A CLT for Empirical Processes Involving Time Dependent Data

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Abstract

For stochastic processes \( \{X_t : t \in E\} \), we establish sufficient conditions for the empirical process based on \( \{I_{X_t \leq y} - P(X_t \leq y) : t \in E, y \in \mathbb{R}\} \) to satisfy the CLT uniformly in \( t \in E, y \in \mathbb{R} \). Corollaries of our main result include examples of classical processes where the CLT holds, and we also show that it fails for Brownian motion tied down at zero and \( E = [0, 1] \).

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1 Introduction.

To form the classical empirical process one starts with iid random variables, \( \{X_j : j \geq 1\} \), with distribution function, \( F \), and with

\[
\mathbb{P}_n(A) = \frac{1}{n} \sum_{j=1}^{n} I_{X_j \in A}
\]  

one considers the process \( \mathbb{F}_n(y) = \mathbb{P}_n((\infty, y]) \).

By the classical Glivenko-Cantelli Theorem

\[ \sup_{y \in \mathbb{R}} |\mathbb{F}_n((\infty, y]) - F(y)| \to 0 \text{ a.s.} \]

By Donsker’s theorem

\[ \left\{ \sqrt{n} (\mathbb{F}_n(y) - F(y)) : y \in \mathbb{R} \right\} \]

covers in distribution in a sense described more completely below. Hence limit theorems for such processes, such as the Law of Large numbers and the Central Limit theorem (CLT), allow one to asymptotically get uniform estimates for the unknown cdf, \( F(y) = \Pr(X \leq y) \) via the sample data.

A more general version of these processes is to replace the indicators of half-lines in (1) by functions of a “random variable” taking values in some abstract space \( (S, \mathcal{S}) \). More specifically,

\[
\left\{ \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left( f(X_j) - \mathbb{E}f(X) \right) : f \in \mathcal{F} \right\},
\]

where the index set, \( \mathcal{F} \), is a subset of \( \mathcal{L}_\infty(S, \mathcal{S}) \) or an appropriate subset of \( \mathcal{L}_2(S, \mathcal{S}, \mathbb{P}) \).

However, even in the case that the class of functions is a class of indicators, unlike the classical case, it is easy to see there are many classes of sets, \( \mathcal{C} \), for which the limit theorem does not hold. As a matter of fact the limiting Gaussian may not be continuous, for example, if \( \mathcal{C} = \) all Borel sets of \( \mathbb{R} \) or
even \( C = \) all finite sets of \( \mathbb{R} \). And further, even if the limiting Gaussian process is continuous, the limit theorem may still fail.

Luckily, in the case of sets, modulo questions of measurability, there are necessary and sufficient conditions for this sequence to converge in distribution to some mean zero Gaussian process. However, all the nasc’s are described in terms of the asymptotic behavior of a complicated function of the sample, \( \{X_n\}_{n=1}^{\infty} \). What we attempt to do in this paper is to obtain additional sufficient conditions that are useful when \( X \) takes values in some function space \( S \), and the sets in \( C \) involve the time evolution of the stochastic process \( X \). Of course, \( C \) is still a class of sets, but a primary goal that emerges here is to provide sufficient conditions for a uniform CLT in terms of the process \( X = \{X(t) : t \in E\} \) that depend as little as possible on the parameter set \( E \). However, classes of sets such as this rarely satisfy the Vapnik-Červonenkis condition. Also, this class of examples arises naturally from the study of the median process for independent Brownian Motions (see [?], [...]), where he studies the limiting quantile process for independent Brownian motions. This was observed by Tom Kurtz, and the followup questions led us to start this study. Here we concentrate on empirical process CLT’s, and our main result is Theorem 3 below. Another theorem and some examples showing the applicability of Theorem 3 appear in section 7. In section 8 there are additional examples which show some obvious conjectures one might be tempted to make regarding the CLT formulated here are false. In particular, the examples in section 8.4 motivate the various assumptions we employ in Theorem 3. As for future work, an upgraded version of [?] would perhaps allow one to relate the results obtained here and those of Swanson, but this is something to be done elsewhere.
2 Previous Results and Some Definitions.

Let \((S, S, P)\) be a probability space and define \((\Omega, \Sigma, Pr)\) to be the infinite product probability space \((S^N, S^N, P^N)\). If \(X_j : \Omega \rightarrow S\) are the natural projections of \(\Omega\) into the \(j^{th}\) copy of \(S\), and \(\mathcal{F}\) is a subset of \(L^2(S, S, P)\) with

\[
\sup_{f \in \mathcal{F}} |f(s)| < \infty, s \in S,
\]
then we define

\[
\nu_n(f) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (f(X_j) - \mathbb{E}f(X)), f \in \mathcal{F}.
\]

Let \(\ell_\infty(\mathcal{F})\) be the bounded real valued functions on \(\mathcal{F}\), with the sup-norm, and recall that a Radon measure \(\mu\) on \(\ell_\infty(\mathcal{F})\) is a finite Borel measure which is inner regular from below by compact sets. Then the functions \(f \rightarrow f(X_j) - E(f(X_j)), j \geq 1\), are in \(\ell_\infty(\mathcal{F})\), and we say \(\mathcal{F} \in CLT(P)\) if the stochastic processes \(\{\nu_n(f), f \in \mathcal{F}\}, n \geq 1\), converge weakly to a centered Radon Gaussian measure \(\gamma_P\) on \(\ell_\infty(\mathcal{F})\). More precisely, we have the following definition.

**Definition 1.** Let \(\mathcal{F} \subset L^2(P)\) and satisfy (3). Then \(\mathcal{F} \in CLT(P)\), or \(\mathcal{F}\) is a \(P\)-Donsker class, if there exists a centered Radon Gaussian measure \(\gamma_P\) on \(\ell_\infty(\mathcal{F})\) such that for every bounded continuous real valued function \(H\) on \(\ell_\infty(\mathcal{F})\) we have

\[
\lim_{n \rightarrow \infty} E^*(H(\nu_n)) = \int H d\gamma_P,
\]

where \(E^*H\) is the usual upper integral of \(H\). If \(\mathcal{C}\) is a collection of subsets from \(S\), then we say \(\mathcal{C} \in CLT(P)\) if the corresponding indicator functions are a \(P\)-Donsker class.

The probability measure \(\gamma_P\) of Definition 1 is obviously the law of the centered Gaussian process \(G_P\) indexed by \(\mathcal{F}\) having covariance function

\[
\mathbb{E}G_P(f)G_P(g) = \mathbb{E}_P fg - \mathbb{E}_P f \mathbb{E}_P g, f, g \in \mathcal{F},
\]
and $L^2$ distance

$$\rho_P(f, g) = \mathbb{E}_P(\{(f - g) - \mathbb{E}_P(f - g)\}^2)^{1/2}, f, g \in \mathcal{F}. $$

Moreover, if $\gamma_P$ is as in Definition 1, then it is known that the process $G_P$ admits a version all of whose trajectories are bounded and uniformly $\rho_P$ continuous on $\mathcal{F}$. Hence we also make the following definition.

**Definition 2.** A class of functions $\mathcal{F} \subset L^2(P)$ is said to be $P$-pregaussian if the mean zero Gaussian process $\{G_P(f) : f \in \mathcal{F}\}$ with covariance and $L^2$ distance as indicated above has a version with all the sample functions bounded and uniformly continuous on $\mathcal{F}$ with respect to the $L^2$ distance $\rho_P(f, g)$.

Now we state some results which are useful for what we prove in this paper. The first is important as it helps us establish counterexamples to natural conjectures one might make in connection to our main result appearing in Theorem 3 below.

**Theorem 1.** (Giné and Zinn [?] for sufficiency and Talagrand [?] for necessity of the $\Delta^C$ condition in (ii)) Let $\Delta^C(A)$ denote the number of distinct subsets of $A$ obtained when one intersects all sets in $C$ with $A$. Then, modulo measurability assumptions, conditions, (i) and (ii) below are equivalent.

(i) The central limit theorem holds for the process

$$\left\{ \frac{1}{\sqrt{n}} \sum_{j=1}^{n} [I_{X_j \in C} - \Pr(X \in C)] : C \in \mathcal{C} \right\}
$$

or more briefly $\mathcal{C} \in CLT(P)$.

(ii)

(a) $\frac{\ln \Delta^C(X_1, \ldots, X_n)}{\sqrt{n}} \to 0$ in (outer) probability, and

(b) $\mathcal{C}$ is $P$-pregaussian.

(5)
A sufficient condition for the empirical CLT, which is used in the proof of our main theorem, is given in Theorem 4.4 of [?].

**Theorem 2.** [?] Let \( F \subset \mathcal{L}_2(S, S, P) \) and let \( F = \sup_{f \in \mathcal{F}} |f(X)| \). Assume that \( \| Pf \|_\mathcal{F} < \infty \) and

(i) \( u^2 \Pr^*(F > u) \to 0 \) as \( u \to \infty \),

(ii) \( F \) is \( P \)-pregaussian, and

(iii) there exists a centered Gaussian process \( \{ G(f) : f \in \mathcal{F} \} \) with \( L_2 \) distance \( d_G \) such that \( G \) is sample bounded and uniformly \( d_G \) continuous on \( \mathcal{F} \), and for some \( K > 0 \), all \( f \in \mathcal{F} \), and all \( \epsilon > 0 \),

\[
\left[ \sup_{u > 0} u^2 \Pr^*(\sup_{\{ g : d_G(g, f) < \epsilon \}} |f - g| > u) \right]^{1/2} \leq K \epsilon.
\]

Then \( \mathcal{F} \in CLT(P) \).

In this paper we take iid copies \( \{ X_j \}_1^\infty \) of a process \( \{ X(t) : t \in E \} \), and consider

\[
\left\{ \frac{1}{\sqrt{n}} \sum_{j=1}^n [I_{X_j(t) \leq y} - \Pr(X(t) \leq y)] : t \in E, y \in \mathbb{R} \right\}, \tag{7}
\]

with the goal of determining when these processes converge in distribution in some uniform sense to a mean zero Gaussian process. For example, if the process \( X \) has continuous sample paths on \( E \), then \( S = C(E) \) and the class of sets \( C \) in (i) of Theorem 1 consists of the sets \( \{ f \in C(E) : f(t) \leq y \} \) for \( t \in E \) and \( y \in \mathbb{R} \), and we examine when \( C \in CLT(P) \). As a result of the previous definitions, the limiting centered Gaussian process has a version with sample paths in a separable subspace of \( \ell_\infty(C) \), and as a consequence of the addendum to Theorem 1.5.7 of [?], p. 37, almost all sample paths are uniformly \( L_2 \) continuous on \( C \) provided we identify the indicator functions of
sets in $\mathcal{C}$ with $\mathcal{C}$ itself. Furthermore, we also have that $\mathcal{C}$ is totally bounded in the $L_2$ distance with this identification. In addition, the following remark is important in this setting.

*Remark 1.* The assumption that a centered process $\{X(t) : t \in T\}$ with $L_2$ distance $d$ is sample bounded and uniformly continuous on $(T,d)$ is easily seen to follow if $(T,d)$ is totally bounded and the process is uniformly continuous on $(T,d)$. Moreover, if the process is Gaussian, the converse also holds using Sudakov’s result as presented in Corollary 3.19 of [?].

To state and prove our result we will make use of a distributional transform that appears a number of places in the literature, see [?]. The paper [?] provides an excellent introduction to its history, and some uses. In particular, it is used there to obtain an elegant proof of Sklar’s theorem, see [?], and also in some related applications.

Given the distribution function $F$ of a real valued random variable, $Y$, let $V$ be a uniform random variable independent of $Y$. In this paper we use the distributional transform of $F$ defined as

$$\tilde{F}(x,V) = F(x^-) + V(F(x) - F(x^-)),$$

and Proposition 1 in [?] shows that

$$\tilde{F}(Y,V) \text{ is uniform on } [0,1]. \quad (8)$$

Rüschendorf calls $\tilde{F}(Y,V)$ the distributional transform of $Y$, and we also note that $\tilde{F}(x,V)$ is non-decreasing in $x$.

## 3 The Main Conditions

Let $\{X(t) : t \in E\}$ be a stochastic process as in [7], and assume

$$\rho(s,t) = (E(H_t - H_s)^2)^{\frac{1}{2}}, \quad s,t \in E,$$
where \( \{H(t) : t \in E\} \) is some Gaussian process which is sample bounded and \( \rho \) uniformly continuous on \( E \). In our main result, see Theorem 3 below, we hypothesize the relationship between \( \{X(t) : t \in E\} \) and \( \rho(s, t), s, t \in E \) given in (10). The importance of this condition in the proof of our theorem is 2-fold. First, it allows one to establish the limiting Gaussian process for our CLT actually exists. This verifies condition (ii) in Theorem 2 above, and is accomplished via a subtle application of the necessary and sufficient conditions for the existence and sample function regularity of a Gaussian process given in ([?]). Secondly, it also allows us to verify that the remaining non-trivial condition sufficient for our CLT, namely condition (iii) of Theorem 2, applies in this setting. This is useful in applications as condition (iii) is in terms of a single random element involved in our CLT, and hence is far easier to verify than the typical condition which depends on the full sample of the random elements as in [3]. The reader should also note that Theorem 2 is a refinement of a number of previous results sufficient for the CLT, and covers a number of important earlier empirical process papers.

The existence of such a \( \rho(\cdot, \cdot) \) is obtained for a number of specific processes \( \{X(t) : t \in E\} \) in section 7. Nevertheless, given the process \( X \), determining whether a suitable \( \rho \) satisfying (10) exists, or does not exist, may be quite difficult. However, using (12)) we may limit our choice for \( \rho \) in (10) to be such that

\[
\rho(s, t) \geq c^{-1} \sup_{x \in \mathbb{R}} (\mathbb{E}(|I_{X_t \leq x} - I_{X_s \leq x}|^2))^{\frac{1}{2}}
\]

for all \( s, t \in E \) and some constant \( c \in (0, \infty) \).

Throughout, \( \rho(s, t), s, t \in E \), denotes the \( L_2 \) metric of a Gaussian process indexed by \( E \) and, to simplify notation, we let

\[
\tilde{F}_t(x) \equiv (\tilde{F}_t)(x, V), x \in \mathbb{R}
\]

be the distributional transform of \( F_t \), the distribution function of \( X_t \). Note that this simplification of notation also includes using \( X_t \) for \( X(t) \) when
the extra parenthesis make the latter clumsy or unnecessary. Moreover, this variable notation is also employed for other stochastic processes in the paper. In addition, for each \( \epsilon > 0 \), let

\[
\sup_{\{s,t \in E: \rho(s,t) \leq \epsilon\}} \Pr^*(|\tilde{F}_t(X_s) - \tilde{F}_t(X_t)| > \epsilon^2) \leq L \epsilon^2, \tag{9}
\]

(the weak L condition)

and

\[
\sup_{t \in E} \Pr^*(\sup_{s: \rho(s,t) \leq \epsilon} |\tilde{F}_t(X_s) - \tilde{F}_t(X_t)| > \epsilon^2) \leq L \epsilon^2, \tag{10}
\]

(the L condition)

Remark 2. In the L conditions the probabilities involve an \( \epsilon^2 \). However, since for any constant, \( C \in (0, \infty) \), an \( L_2 \) metric, \( \rho \), is pre-Gaussian iff \( C\rho \) is pre-Gaussian, WLOG we can change to \( C\epsilon^2 \). Moreover, note that any constant \( L \) sufficient for (10) will also suffice for (9), and hence to simplify notation we do not distinguish between them.

Lemma 1. Let \( L \) be as in (9), and take \( s, t \in E \). Then, for all \( x \in \mathbb{R} \)

\[
\Pr(X_s \leq x < X_t) \leq (L + 1)\rho^2(s,t). \tag{11}
\]

and by symmetry

\[
\mathbb{E}|I_{X_t \leq x} - I_{X_s \leq x}| = \Pr(X_t \leq x < X_s) + \Pr(X_s \leq x < X_t) \tag{12}
\]

\[
\leq 2(L + 1)\rho^2(s,t).
\]

Further, we have

\[
\sup_x |F_t(x) - F_s(x)| \leq 2(L + 1)\rho^2(s,t). \tag{13}
\]
Remark 3. As in Lemma 1, Lemmas 2 and 3 below use only the weak L condition (9). Actually for Lemmas 2 and 3 all we need is Lemma 1. However in Lemma 4 we need the stronger form as stated in (10).

Proof. Since \( \tilde{F}_t \) is non-decreasing and \( x < y \) implies \( F_t(x) \leq \tilde{F}_t(y) \), we have

\[
P(X_s \leq x < X_t) \leq P(\tilde{F}_t(X_s) \leq \tilde{F}_t(x), F_t(x) \leq \tilde{F}_t(X_t)).
\]

Thus

\[
P(X_s \leq x < X_t) \leq P(F_t(x) \leq \tilde{F}_t(X_t) \leq F_t(x) + \rho^2(s,t), \tilde{F}_t(X_s) \leq \tilde{F}_t(x)) + \\
P(\tilde{F}_t(X_t) > F_t(x) + \rho^2(s,t), \tilde{F}_t(X_s) \leq \tilde{F}_t(x)),
\]

and hence

\[
P(X_s \leq x < X_t) \leq P(F_t(x) \leq \tilde{F}_t(X_t) \leq F_t(x) + \rho^2(s,t)) + \\
P(|\tilde{F}_t(X_t) - \tilde{F}_t(X_s)| > \rho^2(s,t)).
\]

Now (9) implies for all \( s, t \in E \) that

\[
P(|\tilde{F}_t(X_t) - \tilde{F}_t(X_s)| > \rho^2(s,t)) \leq L\rho^2(s,t),
\]

since its failure for \( s_0, t_0 \in E \) and \( \epsilon = \rho(s_0, t_0) \) in (9) implies a contradiction. Therefore, since \( \tilde{F}_t(X_t) \) is uniform on \([0, 1]\), we have

\[
P(X_s \leq x < X_t) \leq \rho^2(s,t) + L\rho^2(s,t).
\]

The last conclusion follows by moving the absolute values outside the expectation. \( \square \)

4 The Main Result.

Recall the relationship of the \( X \) process and \( \rho \) as described at the beginning of section 3. Then we have:
Theorem 3. Let $\rho$ be given by $\rho^2(s, t) = \mathbb{E}(H(s) - H(t))^2$, for some centered Gaussian process $H$ that is sample bounded and uniformly continuous on $(E, \rho)$ with probability one. Furthermore, assume that for some $L < \infty$, and all $\epsilon > 0$, the $L$ condition (10) holds, and $D(E)$ is a collection of real valued functions on $E$ such that $P(X(\cdot) \in D(E)) = 1$. If

$$C = \{C_{s,x} : s \in E, x \in \mathbb{R}\},$$

where

$$C_{s,x} = \{z \in D(E) : z(s) \leq x\}$$

for $s \in E, x \in \mathbb{R}$, then $C \in CLT(P)$.

Remark 4. Note that a sample function of the $X$-process is in the set $C_{s,x}$ iff $X_s \leq x$. Hence, if we identify a point $(s, x) \in E \times \mathbb{R}$ with the set $C_{s,x}$, then instead of saying $C \in CLT(P)$, we will often say

$$\{I_{X_s \leq x} - \Pr(X_s \leq x) : s \in E, x \in \mathbb{R}\}$$

satisfies the CLT in $\ell_\infty(C)$ (or in $\ell_\infty(E \times \mathbb{R})$).

Remark 5. At this point one might guess the reader is questioning the various assumptions in Theorem 3. First we mention that $D(E)$ is some convenient function space. For example, typically the process $X$ has continuous sample paths on $E$, so $D(E) = C(E)$ in these situations. More perplexing, at least for most readers, is probably the appearance of the distributional transforms $\{\tilde{F}_t : t \in E\}$ in the $L$-condition (10). If the distribution functions $F_t$ are all continuous, then $F_t = \tilde{F}_t, t \in E$, and our proof obviously holds with $F_t$ replacing $\tilde{F}_t$ in the $L$ condition. However, without all the distribution functions $F_t$ assumed continuous, the methods required in our proof fail with this substitution. An interesting case where the distributional transforms are useful occurs when one has a point $t_0 \in E$ such that $P(X(t) = X(t_0) \text{ for all } t \in E) = 1$, and $F_{t_0}$ is possibly discontinuous. In
this situation, the $L$ condition \([10]\) holds for the Gaussian process $H(t) = g$ for all $t \in E$, $g$ a standard Gaussian random variable, and $X(t_0)$ having any distribution function $F_{t_0}$. Thus Theorem 3 applies, and yields the classical empirical CLT when the set $S$ is the real line and the class of sets consists of half-lines for all laws $F_{t_0}$. A similar result also applies if $E$ is a finite disjoint union of non-empty sets, and the process $\{X_t : t \in E\}$ is constant on each of the disjoint pieces of $E$ regardless of the distribution functions $F_t, t \in E$. More importantly, however, is not only does our method of proof fail when the $L$-condition is given in terms of at least one discontinuous $F_t$, but even when the empirical process is assumed to be pregaussian, the example in section 8.4 shows that the $L$-condition for the process $\{F_s(X_s) : s \in E\}$ is not sufficient for the CLT of Theorem 3. One needs to assume something more, and our results show the $L$-condition for the process $\{\tilde{F}_s(X_s) : s \in E\}$ is sufficient.

In section 7 we will provide another theorem showing how Theorem 3 can be applied, and hence the examples obtained there are motivation for its formulation in terms of the $L$ condition \([10]\). The following remark also motivates the presence of the process $\{\tilde{F}_t(X_s) : s \in E\}$ and the $L$-condition in our CLT. In particular, we sketch an argument that for each $t \in E$ a symmetric version of this process satisfies a CLT in $\ell_\infty(E)$. This remark is meant only for motivation, and in its presentation we are unconcerned with a number of details.

**Remark 6.** Let

$$\{I_{X_s \leq x} - \Pr(X_s \leq x) : s \in E, x \in \mathbb{R}\}$$

satisfy the Central Limit Theorem in the closed subspace of $\ell_\infty(E \times \mathbb{R})$ consisting of functions whose $s$-sections are Borel measurable on $\mathbb{R}$. We denote this subspace by $\ell_\infty,m(E \times \mathbb{R})$, and also assume the distribution functions $F_t$ are all continuous. Then, for each fixed $t \in E$ we define the bounded linear
operator $\phi : \ell_\infty,m(E \times \mathbb{R}) \rightarrow \ell_\infty(E)$ given by

$$\phi(f)(s) = \int f(s,x)F_t(dx).$$

Now by the symmetrization Lemma (Lemma 2.7 in [?]) we have for a Rademacher random variable $\epsilon$ independent of the empirical process variables that \{$\epsilon I_{C_{s,x}} : s \in E, x \in \mathbb{R}$\} satisfy the CLT in $\ell_\infty(E \times \mathbb{R})$. Taking $f(s,x) = I_{C_{s,x}}$, we have for all $t \in E$ fixed that

$$\phi(f)(s) = 1 - F_t(z(s)^-) = 1 - F_t(z(s)),$$

as we are assuming the $F_t$ are continuous. Therefore the continuous mapping theorem (see, e.g., Theorem 1.3.6 in [?]) implies that for each $t \in E$,

$$Z_n(s) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \epsilon_j (1 - F_t(X_s)), s \in E,$$

satisfies the CLT in $\ell_\infty(E)$. In addition, since we are assuming $F_t = \tilde{F}_t$, we should then have “asymptotically small oscillations”, namely, for every $\delta > 0$ there exists $\epsilon > 0$ such that

$$\Pr^*(\sup_{\rho(s,t) \leq \epsilon} \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \epsilon_j (\tilde{F}_t(X_s) - \tilde{F}_t(X_t)) > \sqrt{n}\delta) \leq \delta.$$

By using standard symmetry arguments this last probability dominates

$$\frac{1}{2} \Pr^*(\max_{j \leq n} \sup_{\rho(s,t) \leq \epsilon} |\tilde{F}_t(X_s) - \tilde{F}_t(X_t)| > \sqrt{n}\delta) \leq \delta,$$

which (again by standard arguments) implies (modulo multiplicative constants)

$$n\Pr^*(\sup_{\rho(s,t) \leq \epsilon} |\tilde{F}_t(X_s) - \tilde{F}_t(X_t)| > \sqrt{n}\delta) \leq \delta.$$

While this is different from the hypotheses in our Theorem, it indicates that the quantity $\sup_{\rho(s,t) \leq \epsilon} |\tilde{F}_t(X_s) - \tilde{F}_t(X_t)|$ is relevant to any such theorem.
5 Preliminaries for generic chaining.

Let $T$ be an arbitrary countable set. Then, following Talagrand in [?] we have:

**Definition 1.** An admissible sequence is an increasing sequence $(A_n)$ of partitions of $T$ such that

$$\text{Card} A_n \leq N_n = 2^{2^n}.$$ 

The partitions $(A_n)$ are increasing if every set in $A_{n+1}$ is a subset of some set of $A_n$.

We also have

**Definition 2.** If $t \in T$, we denote by $A_n(t)$ the unique element of $A_n$ that contains $t$. For a psuedo-metric $e$ on $T$, and $A \subseteq E$, we write $\Delta_e(A)$ to denote the diameter of $A$ with respect to $e$.

Using generic chaining and the previous definitions, Theorem 1.4.1 of [?] includes the following:

**Theorem 4.** Let $\{H(t) : t \in T\}$ be a centered Gaussian process with $L_2$ distance $e(s, t)$ on $T$, and assume $T$ is countable. Then the process $\{H(t) : t \in T\}$ is uniformly continuous and sample bounded on $(T, e)$ with probability one if and only if there is an admissible sequence of partitions of $T$ such that

$$\lim_{r \to \infty} \sup_{t \in T} \sum_{n \geq r} 2^{n/2} \Delta_e(A_n(t)) = 0.$$

(14)

Under the assumption that $H$ is centered Gaussian and uniformly continuous on $(T, e)$, then, recalling remark 1, it follows that $H$ is sample bounded on $T$ is equivalent to $(T, e)$ being totally bounced. Also, an immediate corollary of this result used below is as follows.
Proposition 1. Let $H_1$ and $H_2$ be mean zero Gaussian processes with $L_2$ distances $e_1, e_2$, respectively, on $T$. Furthermore, assume $T$ is countable, and $e_1(s, t) \leq e_2(s, t)$ for all $s, t \in T$. Then, $H_2$ sample bounded and uniformly continuous on $(T, e_2)$ with probability one, implies $H_1$ is sample bounded and uniformly continuous on $(T, e_1)$ with probability one.

Remark 7. One can prove this using Slepian’s Lemma (see, e.g., [?]). However, the immediate conclusion is that $H_1$ is sample bounded and uniformly continuous on $(T, e_2)$. Then, a separate argument is needed to show the statement in this proposition. Using the more classical formulation for continuity of Gaussian processes involving majorizing measures (see, e.g., Theorem 12.9 of [?]), the result also follows similarly to what is explained below.

Proof. By the previous theorem $\{H_2(t) : t \in T\}$ is sample bounded and uniformly continuous on $(T, e_2)$ with probability one if and only if there exists an admissible sequence of partitions of $T$ such that

$$
\lim_{r \to \infty} \sup_{t \in T} \sum_{n \geq r} 2^{n/2} \Delta_{e_2}(A_n(t)) = 0.
$$

Since $\Delta_{e_1}(A_n(t)) \leq \Delta_{e_2}(A_n(t))$, we have

$$
\lim_{r \to \infty} \sup_{t \in T} \sum_{n \geq r} 2^{n/2} \Delta_{e_1}(A_n(t)) = 0,
$$

and hence Theorem 4 implies $H_1$ is sample bounded and uniformly continuous on $(T, e_1)$ with probability one. Thus the proposition is proven. \qed

6 The Proof of Theorem 3.

First we establish some necessary lemmas, and the section ends with the proof of Theorem 3. Throughout we take as given the assumptions and notation of that theorem.
6.1 Some Additional Lemmas.

In order to simplify notation, we denote the $L_2$ distance on the class of indicator functions

$$\mathcal{F} = \{I_{X_s \leq x} : s \in E, x \in \mathbb{R}\}$$

by writing

$$\tau((s, x), (t, y)) = \{E((I_{X_s \leq x} - I_{X_t \leq y})^2)\}^{\frac{1}{2}},$$

and identifying $\mathcal{F}$ with $E \times \mathbb{R}$. Our next lemma relates the $\tau$-distance and the $\rho$-distance. It upgrades (12) when $x \neq y$.

**Lemma 2.** Assume that (9) holds. Then

$$\tau^2((s, x), (t, y)) \leq \min_{u \in \{s, t\}} |F_u(y) - F_u(x)| + (2 + 2L + 2)^2 \rho^2(t, s). \quad (15)$$

Moreover, if $Q$ denotes the rational numbers, there is a countable dense set $E_0$ of $(E, \rho)$ such that $\mathcal{F}_0 = \{I_{X_s \leq x} : (s, x) \in E_0 \times Q\}$ is dense in $(\mathcal{F}, \tau)$.

**Proof.** First observe that by using the symmetry in $s$ and $t$ of the right hand term of (15), we have by applying (11) in the second inequality below that

$$\tau^2((s, x), (t, y)) = \mathbb{E}|I_{X_t \leq y} - I_{X_s \leq x}| \leq \mathbb{E}|I_{X_t \leq y} - I_{X_t \leq x}| + \mathbb{E}|I_{X_t \leq x} - I_{X_s \leq x}|$$

$$= |F_t(y) - F_t(x)| + \Pr(X_s \leq x < X_t) + \Pr(X_t \leq x < X_s)$$

$$\leq |F_t(y) - F_t(x)| + (2 + 2L)\rho^2(s, t).$$

Similarly, we also have by applying (11) again that

$$\tau^2((s, x), (t, y)) \leq |F_s(y) - F_s(x)| + (2 + 2L)\rho^2(s, t).$$

Combining these two inequalities for $\tau$, the proof of (15) holds. Since $(E, \rho)$ is assumed totally bounded, there is a countable dense set $E_0$ of $(E, \rho)$, and hence the right continuity of the distribution functions and (15) then imply the final statement in Lemma 2. \qed
Using Lemma 2 and the triangle inequality we can estimate the \( \tau \)-diameter of sets as follows.

**Corollary 1.** If \( t_B \in B \subseteq E \) and \( D \subseteq \mathbb{R} \), then

\[
\text{diam}_{\tau}(B \times D) \leq 2\{(2L + 2)^{1/2}\text{diam}_{\rho}(B) + \sup_{x,y \in D} |F_{t_B}(y) - F_{t_B}(x)|^{1/2}\}.
\]

**Lemma 3.** Assume that \((s, x)\) satisfies

\[
\tau((s, x), (t, y)) = \|I_{X_t \leq x} - I_{X_t \leq y}\|_2 \leq \epsilon,
\]

\(\rho(s, t) \leq \epsilon\), and \([9]\) holds. Then, for \( c = (2L + 2)^{1/2} + 1\),

\[
|F_t(x) - F_t(y)| \leq c\epsilon^2,
\]

or in other words

\[
|F_t(x) - F_t(y)| \leq (c \max\{\tau((s, x), (t, y)), \rho(s, t)\})^2.
\]

**Proof.** Using \([11]\) in the second inequality below we have

\[
|F_t(y) - F_t(x)|^{1/2} = \|I_{X_t \leq x} - I_{X_t \leq y}\|_2 \leq \|I_{X_t \leq x} - I_{X_t \leq y}\|_2 + \|I_{X_t \leq x} - I_{X_t \leq x}\|_2
\]

\[
\leq \epsilon + \left(\Pr(X_s \leq x < X_t) + \Pr(X_t \leq x < X_s)\right)^{1/2}
\]

\[
\leq \epsilon + (2L\epsilon^2 + 2\epsilon^2)^{1/2} = [(2L + 2)^{1/2} + 1]\epsilon \equiv c\epsilon.
\]

Hence the lemma is proven. \(\square\)

The next lemma is an important step in verifying the weak-\(L_2\) condition in item (iii) of Theorem 2 above, i.e. see Theorem 4.4 of \([?]\).

**Lemma 4.** If \([10]\) holds, \(c\) is as in Lemma 3, and

\[
\lambda((s, x), (t, y)) = \max\{\tau((s, x), (t, y)), \rho(s, t)\},
\]

then for all \((t, y)\) and \(\epsilon > 0\)

\[
\Pr^*\left(\sup_{\{(s, x): \lambda((t, y), (s, x)) \leq \epsilon\}} |I_{X_t \leq y} - I_{X_t \leq x}| > 0\right) \leq (2c^2 + 2L + 1)\epsilon^2.
\]

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Proof. First we observe that

\[
\Pr^\ast\left( \sup_{\{(s,x):\lambda((t,y),(s,x))\leq \epsilon\}} |I_{X_t \leq y} - I_{X_s \leq x}| > 0 \right) = \Pr^\ast\left( \sup_{\{(s,x):\lambda((t,y),(s,x))\leq \epsilon\}} I_{X_t \leq y, X_s > x} + I_{X_s \leq x, X_t > y} > 0 \right).
\]

Again, using the fact that \(x < y\) implies \(F_t(x) \leq \tilde{F}_t(y)\), we have

\[
\Pr^\ast\left( \sup_{\{(s,x):\lambda((t,y),(s,x))\leq \epsilon\}} |I_{X_t \leq y} - I_{X_s \leq x}| > 0 \right) \leq \Pr^\ast\left( \sup_{\{(s,x):\lambda((t,y),(s,x))\leq \epsilon\}} I_{\tilde{F}_t(X_t) \leq \tilde{F}_t(y), F_t(x) \leq \tilde{F}_t(X_s) > 0} \right)
\]

\[ + \Pr^\ast\left( \sup_{\{(s,x):\lambda((t,y),(s,x))\leq \epsilon\}} I_{\tilde{F}_t(X_s) \leq \tilde{F}_t(x), F_t(y) \leq \tilde{F}_t(X_t) > 0} \right) = I + II,
\]

where

\[
I = \Pr^\ast\left( \sup_{\{(s,x):\lambda((t,y),(s,x))\leq \epsilon\}} I_{\tilde{F}_t(X_t) \leq \tilde{F}_t(y), F_t(x) \leq \tilde{F}_t(X_s) > 0} \right) \quad (16)
\]

and

\[
II = \Pr^\ast\left( \sup_{\{(s,x):\lambda((t,y),(s,x))\leq \epsilon\}} I_{\tilde{F}_t(X_s) \leq \tilde{F}_t(x), F_t(y) \leq \tilde{F}_t(X_t) > 0} \right). \quad (17)
\]

At this point we use Lemma 3 to see that in (16) we can use

\[
\inf_{\{(s,x):\lambda((t,y),(s,x))\leq \epsilon\}} F_t(x) \geq F_t(y) - (c\epsilon)^2.
\]
Therefore, by again using (10)

\[ I \leq \Pr^*(\sup_{\{(s,x):\lambda((t,y),(s,x)) \leq \epsilon\}} I_{\tilde{F}_t(X_t) \leq F_t(y), F_t(y) - (c\epsilon)^2 \leq \tilde{F}_t(X_t) > 0}) \]

\[ \leq \Pr^*(\sup_{\{(s,x):\lambda((t,y),(s,x)) \leq \epsilon\}} I_{\tilde{F}_t(X_t) \leq F_t(y), F_t(y) - (c\epsilon)^2 \leq \tilde{F}_t(X_t) + \epsilon^2 > 0}) + L\epsilon^2 \]

\[ \leq \Pr(F_t(y) - (c\epsilon)^2 - \epsilon^2 \leq \tilde{F}_t(X_t) \leq F_t(y)) + L\epsilon^2 \]

\[ \leq (c^2 + L + 1)\epsilon^2 \text{ by (8)}. \]

Now, we estimate II in (17). Using the fact that \( \tilde{F}_t(x) \leq F_t(x) \) for all \( x \), Lemma 3, and our definition of \( L \), we therefore have

\[ \Pr^*(\sup_{\{(s,x):\lambda((t,y),(s,x)) \leq \epsilon\}} I_{\tilde{F}_t(X_t) \leq \tilde{F}_t(X_t), F_t(y) \leq \tilde{F}_t(X_t) > 0}) \]

\[ \leq \Pr(\tilde{F}_t(X_t) - \epsilon^2 \leq F_t(y) + (c\epsilon)^2, F_t(y) \leq \tilde{F}_t(X_t)) + L\epsilon^2 \]

\[ \leq (c^2 + L + 1)\epsilon^2. \]

\[ \square \]

### 6.2 The Construction and the proof of Theorem 3.

Since \((E, \rho)\) is totally bounded by Remark 1, take \( E_0 \) to be any countable dense subset of \( E \) in the \( \rho \) distance. Then by Theorem 4, Talagrand’s continuity theorem, there exists an admissible sequence of partitions, \( B_n \) of \( E_0 \) for which

\[ \limsup_{r \to \infty} \sup_{t \in E_0} \sum_{n \geq r} 2^{n/2} \Delta_\rho(B_n(t)) = 0. \]
Fix $n$. Then, for each $B \in \mathcal{B}_{n-1}$ choose $t_B \in B$. Fix the distribution function $F_B := F_{t_B}$ and $\mu_B$ the associated probability measure. Put $\alpha = (\Delta \rho(B) + 2^{-n})^2$ and set $z_1 = \sup\{x \in \mathbb{R} : F_B(x) < \alpha\}$. We consider two cases.

- $F_B(z_1) \leq \alpha$ and
- $F_B(z_1) > \alpha$.

In the first case $F_B(z_1) = \alpha$. If $F_B(z_1) < \alpha$, then by right continuity there exist $w > z_1$ such that $F_B(w) < \alpha$, which contradicts the definition of $z_1$. In this case we consider $C_1 = (-\infty, z_1]$ and $D_1 = \emptyset$.

In the second case we let $C_1 = (-\infty, z_1)$ and $D_1 = \{z_1\}$. In either case $\mu_B(C_1 \cup D_1) \geq \alpha$.

Now assume that we have constructed $C_1, \ldots, C_k$ and $D_1, \ldots, D_k$. Therefore we have $z_k$. If $\mu_B((z_k, \infty)) \geq \alpha$, let $z_{k+1} = \sup\{x > z_k : F_B(x) - F_B(z_k) < \alpha\}$. If $z_{k+1} = \infty$, we set $C_{k+1} = (z_k, \infty)$ and $D_{k+1} = \emptyset$. Otherwise, if $z_{k+1} < \infty$ there are two cases.

Again, we have two cases.

- $F_B(z_{k+1}) - F_B(z_k) \leq \alpha$ and
- $F_B(z_{k+1}) - F_B(z_k) > \alpha$.

In the first case we consider $C_{k+1} = (z_k, z_{k+1}]$ and $D_{k+1} = \emptyset$. In the second case we let $C_{k+1} = (z_k, z_{k+1})$ and $D_{k+1} = \{z_{k+1}\}$. As before, $\mu_B(C_{k+1} \cup D_{k+1}) \geq \alpha$. Hence, there can be at most $\frac{1}{\alpha}$ steps before $\{C_k, D_k\}_k$ cover $\mathbb{R}$.

Therefore, after eliminating any empty set, we have a cover with at most $\frac{2}{\alpha}$ sets. By our choice of $\alpha$ the cover has at most $2^{2n+1}$ sets. Hence since we have $B \in \mathcal{B}_{n-1}$, the number of sets used to cover $E_0 \times \mathbb{R}$ is less than or equal to $2^{2n-1}2^{2n+1}$. The reader should note that the points $\{z_k\}$ depend on the set $B$, but we have suppressed that to simplify notation. We now check the $\tau$-diameters of the non-empty $B \times C_k$ and $B \times D_k$. 
Estimating these diameters by doubling the radius of the sets, the triangle inequality allows us to upper bound their radius using one of \( s \) and \( t \) to be \( t_B \). Also note that in Lemma 2, or Corollary 1, the term which contains \(|F_{t_B}(y) - F_{t_B}(x)|\) would cause trouble in the case \( D_k \neq \emptyset \), since this is only known to be \( \geq \alpha \). Luckily it doesn’t appear when \( D_k \neq \emptyset \).

First we consider the \( \tau \)-diameter of sets of the form \( B \times C_k \) when \( D_k = \emptyset \). Then \( C_k = (z_{k-1,B}, z_{k,B}] \). Hence for \((s,x), (t,y) \in B \times C_k \), Corollary 1 implies

\[
\Delta_{\tau}(B \times (z_{B,k-1}, z_{B,k}]) \leq 2 \left( (2L + 2)^{1/2} \Delta_{\rho}(B) + \Delta_{\rho}(B) + \frac{1}{2^n} \right).
\]

When \( D_k \neq \emptyset \), then \( C_k = (z_{k-1,B}, z_{k,B}] \), so again by Corollary 1 the \( \tau \)-diameter of \( B \times C_k \) has an upper bound as in the previous case.

If \( D_k \neq \emptyset \), then the only element of \( D_k \) is \( z_{k,B} \) and by Corollary 1 we have

\[
\Delta_{\tau}(B \times D_k) \leq 2(2L + 2)^{1/2} \text{diam}_{\rho}(B).
\]

So, in either case

\[
\Delta_{\tau}(B \times C_{B,k} \text{ or } D_{B,k}) \leq 2 \left( (2L + 2)^{1/2} \Delta_{\rho}(B) + \Delta_{\rho}(B) + \frac{1}{2^n} \right). \tag{19}
\]

**Lemma 5.** Let \( \mathcal{G}_n \) be a sequence of partitions of an arbitrary parameter set \( T \) with pseudo metric \( e \) on \( T \) satisfying both

1. \( \text{Card}(\mathcal{G}_n) \leq 2^n \) and
2. \( \lim_{r \to \infty} \sup_{t \in T} \sum_{n \geq r} 2^{n/2} \Delta_e(G_n(t)) = 0 \),

and set \( \mathcal{H}_n := \mathcal{P}(\bigcup_{1 \leq k \leq n-1} \mathcal{G}_k) \), where \( \mathcal{P}(\mathcal{D}) \) denotes the minimal partition generated by the sets in \( \mathcal{D} \). Then the sequence \( \mathcal{H}_n \) (Notice the \( n - 1 \) in the union) also satisfies those conditions.

**Proof.** The first condition holds since a simple induction on \( n \) implies the minimal partition

\[
\mathcal{H}_n = \mathcal{P}(\bigcup_{1 \leq k \leq n-1} \mathcal{G}_k)
\]
has cardinality at most $\prod_{k=1}^{n-1} 2^{2^k} \leq 2^{2^n}$. The second condition holds since the partitions are increasing collections of sets, and hence $\text{diam}_e(H_n(t)) \leq \text{diam}_e(G_{n-1}(t))$. \hfill \Box

**Lemma 6.** Let $E_0$ be a countable dense subset of $(E, \rho)$. Then there exists an admissible sequence of partitions $\{A_n : n \geq 0\}$ of $E_0 \times \mathbb{R}$ such that

$$\lim_{r \to \infty} \sup_{(t,y) \in E_0 \times \mathbb{R}} \sum_{n \geq r} 2^{n/2} \Delta_\tau(G_n((t,y))) = 0. \quad (20)$$

**Proof.** We construct the admissible sequence of partitions $A_n$ as above. More precisely, let $\{B_n : n \geq 0\}$ be an increasing sequence of partitions of $E_0$ such that (18) holds, and after the construction above we also have (19). That is, for $k \geq 1$ let

$$G_k = \{B \times F : B \in B_{k-1}, F \in \mathcal{E}_B\}$$

where

$$\mathcal{E}_B = \{C_{j,B}, D_{j,B} \text{ all sets nonempty}\}$$

and $C_{j,B}, D_{j,B}$ are constructed from $B \in B_{k-1}$ as above. Then, for $n \geq 3$ set

$$A_n = \mathcal{P}(\cup_{2 \leq k \leq n-1} G_k),$$

where $\mathcal{P}(\mathcal{D})$ is the minimal partition generated by the sets in $\mathcal{D}$, and for $n = 1, 2$ we take $A_n$ to be the single set $E_0 \times \mathbb{R}$. Since the cardinality of the partitions $G_k$ defined above is less than or equal to $2^{2^{k-1}} 2^{2^k+1}$, then a simple induction on $n$ implies the minimal partition

$$A_n = \mathcal{P}(\cup_{2 \leq k \leq n-1} G_k)$$

has cardinality at most $\prod_{k=2}^{n-1} 2^{2^{k-1}} 2^{2^k+1} \leq 2^{2^n}$. By (19) and Lemma 5 we have

$$\sup_{(t,y) \in E_0 \times \mathbb{R}} \sum_{n \geq r} 2^{n/2} \Delta_\tau(A_n(t,y)) \leq C\{\sup_t \sum_{n \geq r} 2^{n/2} \Delta_\rho(B_n(t)) + \sum_{n \geq r} 2^{n/2} 2^{-n}\}.$$ 

Thus (18) implies $\tau$ satisfies (20) with respect to the sequence of admissible partitions $A_n$ on $E_0 \times \mathbb{R}$. \hfill \Box
Proof of Theorem 3. Let $Q$ denote the rational numbers. Then, if we restrict the partitions $\mathcal{A}_n$ of $E_0 \times \mathbb{R}$ in Lemma 6 to $E_0 \times Q$, we immediately have

$$\lim_{r \to \infty} \sup_{(t,y) \in E_0 \times Q} \sum_{n \geq r} 2^{n/2} \Delta_r(G_n((t,y))) = 0, \quad (21)$$

and $(E_0 \times Q, \tau)$ is totally bounded. Now let $\{G_{(s,x)} : (s, x) \in E \times \mathbb{R}\}$ be a centered Gaussian process with $E(G_{(s,x)}G_{(t,y)}) = P(X_s \leq x, X_t \leq y)$. Then, $G$ has $L_2$ distance $\tau$, and by (21) and Theorem 4, it is uniformly continuous on $(E_0 \times Q, \tau)$. Hence if $\{H_{(s,x)} : (s, x) \in E_0 \times Q\}$ is a centered Gaussian process with

$$E(H_{(s,x)}H_{(t,y)}) = P(X_s \leq x, X_t \leq y) - P(X_s \leq x)P(X_t \leq y),$$

then

$$E((H_{(s,x)} - H_{(t,y)})^2) = \tau^2((s, x), (t, y)) - (P(X_s \leq x) - P(X_t \leq y))^2).$$

Hence the $L_2$ distance of $H$ is smaller than that of $G$, and therefore Proposition 1 implies the process $H$ is uniformly continuous on $(E_0 \times Q, d_H)$. By Lemma 2 the set $E_0 \times Q$ is dense in $(E \times \mathbb{R}, \tau)$, and since

$$d_H((s, x), (t, y)) \leq \tau((s, x), (t, y))$$

we also have $E_0 \times Q$ is dense in $(E \times \mathbb{R}, d_H)$. Thus the Gaussian process $\{H_{(s,x)} : (s, x) \in E \times \mathbb{R}\}$ has a uniformly continuous version, which we also denote by $H$, and since $(E, d_H)$ is totally bounded the sample functions are bounded on $E$ with probability one.

If $\mathcal{F} = \{I_{X_s \leq x} : (s, x) \in E \times \mathbb{R}\}$, then since

$$d_H((s, x), (t, y)) = \rho_P(I_{X_s \leq x}, I_{X_t \leq y}),$$

the continuity of $H$ on $(E \times \mathbb{R}, d_H)$ implies condition (ii) in Theorem 2 is satisfied. Since $I_{X_t \leq y}$ is bounded, condition (i) in Theorem 2 is also satisfied. Therefore, Theorem 3 follows once we verify condition (iii) of Theorem 2.
To verify (iii) we use Lemma 4. As before, we identify the function 
\( f = I_{X_s \leq x} \in \mathcal{F} \) with the point \((s, x) \in E \times \mathbb{R}\). Hence, for the centered Gaussian process

\[ \{G_f : f \in \mathcal{F}\} \]

in (iii) of Theorem 2, for \((s, x) \in E \times \mathbb{R}\), we take the process

\[ \tilde{G}_{(s,x)} = G_{(s,x)} + \tilde{H}_s. \]

In our definition of \( \tilde{G} \) we are assuming:

(a) \( \{\tilde{H}_s : s \in E\} \) is a Gaussian process whose law is that of the process \( \{H_s : s \in E\} \) given in the theorem, and independent of everything in our empirical model, and

(b) \( \{G_{(s,x)} : (s, x) \in E \times \mathbb{R}\} \) is a uniformly continuous and sample bounded version of the Gaussian process, also denoted by \( G_{(s,x)} \), but defined above on \( E_0 \times Q \). The extension to all of \( E \times \mathbb{R} \) again follows by the fact that \( E_0 \times Q \) is dense in \( (E \times \mathbb{R}, \tau) \).

Therefore, \( \tilde{G} \) is sample bounded and uniformly continuous on \( E \times \mathbb{R} \) with respect to its \( L_2 \) distance

\[ d_{\tilde{G}}((s, x), (t, y)) = \{\tau^2((s, x), (t, y)) + \rho^2(s, t)\}^{1/2}. \]

Condition (iii) of Theorem 2 now follows easily from Lemma 4 since for \((t, y)\) fixed,

\[ \{(s, x) : \lambda((s, x), (t, y)) \leq \epsilon/2\} \subseteq \{(s, x) : d_{\tilde{G}}((s, x), (t, y)) \leq \epsilon\}, \]

and for a random variable \( Z \) bounded by one we have

\[ \sup_{t > 0} t^2 P(|Z| > t) \leq P(|Z| > 0). \]

7 Another Theorem and some Examples.

Let \( \{X_t : t \in E\} \) be a sample continuous process such that
(I) \( \sup_{t \in E} |F_t(x) - F_t(y)| \leq k|x - y|^{\beta} \) for all \( x, y \in \mathbb{R} \), \( k < \infty \), and some \( \beta \in (0, 1] \).

(II) \( |X_t - X_s| \leq \Gamma \phi(s, t) \) for all \( s, t \in E \), and for some \( \eta > 0 \) and all \( x \geq x_0 \)

\[ P(\Gamma \geq x) \leq x^{-\eta}. \]

(III) For \( \alpha \in (0, \alpha_0) \) we have \( \phi(s, t) \leq \rho_\alpha(s, t) \), \( s, t \in E \), where \( \rho_\alpha(s, t) \) is the \( L_2 \) distance of a sample bounded, uniformly continuous, centered Gaussian process on \((E, \rho_\alpha)\), which we denote by \( \{H_\alpha(t) : t \in E\}\).

**Theorem 5.** Let \( \{X_t : t \in E\} \) be a sample continuous process satisfying (I-III) above. Then

\[ \{I_{X_s \leq x} - \Pr(X_s \leq x) : s \in E, x \in \mathbb{R}\} \]

satisfies the Central Limit Theorem in \( \mathcal{L}^\infty(E \times \mathbb{R}) \). This CLT also holds under (I-III) provided (II) is strengthened to hold for all \( \eta > 0 \) and \( x \geq x_\eta \), and (III) holds for some \( \alpha \in (0, 2/\beta) \).

**Remark 8.** If the process \( \{X_t : t \in E\} \) in (III) is a Gaussian process, then the CLT of Theorem 5 holds provided (I) is satisfied, (II) is such that \( |X_t - X_s| \leq \Gamma \phi(s, t) \) for all \( s, t \in E \) and \( \Gamma < \infty \), and (III) holds only for some \( \alpha \in (0, 2/\beta) \).

This follows from the final conclusion of the theorem, and the Fernique-Landau-Shepp result, which implies exponential estimates for the probability in (II).

**Proof.** The theorem follows by taking \( \{H(t) : t \in E\} \) in Theorem 3 to be a suitably chosen \( \{H_\alpha(t) : t \in E\} \) as in (III), and then verifying condition (10). Since (I) implies the distribution functions \( F_t \) are all continuous, the distributional transforms in (10) are simply the distributions themselves. Therefore, applying (I), with \( \alpha \in (0, \alpha_0) \) to be chosen later, we have

\[ \Pr^* \left( \sup_{\rho_\alpha(s, t) \leq \epsilon} |F_t(X_s) - F_t(X_t)| \geq \epsilon^2 \right) \leq \Pr^* \left( \sup_{\rho_\alpha(s, t) \leq \epsilon} |X_s - X_t| \geq (\frac{\epsilon^2}{k})^{\beta} \right). \] (22)
Hence (II) implies

$$
\Pr^*(\sup_{\rho_{\alpha}(s,t) \leq \epsilon} |F_t(X_s) - F_t(X_t)| \geq \epsilon^2) \leq P(\Gamma \geq \left(\frac{\epsilon^2}{k}\right)^{\frac{1}{\beta}}(\sup_{\rho_{\alpha}(s,t) \leq \epsilon} \phi(s,t))^{-1}),
$$

and since

$$
(\sup_{\rho_{\alpha}(s,t) \leq \epsilon} \phi(s,t))^{-1} \geq (\sup_{\rho_{\alpha}(s,t) \leq \epsilon} \rho_{\alpha}(s,t))^{\frac{1}{\alpha}} \geq \epsilon^{\frac{1}{\alpha}},
$$

(III) therefore implies

$$
\Pr^*(\sup_{\rho_{\alpha}(s,t) \leq \epsilon} |F_t(X_s) - F_t(X_t)| \geq \epsilon^2) \leq k^{\alpha/\beta} \epsilon^{\frac{2\alpha}{\beta} - \frac{1}{\alpha}} \leq k^{\alpha/\beta} \epsilon^2,
$$

provided $\alpha > 0$ is chosen sufficiently small so that $\eta(\frac{1}{\alpha} - \frac{2}{\beta}) \geq 2$ and $0 < \epsilon < \epsilon_0$ is sufficiently small to imply $k^{\frac{1}{\alpha}} \epsilon^{\frac{2\alpha}{\beta} - \frac{1}{\alpha}} > x_0$. To obtain the final conclusion of the theorem take $\alpha \in (0, 2/\beta)$ and $\eta$ sufficiently large that $\eta(1/\alpha - 2/\beta) > 2$. Then, for $0 < \epsilon < \epsilon_\eta$ sufficiently small that $k^{\frac{1}{\alpha}} \epsilon^{\frac{2\alpha}{\beta} - \frac{1}{\alpha}} > x_\eta$ we again have (24).

**Corollary 2.** Let $\{Y_t : t \in [0,T]\}$ be a sample continuous process $\gamma$-fractional Brownian motion for $0 < \gamma < 1$ such that $Y_0 = 0$ with probability one, and set $X_t = Y_t + Z$, where $Z$ is independent of $\{Y_t : t \in [0,T]\}$ and has a bounded density function. Then,

$$
\{I_{X_s \leq x} - \Pr(X_s \leq x) : s \in [0,T], x \in \mathbb{R}\}
$$

satisfies the Central Limit Theorem in $L^\infty([0,T] \times \mathbb{R})$.

**Remark 9.** The addition of the random variable $Z$ in the previous corollary suffices to make condition (I) hold, and is not used in any other way. Furthermore, below we will see that something of this sort is necessary, since we will show that the CLT of the previous corollary fails for the fractional Brownian motion process $Y$, itself.
Proof. The $L_2$ distance for $\{X_t : t \in [0,T]\}$ is given by
\[
E((X_s - X_t)^2)^{\frac{1}{2}} = c_\gamma |s - t|^{\gamma}, s, t \in [0,T],
\] (25)
and without loss of generality we may assume the process to be normalized so that $c_\gamma = 1$. Furthermore, it is well known that these processes are Hölder continuous on $[0,T]$, i.e. for every $\theta < \gamma/2$ we have
\[
|X_t - X_s| \leq \Gamma|t - s|^\theta, s, t \in [0,T],
\] (26)
where
\[
E(\exp\{c\Gamma^2\}) < \infty,
\] (27)
for some $c > 0$. That $\Gamma$ has exponential moments is due to the Fernique-Landau-Shepp Theorem, and hence conditions (I-III) hold provided we take the Gaussian process $H^\alpha$ of condition (III) to be an $\alpha\theta$ fractional Brownian motion for any $\theta < \gamma/2$. Hence the corollary follows from Theorem 5. \hfill \Box

**Corollary 3.** Let $I = [0,T]$ and $\{Y(s,t) : (s, t) \in I \times I\}$ be a sample continuous Brownian sheet such that $Y(0,t) = Y(s,0) = 0, 0 \leq s, t \leq T$ with probability one, and set $X(s,t) = Y(s,t) + Z$, where $Z$ is independent of $\{Y(s,t) : (s, t) \in I \times I\}$ and has a bounded density function. Then,
\[
\{I_{X(s,t) \leq x} - \Pr(X(s,t) \leq x) : (s, t) \in I \times I\}, x \in \mathbb{R}
\]
satisfies the Central Limit Theorem in $\mathcal{L}^\infty((I \times I) \times \mathbb{R})$.

**Proof.** First of all observe that since $Z$ has a bounded density, and is independent of the Brownian sheet $Y$, we have (I) holding. Furthermore, from Theorem 1 in the paper [?], these processes are Hölder continuous on $I \times I$, i.e. for $(s, t), (u, v) \in I \times I$ and $0 < \gamma < 1/2$ we have
\[
|X(s,t) - X(u,v)| \leq \Gamma((\frac{u - s}{T})^2 + (\frac{v - t}{T})^2)^{\frac{\gamma}{2}}.
\] (28)
where

\[ E(\exp\{c\Gamma^2\}) < \infty, \]  

(29)

for some \( c > 0 \). That \( \Gamma \) has exponential moments is due to the Fernique-Landau-Shepp Theorem, and hence conditions (I-III) hold provided we take the Gaussian process \( H^\alpha_{(s,t)} \) of condition (III) to be

\[ H^\alpha_{(s,t)} = Y_s + Z_t + Z, \quad (s,t) \in I \times I, \]  

(30)

where the processes \( \{Y_s : s \in I\} \) and \( \{Z_t : t \in I\} \) are independent \( \theta \)-fractional Brownian motions, independent of the normal zero-one random variable \( Z \).

To determine \( \theta \) we set

\[ \phi((s,t),(u,v)) = ((u - s T)^2 + (v - t T)^2)^\gamma, \]

where \( 0 < \gamma < 1/2 \). Then,

\[ \phi((s,t),(u,v)) \leq (|u - s T| + |v - t T|)^\gamma \leq [|u - s T|^\gamma + |v - t T|^\gamma] \]

Normalizing the \( Y_s \) and \( Z_t \) processes suitably, we have

\[ \rho^2_\alpha((s,t),(u,v)) = E((H^\alpha_{(u,v)} - H^\alpha_{(s,t)})^2) = (|u - s T|^{2\beta} + |v - t T|^{2\beta}). \]

Hence for \( 0 < \alpha < 1 \), \( \gamma = 1/4 \), and \( \theta < \alpha/8 \) we have

\[ \phi^\alpha((s,t),(u,v)) \leq \rho_\alpha((s,t),(u,v)) \]

Hence the corollary follows from Theorem 5.

\[ \square \]

8 Examples where our CLT fails.

8.1 Fractional Brownian Motions.

Since the class of sets in our CLT arises using the Vapnik-Cervonenkis class of half lines, one might think that perhaps if i.i.d copies of the process \( \{X_t :
Let $t \in E$ satisfied the CLT in $C(E)$, then the class of sets $\mathcal{C}$ of Theorem 3 would satisfy the $CLT(P)$. Our first example shows this fails, even if the process $X_t$ is Brownian motion on $[0, 1]$ tied down at $t = 0$. In this example the process fails condition (I) in Theorem 5 since $P(X_0 = 0) = 1$. To prove this we show the necessary condition for $\mathcal{C}$ to satisfy the $CLT(P)$ appearing in (ii-b) of Theorem 1 fails. More precisely, since measurability is an issue here, the next lemma shows that there is a countable subclass $\mathcal{C}_Q$ of sets in $\mathcal{C}$ such that by Theorem 3 of [?], $\mathcal{C}_Q \not\subset CLT(P)$. Thus $\mathcal{C}$ fails the $CLT(P)$, as otherwise all subclasses also are in $CLT(P)$.

**Lemma 7.** Let $\mathcal{C} = \{C_{t,x} : 0 \leq t \leq 1, \infty < x < \infty\}$, where $C_{t,x} = \{z \in C[0, 1] : z(t) \leq x\}$, and assume $\{X_t : t \in [0, 1]\}$ is a sample continuous Brownian motion tied down at zero. Also, let $\mathcal{C}_Q$ denote the countable subclass of $\mathcal{C}$ given by $\mathcal{C}_Q = \{C_{t,y} \in \mathcal{C} : t, y \in Q\}$, where $Q$ denotes the rational numbers. Then, for each integer $n \geq 1$, with probability one

$$\Delta^{\mathcal{C}_Q}(B_1, \ldots, B_n) = 2^n,$$

where $\Delta^{\mathcal{C}_Q}(B_1, \ldots, B_n) = \text{card}\{C \cap \{B_1, \ldots, B_n\} : C \in \mathcal{C}_Q\}$, and $B_1, \ldots, B_n$ are independent Brownian motions on $[0, 1]$ starting at zero.

**Proof.** Suppose one wants the $k$ functions $\{B_{j_1}, \ldots, B_{j_k}\}$ with probability one, where $1 \leq j_1 < j_2 < \cdots < j_k \leq n$. Let

$$u = (u_1, \ldots, u_n)$$

where $u_{j_1} = u_{j_2} = \cdots = u_{j_k} = 1$ and all other $u_j = 2$. Then $||u|| = (\sum_{j=1}^n u_j^2)^{1/2} = (4n - 3k)^{1/2}$. Now set $v = (v_1, \ldots, v_n)$, where

$$v_{j_1} = v_{j_2} = \cdots = v_{j_k} = \frac{1}{2(4n - 3k)^{1/2}}$$

and all other $v_j = \frac{1}{(4n-3k)^{1/2}}$. Then $v = u/(2||u||)$ and $||v|| = 1/2$. 

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Let

\[ W(s) = \frac{(B_1(s), \cdots, B_n(s))}{(2sLL(1/s))^{1/2}} \]

for \(0 < s \leq 1\). Then the LIL implies with probability one that

\[ \lim \inf_{s \downarrow 0} ||v - W(s)|| = 0, \]

and hence with probability one there are infinitely many rational numbers \(t \downarrow 0\) such that

\[ C_{t,x(t)} \cap \{B_1, \cdots, B_n\} = \{B_{j_1}, \cdots, B_{j_k}\}, \]

where \(x(t) \in Q\) for \(t \in Q\) and

\[ |x(t) - \frac{3(2tLL(1/t))^{1/2}}{4(4n - 3k)^{1/2}}| < \frac{(2tLL(1/t))^{1/2}}{16(4n - 3k)^{1/2}}, \]

Since \(k\) and the set \(\{j_1, \cdots, j_k\}\) were arbitrary, and with probability one we can pick out the subset \(\{B_{j_1}, \cdots, B_{j_k}\}\), the lemma follows as the intersection of \(2^n\) subsets of probability one has probability one.

The failure of the CLT also holds for all sample continuous fractional Brownian motions \(\{X_H(t) : t \in [0, 1]\}\) which are tied down at zero. The proof of this again depends on the law of the iterated logarithm for \(n\) independent copies of this process at \(t = 0\), which then allows us to prove an analogue of the previous lemma. The LIL result at \(t = 0\) for a single copy follows, for example, by Theorem 4.1 of [?], and then one can extend that result to \(n\) independent copies by classical proofs as in [?]. The details of this last step are lengthy, but at this stage are more or less routine in the subject, and hence are omitted. Of course, the CLT for i.i.d. copies of these processes is obvious as they are Gaussian.
8.2 A uniform CLT Example.

In the previous examples, when the distribution function $F_t$ of $X_t$ jumped, the oscillation of the processes at that point caused a failure of our CLT. Hence one other possible idea is that if the process $\{X_t : t \in [0, 1]\}$ is Lip-1 on $[0, 1]$, then our CLT might hold. For example, this is true for the Lip-1 process $X_t = tU, t \in [0, 1]$, where $U$ is uniform on $[0, 1]$. Moreover, in this example the densities of $F_t$ still are unbounded near $t = 0$.

To see this let $X_{t,j}, j = 1, \cdots, n, t \in [0, 1]$ be i.i.d. copies of $X_t = tU, t \in [0, 1]$, and define

$$W_n(C_{t,y}) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} [I(X_{t,j} \in C_{t,y}) - P(X_{t,j} \in C_{t,y})],$$

where $C = \{C_{t,y} : t \in [0, 1], y \in \mathbb{R}\}$ and $C_{t,y} = \{z \in C[0, 1] : z(t) \leq y\}$. Therefore, $W_n(C_{t,y}) = 0$ for all $y \in \mathbb{R}$ when $t = 0$, and also when $y/t \geq 1$. Moreover, if we define $\mathcal{G} = \{(-\infty, r] : 0 \leq r \leq 1\}$, and

$$\phi(I_{C_{t,y}}) = I_{(-\infty, 1]} \text{ if } y/t > 1, 0 \leq t \leq 1, \text{ or } y = 0 \text{ and } t = 0,$$

$$\phi(I_{C_{t,y}}) = I_{(\infty, 0]} \text{ if } y/t \leq 0 \text{ but not } y = 0 \text{ and } t = 0,$$

and

$$\phi(I_{C_{t,y}}) = I_{(-\infty, y/t]} \text{ if } 0 < y/t < 1, 0 < y < 1, 0 < y < 1.$$

Then $\phi(\mathcal{C}) = \{I_{U \leq r} : 0 \leq r \leq 1\} \equiv \mathcal{G}$ and $\phi$ maps $L_2$ equivalence classes of $\mathcal{C}$ onto $\mathcal{G}$ with respect to the law of $\{X_t : t \in [0, 1]\}$ for sets in $\mathcal{C}$, and the law of $U$ for sets in $\mathcal{G}$. Now $\mathcal{G}$ satisfies the $CLT(\mathcal{L}(U))$ by the classical empirical CLT, i.e. see Theorem 16.4 of [?], and since $\phi$ preserves covariances we thus have $W_n(C_{t,y})$ converges weakly to the Gaussian centered process $W(C_{t,y}) = Y(\phi(C_{t,y}))$ on $\mathcal{C}$, where $Y((-\infty, s]) = B(s) - sB(1)$ is the tied down Wiener process on $[0, 1]$, i.e. $B(\cdot)$ is a Brownian motion.
8.3 A Lip-1 Example without the CLT.

In this example we see that the Lip-1 property for \( \{X_t : t \in [0, 1]\} \) is not always sufficient for our CLT. Here \( X_0 = 0 \), and for \( 0 < t \leq 1 \) we define

\[
X_t = t \sum_{j=1}^{\infty} (\alpha_j(t) + 2) I_{E_j}(t) U,
\]

where

(i) \( E_j = (2^{-j}, 2^{-(j-1)}) \) for \( j \geq 1 \).

(ii) \( \{\alpha_j(t) : j \geq 1\} \) are independent random processes with \( \alpha_j(\cdot) \) defined on \( E_j \) such that for \( j \geq 1 \)

\[
P(\alpha_j(t) = \sin(2\pi 2^j t), t \in E_j) = 1/2,
\]

and

\[
P(\alpha_j(t) = \sin(2\pi 2^{j+1} t), t \in E_j) = 1/2.
\]

(iii) \( U \) is a uniform random variable on \([3/2, 2]\), independent of the \( \{\alpha_j\} \).

Since the \( \alpha_j \)'s are zero at endpoints of the \( E_j \) and we have set \( X(0) = 0 \), it is easy to see \( X(t) \) has continuous paths on \([0, 1]\). Moreover, \( X(t) \) is Lip-1 on \([0, 1]\) and \( X(t) \) has a density for each \( t \in (0, 1] \), but our CLT fails.

The failure of the empirical CLT can be shown by verifying a lemma of the sort we have above for Brownian motion, and again we see the lack of uniformly bounded densities is a determining factor.

For each integer \( n \geq 1 \), let \( X_1, \ldots, X_n \) be independent copies of \( X \), and again take \( C = \{C_{t,x} : 0 \leq t \leq 1, \infty < x < \infty\} \), where \( C_{t,x} = \{z \in C[0, 1] : z(t) \leq x\} \). Also, define \( C_Q \) as in Lemma 7. Then, we have the following lemma, and combined with the argument in section 8.1 we see the empirical CLT fails when this \( X(\cdot) \) is used.
Lemma 8. For each integer $n \geq 1$, with probability one

$$\Delta_{CQ}(X_1, \cdots, X_n) = 2^n,$$

where $\Delta_{CQ}(X_1, \cdots, X_n) = \text{card}\{C \cap \{X_1, \cdots, X_n\} : C \in \mathcal{C}_Q\}$.

Proof. Suppose one wants the $k$ functions $\{X_{i_1}, \cdots, X_{i_k}\}$ with probability one, where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. If we write

$$X_i(t) = t \sum_{j=1}^{\infty} (\alpha_{i,j}(t) + 2) I_{E_j}(t) U_i$$

where the $\{\alpha_{i,j} : j \geq 1\}$ are independent copies of $\{\alpha_j(t) : j \geq 1\}$ and $\{U_i : i \geq 1\}$ are independent copies of $U$, independent of all the $\alpha_{i,j}$'s, then this can be arranged by taking

$$\alpha_{i,j}(t) = \sin(2\pi 2^{j+1} t), i \in \{i_1, \cdots, i_k\},$$

and

$$\alpha_{i,j}(t) = \sin(2\pi 2^j t), i \in \{1, \cdots, n\} \cap \{i_1, \cdots, i_k\}^c,$$

provided we set $t = t_j = 2^{-j} + \frac{1}{2}(2^{-(j+1)} - 2^{-j})$. The probability of this configuration on the interval $E_j$ is $1/2^n$ and hence with probability one the Borel-Cantelli lemma implies there are infinitely many (random) $\{t_j \downarrow 0\}$ such that

$$\alpha_{i_1,j}(t_j) = \cdots = \alpha_{i_k,j}(t_j) = 0,$$

and

$$\alpha_{i,j}(t_j) = 1$$

for all other $i \in \{1, \cdots, n\}$. Thus with probability one there are infinitely many rational numbers $t \downarrow 0$ such that

$$C_{t,x(t)} \cap \{X_1, \cdots, X_n\} = \{X_{i_1}, \cdots, X_{i_k}\},$$

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provided

\[ x = x(t) = \frac{17t}{4}. \]

Of course, \( x(t) \) is then also in \( Q \), and since \( k \) and the set \( \{i_1, \cdots, i_k\} \) were arbitrary, and we can pick out \( \{X_{i_1}, \cdots, X_{i_k}\} \) with probability one using sets in \( \mathcal{C}_Q \), the lemma follows as the intersection of \( 2^n \) subsets of probability one has probability one.

To see \( X(t) \) is Lip-1 on \([0, 1]\), observe that since the intervals \( \{E_j : j \geq 1\} \) are disjoint, \( X(t) \) is differentiable on there interiors, and an easy computation implies

\[
\sup_{j \geq 1} \sup_{2^{-j} < t < 2^{-(j-1)}} |X'(t)| \leq [4\pi + 3]U.
\]

Furthermore, \( X(t) \) is continuous on \([0, 1]\), so we can apply the mean value theorem on each of these intervals.

If \( 0 < s < t \leq 1 \) we either have \( s, t \in E_j \) for some \( j \geq 1 \) or \( s \in E_j \) and \( t \in E_k \) for \( j > k \). If \( 0 < s < t \leq 1 \), then since \( t(\alpha_j(t) + 2)U \) is differentiable on the interior of \( E_j \) and continuous on \( E_j \) the mean value theorem implies

\[
|X(t) - X(s)| \leq (4\pi + 3)U|t - s|.
\]

If \( s \in E_j \) and \( t \in E_k \) with \( j > k \), then by applying the mean value theorem in each interval we obtain

\[
|X(t) - X(s)| \leq |X(s) - X(2^{-j-1})| \\
+ \sum_{r=1}^{j-k-1} |X(2^{-(j-1)}) - X(2^{-(j-1-r)})| + |X(2^{-k}) - X(t)| \\
\leq (4\pi + 3)U|t - s|.
\]

If \( s = 0 \), then since \( X(0) = 0 \), the proof is immediate. Hence \( X(t) \) is Lip-1 as claimed, and the Lipschitz constant is bounded by \((4\pi + 3)U\) with probability one. Furthermore, since \( U \) is uniform on \([3/2, 2]\) and independent of the \( \{\alpha_j\} \), then \( X(t) \) has a density for each \( t \in (0, 1] \).
8.4 Variations of the L-condition and the CLT.

Here we produce examples where the sets $C$, or more precisely the class
of indicator functions given by $C$, are $P$-pregaussian, and yet $C \not\subset CLT(P)$.
More importantly, they also satisfy the modified L-condition, i.e. we say
$\{X_s : s \in E\}$ satisfies the modified L-condition if for all $t \in E$ and $\epsilon > 0$
\[
\Pr^*(\sup_{\rho(s,t) \leq \epsilon} |F_t(X_s) - F_t(X_t)| > \epsilon^2) \leq L\epsilon^2.
\] (31)

Hence these examples also provide motivation for the use of the distributional
transforms in the L-condition of (10) used in Theorem 3.

Notation for the examples in this subsection is as follows. Let $E = \{1, 2, 3, \ldots\}$ and assume $D(E) = \{z : z(t) = 0$ or $1, t \in E\}$ with $C = \{C_{t,y} : t \in E, y \in \mathbb{R}\}$, where $C_{t,y} = \{z \in D(E) : z(t) \leq y\}$. Then, since the
functions in $D(E)$ take only the values zero and one, we have
\[
C = C_0 \cup C_1,
\]
where $C_a = \{\tilde{C}_{t,a} : t \in E\}$ and for $t \in E, a = 0, 1$, $\tilde{C}_{t,a} = \{z \in D(E) : z(t) = a\}$. Also, let $\Sigma$ denote the minimal sigma-field of subsets of $D(E)$ containing $C$, and let $P$ denote the probability on $(D(E), \Sigma)$ such that $P(\tilde{C}_{t,1}) = p_t$ and
the events $\{\tilde{C}_{t,1} : t \in E\}$ are independent events, i.e. $P$ is a product measure
on the coordinate spaces of $D(E) = \{0, 1\}^N$ with the $t^{th}$ coordinate of $D(E)$
having the two point probability that puts mass $p_t$ on one, and $1 - p_t$ on
zero.

Proposition 2. Let $C$ be defined as above. Then,
(i) $C$ is $P$-pregaussian whenever $p_t = o((\log(t + 2))^{-1})$ as $t \to \infty$.
(ii) $C \subset CLT(P)$ if and only if for some $r > 0$
\[
\sum_{t=1}^{\infty} (p_t(1 - p_t))^r < \infty.
\]
(iii) If \( p_t = (\log(t + 2))^{-2} \), and \( \{H(t) : t \in E\} \) consists of centered independent Gaussian random variables with \( E(H(t)^2) = (\log(t + 2))^{-\frac{3}{2}} \), then \( \{X_s : s \in E\} \) satisfies the modified L-condition, \( \mathcal{C} \) is \( P \)-pregaussian, and \( \mathcal{C} \notin CLT \). In particular, in view of Theorem 3, it does not satisfy the L-condition.

Proof. Since the events of \( \mathcal{C}_0 \) are complements of those in \( \mathcal{C}_1 \), it is easy to see that \( \mathcal{C}_0 \in CLT(P) \) if and only if \( \mathcal{C}_1 \in CLT(P) \), and since Theorem 3.8.1 of [?] implies finite unions of classes of sets that satisfy the \( CLT(P) \) are in \( CLT(P) \), we have \( \mathcal{C} \in CLT(P) \) if and only if \( \mathcal{C}_1 \in CLT(P) \). Therefore, since the events of \( \mathcal{C}_1 \) are independent, Theorem 3.9.1 in [?], p. 122, implies that

\[
\sum_{t=1}^{\infty} (p_t(1 - p_t))^r < \infty.
\]

Hence (ii) holds.

Now the centered Gaussian process \( \{G_P(C) : C \in \mathcal{C}_1\} = \{G_P(C_{t,1}) : t \in E\} \), and since the random variables \( \{G_P(C_{t,1}) : t \in E\} \) are mean zero and \( E(G_P(C_{t,1})^2) = p_t(1 - p_t) \), we have \( \mathcal{C}_1 \) is \( P \)-pregaussian provided

\[
p_t = o((\log(t + 2))^{-1}) \text{ as } t \to \infty.
\]

Similarly, taking complements we have \( \{G_P(C) : C \in \mathcal{C}_0\} = \{G_P(C_{t,0}) : t \in E\} \) is also \( P \)-pregaussian under this condition for \( p_t \), and hence \( \mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1 \) is \( P \)-pregaussian whenever \( p_t = o((\log(t + 2))^{-1}) \) as \( t \to \infty \). Hence (i) holds.

To verify (iii) we take \( p_t = (\log(t + 2))^{-2} \), and \( \{H(t) : t \in E\} \) to be centered independent Gaussian random variables with \( E(H(t)^2) = (\log(t + 2))^{-\frac{3}{2}} \). If \( \rho^2(s,t) = E((H(s) - H(t))^2) \), then, for \( s \neq t \),

\[
\rho^2(s,t) = (\log(s+2))^{-\frac{3}{2}} + (\log(t+2))^{-\frac{3}{2}} \geq \max\{(\log(s+2))^{-\frac{3}{2}},(\log(t+2))^{-\frac{3}{2}}\}.
\]

In addition, we have \( |F_t(X_s) - F_t(X_t)| = 0 \) if \( X_s = X_t = 0 \) or \( X_s = X_t = 1 \), and \( |F_t(X_s) - F_t(X_t)| = p_t \) if \( X_t \neq X_s \).
Therefore, for all $t \in E$ fixed and $\epsilon > 0$ we have

$$\Pr^*(\sup_{\{s: \rho(s,t) \leq \epsilon\}} |F_t(X_s) - F_t(X_t)| > \epsilon^2) = 0 \text{ if } p_t \leq \epsilon^2,$$

and

$$\Pr^*(\sup_{\{s: \rho(s,t) \leq \epsilon\}} |F_t(X_s) - F_t(X_t)| > \epsilon^2) \leq \Pr^*(\sup_{\{s: \rho(s,t) \leq \epsilon\}} I_{X_t \neq X_s} > 0) \text{ for } p_t > \epsilon^2.$$

Of course, $E$ countable makes the outer probabilities in the above ordinary probabilities, but for simplicity we retained the outer probability notation used in (9) and (10). Now $\rho(s, t) \leq \epsilon$ implies

$$\max\{(\log(s + 2))^{-\frac{3}{4}}, (\log(t + 2))^{-\frac{3}{4}}\} \leq \epsilon.$$

Thus if $p_t = (\log(t + 2))^{-\frac{3}{2}} > \epsilon^2$ we have $\{\sup_{\{s: \rho(s,t) \leq \epsilon\}} I_{X_t \neq X_s} > 0\} = \emptyset$. Combining the above we have for each fixed $t \in E$ and $\epsilon > 0$ that

$$\Pr^*(\sup_{\{s: \rho(s,t) \leq \epsilon\}} |F_t(X_s) - F_t(X_t)| > \epsilon^2) = 0,$$

and hence the modified L-condition for $\{X_s: s \in E\}$ holds. Thus (iii) follows.

\[\square\]

9 Appendix on Talagrand’s Continuity Result for Gaussian Processes.

The following result appears in [?] without proof. There is a curious wording at the end of the first sentence in its statement there, which after closer inspection suggests that cutting and pasting led to something being omitted. Hence we include a proof, where we have added the necessary assumption of total boundedness of the parameter space to the statement of the theorem.
Theorem 6. Let \( \{X_t : t \in T\} \) be a centered Gaussian process with \( L_2 \) distance \( d(s, t), s, t \in T \), where \( T \) is countable and \((T, d)\) is totally bounded. Then, the following are equivalent:

(i) \( X_t \) is uniformly continuous on \((T, d)\) with probability one.

(ii) We have

\[
\lim_{\epsilon \to 0} E(\sup_{d(s,t) \leq \epsilon} (X_s - X_t)) = 0.
\]

(iii) There exists an admissible sequence of partitions of \( T \) such that

\[
\lim_{k \to \infty} \sup_{t \in T} \sum_{n \geq k} 2^{n/2} \Delta(A_n(t)) = 0.
\]

Proof. First we will show (i) and (ii) are equivalent. If (ii) holds, then by Fatou’s lemma we have

\[
0 = \lim_{n \to \infty} E(\sup_{d(s,t) \leq 1/n} |X_s - X_t|) \geq E(\liminf_{n \to \infty} \sup_{d(s,t) \leq 1/n} |X_s - X_t|).
\]

Thus, with probability one

\[
\liminf_{n \to \infty} \sup_{d(s,t) \leq 1/n} |X_s - X_t| = 0,
\]

and since the random variables \( \sup_{d(s,t) \leq 1/n} |X_s - X_t| \) decrease as \( n \) increases, this implies with probability one

\[
\lim_{n \to \infty} \sup_{d(s,t) \leq 1/n} |X_s - X_t| = 0,
\]

which implies (i).

If we assume (i), then since \((T, d_X)\) is assumed totally bounded, we have

\[
Z = \sup_{t \in T} |X_t| < \infty
\]

with probability one, and the Fernique-Landau-Shepp Theorem implies \( Z \) is integrable. Since

\[
\sup_{d(s,t) \leq \epsilon} |X_s - X_t| \leq 2Z,
\]
and (i) implies
\[
\lim_{\epsilon \to 0} \sup_{d(s,t) \leq \epsilon} |X_s - X_t| = 0
\]
with probability one, the dominated convergence theorem implies (ii). Thus (i) and (ii) are equivalent.

Now we assume (i) and (ii), and choose \( \varepsilon_k \downarrow 0 \) such that
\[
\sup_s E \sup_{\{t: d(t,s) \leq \varepsilon_k\}} (X_t - X_s) \leq \varepsilon_k \sup_s (X_t - X_s) \leq 2^{-k}.
\]
Since \((T,d)\) is totally bounded, and (i) holds, by Theorem 2.1.1 of [?] there exists an admissible sequence of partitions \( \{\tilde{A}_n : n \geq 0\} \) of \((T,d)\) such that for a universal constant \( L \) we have
\[
\frac{1}{2L} \sup_{t \in T} \sum_{n \geq k} 2^{n/2} \Delta(A_n(t)) \leq E(\sup_{t \in T} X_t).
\]

Now choose \( \{n_k : k \geq 1\} \) to be a strictly increasing sequence of integers such that \( n_1 > 4 \) and
\[
2 \sum_{2 \leq j \leq k} 1/\varepsilon_j^2 \leq n_{k-1} \quad (32)
\]
Based on the \( n_k \)'s we define an increasing sequence of partitions, \( B_n \). For \( 0 \leq n \leq n_1 \) we let \( B_n = \tilde{A}_n \). For \( n_1 < n \leq n_2 \) we proceed as follows.

First we choose a maximal set \( \{s_1, \ldots, s_{N(\varepsilon_2)}\} \) of \((T,d)\) for which \( d(s_i, s_j) \geq \varepsilon_2 \). By Sudakov's inequality we know that \( N(\varepsilon_2) \leq 2^{1/\varepsilon_2^2} \). To define the partitions for \( n_1 < n \leq n_2 \) we next consider the partition of \( T \) formed by the sets
\[
C_j = B(s_j, \varepsilon_2) \cap \left( \bigcup_{k=1}^{j-1} B(s_k, \varepsilon_2) \right)^c, \quad 1 \leq j \leq N(\varepsilon_2), \quad (33)
\]
and the sets \( B(s, \varepsilon) \) are \( \varepsilon \) balls centered at \( s \). Then by Theorem 2.1.1 of [?] for every integer \( 1 \leq j \leq N(\varepsilon_2) \) there exists an admissible sequence of
partitions for \((C_j, d)\), which we denote by \(B_{n,k}^{s_j}\), such that

\[
2^{-2} \geq \mathbb{E} \sup_{t \in C_j} X_t \geq \frac{1}{2L} \sup_{t \in C_j} \sum_{n \geq 0} 2^{n/2} \Delta(B_{n,k}^{s_j}(t))
\]

\[
\geq \frac{1}{2L} \sup_{t \in C_j} \sum_{n_1 < n \leq n_2} 2^{n/2} \Delta(A_{n_1}(t) \cap B_{n,k}^{s_j}(t)).
\]

Since the sets \(C_j\) form a partition of \(T\), if \(B_{n,k}^{s_j}\) is one of the sets, \(B_{n,k}^{s_j}(t)\), then

\[
2^{-2} \geq \frac{1}{2L} \sup_{t \in T} \sum_{n_1 < n \leq n_2} 2^{n/2} \Delta(A_{n_1}(t) \cap B_{n,k}^{s_j}(t)),
\]

and we define the increasing sequence of partitions \(B_{n,k}^{s_j}\) to be all sets of the form \(A_{n_1}(t) \cap B_{n,k}^{s_j}(t)\), where \(t \in T\) and \(B_{n,k}^{s_j} \in B_{n,k}^{s_j}(t)\) for some \(j \in [1, N(\epsilon_2)]\). Furthermore, since the \(C_j\)'s are disjoint, for \(n_1 < n \leq n_2\) we have

\[
\text{Card}(B_{n,k}^{s_j}) \leq 2^{n_1}2^{-n} N(\epsilon_2) \leq 2^{n_1}2^{1/\epsilon_2^3} \leq 2^{n_1}2^{n_2} \leq 2^{n_2}2^{n_1} = 2^{n_1 + n_2},
\]

and for \(n_1 < n \leq n_2\) we define \(B_n = B_{n,k}^{s_j}\).

Iterating what we have done for \(n_1 < n \leq n_2\), we have increasing partitions \(B_{n,k-1,n}, n_{k-1} < n \leq n_k\), for which

\[
(2L)^{-k} \geq \sup_{t} \sum_{n \geq 0} 2^{n/2} \Delta(B_{n,k-1,n})
\]

\[
\geq \sup_{t} \sum_{n_{k-1} < n \leq n_k} 2^{n/2} \Delta(B_{n,k-1,n}(t) \cap B_{n_k-2,n_{k-1}}(t) \cdots \cap B_{n_1,n_2}(t) \cap A_{n_1}(t)),
\]

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and for \( n_{k-1} < n \leq n_k \) we define \( \mathcal{B}_n = \mathcal{B}_{n_{k-1},n} \). Therefore, we now have an increasing sequence of partitions \( \{ \mathcal{B}_n : n \geq 0 \} \) such that

\[
(2L) \sum_{k \geq r} 2^{-k} \geq \sum_{k \geq r} \sup_{n_{k-1} < n \leq n_k} \sum_t 2^{n/2} \Delta(\mathcal{B}_{n_{k-1},n}(t) \cap (\cap_{j=2}^{k-1} \mathcal{B}_{n_{j-1},n_j}(t)) \cap \mathcal{A}_n(t))
\]

\[
\geq \sup_{t} \sum_{k \geq r} \sum_{n_{k-1} < n \leq n_k} 2^{n/2} \Delta(\mathcal{B}_{n_{k-1},n}(t) \cap (\cap_{j=2}^{k-1} \mathcal{B}_{n_{j-1},n_j}(t)) \cap \mathcal{A}_n(t)),
\]

and, letting \( \mathcal{B}_n(t) \) denote the generic set of \( \mathcal{B}_n \) containing \( t \), we have

\[
(2L) \sum_{k \geq r} 2^{-k} \geq \sup_{t} \sum_{k \geq r} \sum_{n_{k-1} < n \leq n_k} 2^{n/2} \Delta(\mathcal{B}_n(t)). \tag{35}
\]

Now to count the number of elements in each partition. Since (1) holds, the partitions \( \mathcal{B}_{n_{k-1},n}^{s_j} \) are assumed admissible, and the \( C_j \)'s used at the subsequent iterations are always disjoint, we have for \( n_{k-1} < n \leq n_k \) that

\[
\text{Card}(\mathcal{B}_{n_{k-1},n}) \leq 2^{2^{n_1}} \prod_{j=2}^{k-1} 2^{n_j} N(\epsilon_j) 2^{2^n} N(\epsilon_k) \leq 2^{\sum_{j \leq k-1} 2^{n_j} + 2^n} n_{k-1} \leq 2^{2^{n+2}}.
\]

Given the increasing sequence of partitions \( \{ \mathcal{B}_n : n \geq 0 \} \), we now define the partitions \( \mathcal{A}_n \) to be the single set \( T \) for \( n = 0,1 \) and \( \mathcal{A}_n = \mathcal{B}_{n-2} \) for \( n \geq 2 \). Since we have \( \text{Card}(\mathcal{B}_n) \leq 2^{2^{n+2}} \) for \( n \geq 0 \) we thus have that the \( \mathcal{A}_n \)'s are admissible and using (4) above they satisfy (iii).

\[
\square
\]

References