu_2 \in \mathcal{P}(E)$ such that*

$$\mu_1 = (1 - \rho)\nu_0 + \rho\nu_1, \quad \mu_2 = (1 - \rho)\nu_0 + \rho\nu_2, \quad (5.12)$$

where

$$\rho = \|\mu_1 - \mu_2\|_{TV}.$$

(ii) *Let E_1 and E_2 be complete separable metric spaces and P be a transition function from E_1 to E_2 , and let*

$$P\mu(dy) = \int_{E_1} P(x, dy)\mu(dx), \quad \mu \in \mathcal{P}(E_1).$$

Then, for $\mu_1, \mu_2 \in \mathcal{P}(E_1)$ and ν_1, ν_2 as in (i),

$$\|P\mu_1 - P\mu_2\|_{TV} = \|\mu_1 - \mu_2\|_{TV} \|P\nu_1 - P\nu_2\|_{TV}.$$

Proof. (i) Let

$$l_i := \frac{d\mu_i}{d(\mu_1 + \mu_2)}, \quad i = 1, 2,$$

$$\nu_0(A) := \frac{1}{\int_E (l_1(x) \wedge l_2(x)) (\mu_1 + \mu_2)(dx)} \int_A (l_1(x) \wedge l_2(x)) (\mu_1 + \mu_2)(dx),$$

$$\nu_i(A) := \frac{1}{1 - \int_E (l_1(x) \wedge l_2(x)) (\mu_1 + \mu_2)(dx)} \int_A (l_i(x) - l_1(x) \wedge l_2(x)) (\mu_1 + \mu_2)(dx), \quad i = 1, 2,$$

and

$$\rho := 1 - \int_E (l_1(x) \wedge l_2(x)) (\mu_1 + \mu_2)(dx).$$

Then (5.12) holds. In addition

$$\|\mu_1 - \mu_2\|_{TV} = \rho \|\nu_1 - \nu_2\|_{TV} = \rho,$$

because ν_1 and ν_2 are mutually singular.

(ii) By (i),

$$\|P\mu_1 - P\mu_2\|_{TV} = \|(1 - \rho)P\nu_0 + \rho P\nu_1 - (1 - \rho)P\nu_0 - \rho P\nu_2\|_{TV} = \rho \|P\nu_1 - P\nu_2\|_{TV}.$$

\square

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