Conditional distributions, exchangeable particle systems, and stochastic partial differential equations

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Abstract

Stochastic partial differential equations (SPDEs) whose solutions are probability-measure-valued processes are considered. Measure-valued processes of this type arise naturally as de Finetti measures of infinite exchangeable systems of particles and as the solutions for filtering problems. In particular, we consider a model of asset price determination by an infinite collection of competing traders. Each trader's valuations of the assets are given by the solution of a stochastic differential equation, and the infinite system of SDEs, assumed to be exchangeable, is coupled through a common noise process and through the asset prices. In the simplest, single asset setting, the market clearing price at any time t is given by a quantile of the de Finetti measure determined by the individual trader valuations. In the multi-asset setting, the prices are essentially given by the solution of an assignment game introduced by Shapley and Shubik. Existence of solutions for the infinite exchangeable system is obtained by an approximation argument that requires the continuous dependence of the prices on the determining de Finetti measures which is ensured if the de Finetti measures charge every open set. The solution of the SPDE satisfied by the de Finetti measures can be interpreted as the conditional distribution of the solution of a single stochastic differential equation given the common noise and the price process. Under mild nondegeneracy conditions on the coefficients of the stochastic differential equation, the conditional distribution is shown to charge every open set, and under slightly stronger conditions, it is shown to be absolutely continuous with respect to Lebesgue measure with strictly positive density. The conditional distribution results are the main technical contribution and can also be used to study the properties of the solution of the nonlinear filtering equation within a framework that allows for the signal noise and the observation noise to be correlated.

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1 Introduction

The price process for a risky asset is usually modeled by a stochastic process $\{S_t, t \geq 0\}$. Finding a good model for asset prices plays a central role in mathematical finance. At the turn of the nineteenth century, Bachelier introduced Brownian motion as a model for the price fluctuations of the Paris stock exchange. In the sixties, Samuelson suggested the use of geometric Brownian motion as a suitable model. Since then a variety of other processes have been used to model price processes.

Rather than imposing an ad-hoc model, a large number of works (for example, [1, 9, 10, 11, 12, 26]) have been devoted to the derivation of the price process $\{S_t, t \geq 0\}$ by modeling the evolution and interaction of the agents involved in the market. The primary motivation for our work is the study of a model of this type. In particular, we consider an asset pricing model, introduced in [22], where the price of a single asset ($d = 1$) is determined through a continuous-time *auction system*. Let us assume that there are N traders who compete for n units of the asset, where $n < N$. Each trader owns either one share or no shares. At any point in time, the traders who submit the n highest bid prices each own a share. We denote by X_t^i, the log of the bid price or valuation of the ith trader at time t and by S_t^N the log of the stock price. Consequently, the market clearing condition for the equilibrium log-stock price S_t^N is given by:

$$\begin{aligned} \sum_{i=1}^{N} \mathbf{1}_{\{X_t^i \geq S_t^N\}} &= n \\ (\textit{Demand}) &= (\textit{Supply}). \end{aligned}$$

Define v_t^N to be the empirical measure of $\{X_t^1, X_t^2, ..., X_t^N\}$, that is $v_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^i}$. Then the market clearing condition can alternatively be expressed as:

$$v_t^N[S_t^N, \infty) = \frac{n}{N}.$$

As N tends to infinity and $\frac{n}{N} \to a$, for some $a \in (0,1)$, the stock price S_t becomes the α-quantile process V_t^α of the measure v that is the limit of the empirical distribution v_t^N of the log bids, where $\alpha \equiv 1 - a$. A simple but suggestive model for X_t^i is the following geometric mean-reverting process, motivated by [9],

$$X_t^i = X_0^i + \beta \int_0^t (S_s - X_s^i) ds + \sigma W_t + \bar{\sigma} B_t^i, \tag{1.1}$$

where β, σ and $\bar{\sigma}$ are some positive constants. In (1.1), each investor takes the stock price as a signal for the value of the asset and adjusts his or her valuation upward if it is below the stock price and downward if it is above. The parameter β measures the mean reversion rate toward S_t. The higher this parameter value is, the faster the positions tend to mean-revert. The Brownian motion W models the common market noise, whilst the Brownian motion B^i models the trader's own uncertainty.

More generally, we will consider systems of the form

$$X_t^i = X_0^i + \int_0^t f\left(X_s^i, V_s^\alpha\right) ds + \int_0^t \sigma\left(X_s^i, V_s^\alpha\right) dW_s + \int_0^t \bar{\sigma}\left(X_s^i, V_s^\alpha\right) dB_s^i, \tag{1.2}$$

where

$$V_t^\alpha = \inf\left\{x \in \mathbb{R} | v_t\left(-\infty, x\right] \geq \alpha\right\} \tag{1.3}$$

and

$$v_t = \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^m \delta_{X_t^i}. \tag{1.4}$$

We assume that $\{X_0^i\}$ is exchangeable and require the solutions $\{X^i\}$ to be exchangeable so that the limit in (1.4) exists by de Finetti's theorem. In (1.2), the process W is common to all diffusions, while the processes B^i, $i \geq 1$ are mutually independent Brownian motions.

Similar to the results in Kurtz and Xiong [19], v will be a solution of the stochastic partial differential equation

$$\langle \phi, v_t \rangle = \langle \phi, v_0 \rangle + \int_0^t \langle L(S)\phi\,, v_s \rangle ds + \int_0^t \langle \sigma(\cdot, S_s)\phi'\,, v_s \rangle dW_s, \tag{1.5}$$

where $\langle \phi, v_t \rangle$ denotes

$$\langle \phi, v_t \rangle = \int_{\mathbb{R}} \phi(x) v_t(dx)$$

and

$$L(S)\phi = \frac{1}{2}\left[\sigma(x,S)^2 + \bar{\sigma}(x,S)^2\right]\frac{d^2\phi}{dx^2} + f(x,S)\frac{d\phi}{dx}.$$

Systems of this type have been considered by Kurtz and Protter [18] and Kurtz and Xiong [19, 20] under the assumption that the coefficients are Lipschitz functions of v in the Wasserstein metric on $\mathcal{P}(\mathbb{R}^d)$. This assumption excludes a variety of interesting examples including the models with coefficients depending on quantiles of primary interest here. Unfortunately, we do not have a general uniqueness theorem for (1.2), although uniqueness for (1.1) can be obtained by direct calculation. (See Remark 2.5.)

Note that (1.3) and (1.4) may not uniquely determine prices unless the distribution v_t charges every nonempty open set. Furthermore, convergence of the finite system to the infinite system depends on the convergence of the price process and convergence of quantiles again depends on the limiting distribution charging every open set at least in a neighborhood of the limiting quantile. Consequently, our fundamental problem is to give conditions under which this assertion holds. Our proof depends on the observation that

$$v_t(\varphi) = \lim_{n\to\infty} E\left[\frac{1}{n}\sum_{i=1}^{n} \varphi(X_t^i)|\mathcal{F}_t^{W,S}\right] = E[\varphi(X_t^1)|\mathcal{F}_t^{W,S}] = \pi_t(\varphi),$$

where π_t is the conditional distribution of X_t^1 given $\mathcal{F}_t^{W,S}$ and $v_t(\varphi) = \int \varphi(x)v_t(dx)$. With this observation in mind, we address our fundamental problem in a more general context.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and (E, r) a complete, separable metric space. Let B and W be d and d'-dimensional standard Brownian motions, and let V be a cadlag E-valued process. We assume that B is independent of (W, V) and that W is compatible with V in the sense that for each $t \geq 0$, $W_{t+\cdot} - W_t$ is independent of $\mathcal{F}_t^{W,V}$, where $\mathcal{F}_t^{W,V} = \sigma(W_s, V_s, s \leq t)$. Let X be a d-dimensional stochastic process satisfying the equation

$$X_t = X_0 + \int_0^t f(X_s, V_s)\, ds + \int_0^t \sigma(X_s, V_s)\, dW_s + \int_0^t \bar{\sigma}(X_s, V_s)\, dB_s, \tag{1.6}$$

where the coefficients satisfy one or more of the following:

Conditions $f : \mathbb{R}^d \times E \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \times E \to \mathbb{M}^{d\times d'}$, $\bar{\sigma} : \mathbb{R}^d \times E \to \mathbb{M}^{d\times d}$

C1) f, σ, $\bar{\sigma}$ are continuous functions, uniformly Lipschitz in the first argument.[1]

C2) $\bar{\sigma}(x, y)$ is non-singular for all (x, y).[2]

C3) f, σ and $\bar{\sigma}$ are continuously differentiable in the first variable.

C4) $E = \mathbb{R}^m$ and there exists a constant K such that f, σ, $\bar{\sigma}$ are bounded by $K(1+|x|+|y|)$.

We assume that, given V_0, X_0 is conditionally independent of W, V and B, that is,

$$E[f(X_0)|\mathcal{F}_\infty^{W,V,B}] = E[f(X_0)|V_0]. \tag{1.7}$$

We are interested in the $\mathcal{P}(\mathbb{R}^d)$-valued process $\pi = \{\pi_t,\ t \geq 0\}$, where π_t is the conditional distribution of X_t given $\mathcal{F}_t^{W,V}$,

$$\pi_t(\varphi) = E\left[\varphi(X_t)\,|\mathcal{F}_t^{W,V}\right], \tag{1.8}$$

for any $\varphi \in B(\mathbb{R}^d)$, where $B(\mathbb{R}^d)$ is the set of bounded Borel-measurable functions on $\mathbb{R}^d$.

Under Conditions C1 and C2, we will show that for $t > 0$, π_t charges any nonempty open set $A \subset \mathbb{R}^d$ almost surely (and the null set can be chosen independent of A). Further, under the additional Condition C3, π_t is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$ and, with probability one, its density is strictly positive. We have the following fundamental theorems.

[1] That is, there exists a constant c_1 such that $|f(x_1, y) - f(x_1, y)| \leq c_1 |x_1 - x_2|$ for all $x_1, x_2 \in \mathbb{R}^d$ and $y \in E$ with a similar inequality holding for σ and $\bar{\sigma}$.

[2] For $d = 1$, we will assume without loss of generality that $\bar{\sigma}(x, y)$ is positive.

Theorem 1.1 *Assume in (1.6) that B is independent of (W, V), that W is compatible with V, and that given V_0, X_0 is conditionally independent of (W, V, B). Under Conditions C1 and C2, there exists a set $\widetilde{\Omega} \in \mathcal{F}$ of full measure such that for every $\omega \in \widetilde{\Omega}$ and $t > 0$, π_t^ω charges every open set, i.e., $\pi_t^\omega(A) > 0$ for every nonempty open set A.*

Theorem 1.2 *In addition to the conditions of Theorem 1.1, assume that Condition C3 holds. Then there exists a set $\widetilde{\Omega} \in \mathcal{F}$ of full measure such that for every $\omega \in \widetilde{\Omega}$ and $t > 0$, π_t^ω is absolutely continuous with respect to Lebesgue measure with a strictly positive density.*

Theorem 1.1 provides the essential ingredient for the proof of the following existence theorem.

Theorem 1.3 *Suppose that Conditions C1, C2, and C4 hold, that $\{X_0^i\}$ is exchangeable, and that $E[|X_0^i|] < \infty$. Then there exists a weak solution for the system (1.2) - (1.4) such that $\{X^i\}$ is exchangeable and v satisfies the stochastic partial differential equation (1.5).*

The proof of Theorem 1.3 is given in Section 2. In Section 3, we extend the one-dimensional model to multiple substitutable assets for which the market clearing condition becomes

$$v_t\{x : x_k - S_{t,k} \geq 0 \vee \max_{l \neq k}(x_l - S_{t,l})\} = a_k, \tag{1.9}$$

where v_t is the distribution of valuations among the infinite collection of traders, $S_{t,k}$ denotes the price of the kth asset, $0 < a_k < 1$ measures the availability of the kth asset, and $\sum_k a_k < 1$. The price then is the solution of an infinite version of the assignment game as defined by Shapley and Shubik [25]. It can also be described as the result of a multi-item auction. (See Demange, Gale and Sotomayor [7].)

For the single asset case, the stochastic differential equation satisfied by the price (the α-quantile of v) is derived in Section 4, Proposition 4.1.

Section 5 is devoted to the application of the support results to the solution of stochastic filtering problems. Let (X, Y) be the solution of

$$\begin{aligned} X_t &= X_0 + \int_0^t f(X_s, Y_s)\,ds + \int_0^t \sigma(X_s, Y_s)\,dW_s + \int_0^t \bar{\sigma}(X_s, Y_s)\,dB_s \\ Y_t &= \int_0^t h(X_s, Y_s)\,ds + \int_0^t k(Y_s)\,dW_s. \end{aligned}$$

Here Y plays the role of V, so B is not independent of (W, Y). Assuming that $k(y)$ is invertible and setting

$$\widetilde{W}_t = W_t + \int_0^t k(Y_s)^{-1} h(X_s, Y_s) ds,$$

we have

$$X_t = X_0 + \int_0^t \left(f(X_s, Y_s) + \sigma(X_s, Y_s) k(Y_s)^{-1} h(X_s, Y_s)\right) ds$$

$$+\int_0^t \sigma(X_s,Y_s)d\widetilde{W}_s + \int_0^t \bar{\sigma}(X_s,Y_s)\,dB_s$$
$$Y_t = \int_0^t k(Y_s)\,d\widetilde{W}_s.$$

Under modest assumptions on $h(x,y)/k(y)$, a Girsanov change of measure gives an equivalent probability measure under which B is independent of $(\widetilde{W},Y)$. In this framework we show that the conditional distribution of X_t given $\mathcal{F}_t^Y$ charges any open set (see Corollary 5.1). Moreover, under additional conditions, it is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$ and, with probability one, its density is strictly positive. Corollary 5.1 can be interpreted as a smoothing result of the most basic kind. Essentially we prove under Lipschitz/differentiability conditions on the coefficients that, whilst π_0 is arbitrary, π_t charges every open set and, respectively, has a positive density with respect to the Lebesgue measure for any $t > 0$. We are not aware of a similar result on the density of the conditional distribution of the process X proved under such generality. Most of the existing results assume higher differentiability of the coefficients of the pair process (X,Y). The exception is the recent work of Krylov: In [15], the density of the conditional distribution of X_t given $\mathcal{F}_t^Y$ is analyzed under Lipschitz assumptions on the coefficients of the pair process (X,Y). However, the coefficients are also assumed to be bounded and the initial distribution of π_0 is assumed to have a density belonging to a suitable Bessel potential space. See Remark 5.3 for details.

In Section 6 we prove the two basic Theorems 1.1 and 1.2. The paper concludes with a short appendix containing results on the convergence of the quantiles and the measurability and positivity of random functions given by conditional expectations.

The analysis of the properties of the density of π_t forms the basis of the above results. The method employed here is novel and will lead to further, more refined, results[3]. We do not do this here as it is not the focus of the current work. The basis of the results are the representation formulae (6.6), (6.9) for the case $d = 1$ and (6.25) for the multi-dimensional case. The manner of proof is a Girsanov-based argument that resembles Bismut's approach (see [2]) to deduce integration by parts formulae using Malliavin calculus. Here we do not use Malliavin calculus and obtain the results under very general conditions. A Malliavin calculus approach to analyze the density of π_t is possible along the lines of [21] and [24] (see also [3], [5, 6] and the recent survey [4]), but only at the expense of more stringent smoothness conditions imposed on the coefficients of (1.6).

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[3]For a glimpse of what can be achieved, in Remark 6.3 we deduce Gaussian tail estimates for the conditional density.

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2 Weak existence for SPDEs with coefficients depending on quantiles

To prove Theorem 1.3, we consider the Euler-type approximation of (1.2) - (1.4) defined as follows:

$$X_t^{i,n} = X_0^i + \int_0^t f\left(X_s^{i,n}, V_s^{\alpha,n}\right) ds + \int_0^t \sigma\left(X_s^{i,n}, V_s^{\alpha,n}\right) dW_s + \int_0^t \bar{\sigma}\left(X_s^{i,n}, V_s^{\alpha,n}\right) dB_s^i, \quad (2.1)$$

where

$$V_t^{\alpha,n} = \inf\left\{x \in \mathbb{R} | v^n_{\frac{[tn]}{n}}((-\infty, x]) \geq \alpha\right\}$$

and v^n is defined as in (1.4). Since we are assuming Lipschitz continuity in x, existence and uniqueness of a solution for (2.1) is obtained recursively on intervals $\left[\frac{k}{n}, \frac{k+1}{n}\right]$. On each such interval, the process $V^{\alpha,n}$ is constant and equal to the quantile of the empirical measure of the system at the beginning of the interval. Note that

$$\alpha \wedge (1-\alpha)|V_t^{\alpha,n}| \leq \overline{|X^n_{\frac{[nt]}{n}}|} \equiv \lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^k |X^{i,n}_{\frac{[tn]}{n}}|$$

and hence

$$\alpha \wedge (1-\alpha) \sup_{s\leq t} |V_t^{\alpha,n}| \leq \sup_{s\leq t} \overline{|X_s^n|}. \quad (2.2)$$

We have the following uniform estimates on the growth of the $X^{i,n}$.

Lemma 2.1 *Suppose that $E[|X_0^i|] < \infty$. Then for each $t > 0$, there exists $C(t)$ such that*

$$\sup_n E[\sup_{s\leq t} |X_s^{i,n}|] \leq C(t),$$

which by (2.2) implies $\sup_n E[\sup_{s\leq t} |V_t^{\alpha,n}|] \leq \frac{1}{\alpha\wedge(1-\alpha)} C(t)$.

Proof. Note that for fixed n, the finiteness of $E[\sup_{s\leq t} |X_s^{i,n}|]$ follows by the recursive construction of the solution. By a result of Lenglart, Lépingle, and Pratelli [23] (see Theorem 1 of [13] or Lemma 2.4 of [18]), there exists a $C > 0$ such that

$$E[\sup_{s\leq t} |X_s^{i,n}|] \quad \leq \quad E[|X_0^i|] + \int_0^t E[|f\left(X_s^{i,n}, V_s^{\alpha,n}\right)|] ds + CE\left[\sqrt{\int_0^t \sigma^2\left(X_s^{i,n}, V_s^{\alpha,n}\right) ds}\right]$$

$$+CE\left[\sqrt{\int_0^t \bar{\sigma}^2\left(X_s^{i,n}, V_s^{\alpha,n}\right)ds}\right]$$

$$\leq E[|X_0^{i,n}|] + (Kt + 2KC\sqrt{t})E[1 + \sup_{s\leq t}|X_s^{i,n}| + \sup_{s\leq t}|V_s^{\alpha,n}|]$$

$$\leq E[|X_0^{i,n}|] + (Kt + 2KC\sqrt{t}) + (Kt + 2KC\sqrt{t})(1 + \frac{1}{\alpha\wedge(1-\alpha)})E[\sup_{s\leq t}|X_s^{i,n}|].$$

Selecting t_0 so that

$$(Kt_0 + 2KC\sqrt{t_0})(1 + \frac{1}{\alpha\wedge(1-\alpha)}) = \frac{1}{2},$$

$$E[\sup_{s\leq t_0}|X_s^{i,n}|] \leq 2E[|X_0^i|] + 2(Kt_0 + 2KC\sqrt{t_0}) \equiv 2E[|X_0^i|] + R,$$

and iterating

$$E[\sup_{(m-1)t_0\leq s\leq mt_0}|X_s^{i,n}|] \leq 2^m E[|X_0^i|] + R\sum_{k=0}^{m-1} 2^k = 2^m E[|X_0^i|] + (2^m - 1)R,$$

so

$$E[\sup_{0\leq s\leq mt_0}|X_s^{i,n}|] \leq (2^{m+1} - 2)E[|X_0^i|] + (2^{m+1} - m - 2)R.$$

□

In the following lemma, we drop the assumption that V^n is a quantile, allowing it to take values in any Euclidean space, and only require that the coefficients be continuous.

Lemma 2.2 *Suppose f, σ, and $\bar{\sigma}$ are continuous on $\mathbb{R}^d\times\mathbb{R}^d$, (V^n, W) is independent of B, X_0^n conditionally independent of (V^n, W, B) given V_0^n, and X^nsatisfies*

$$X_t^n = X_0^n + \int_0^t f\left(X_s^n, V_s^n\right)dt + \int_0^t \sigma\left(X_s^n, V_s^n\right)dW_s + \int_0^t \bar{\sigma}\left(X_s^n, V_s^n\right)dB_s.$$

Suppose that for each $t > 0$, $\{\sup_{s\leq t}|X_s^n|\}_{n\geq 1}$ and $\{\sup_{s\leq t}|V_s^n|\}_{n\geq 1}$ are stochastically bounded. Define

$$\Gamma^n(C\times[0,t]) = \int_0^t \mathbf{1}_C(V^n(s))ds, \quad C\in\mathcal{B}(\mathbb{R}^d),$$

$$M_B^n(\varphi, t) = \int_0^t \varphi(V^n(s))dB_s, \quad \varphi\in C_b(\mathbb{R}^d),$$

and

$$M_W^n(\varphi, t) = \int_0^t \varphi(V^n(s))dW_s, \quad C_b(\mathbb{R}^d).$$

Then $\{\Gamma^n\}$ is relatively compact in $\mathcal{L}_m(\mathbb{R}^d)$ and $\{X^n\}$ is relatively compact in $D_{\mathbb{R}^d}[0,\infty)$. (See Appendix A.2.) Selecting a subsequence if necessary, assume $(X^n, B, W, \Gamma^n) \Rightarrow (X, B, W, \Gamma)$ in $D_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^{d'}}[0,\infty) \times \mathcal{L}_m(\mathbb{R}^d)$. Then for $\varphi_1^B, \ldots, \varphi_k^B, \varphi_1^W, \ldots, \varphi_l^W \in C_b(\mathbb{R}^d)$,

$$\{(\Gamma^n, M_B^n(\varphi_1^B), \ldots, M_B^n(\varphi_k^B), M_W^n(\varphi_1^W), \ldots, M_W^n(\varphi_l^W)\}$$

is relatively compact in $\mathcal{L}_m(\mathbb{R}^d) \times D_{\mathbb{R}^{k+l}}[0,\infty)$, and a subsequence can be selected along which convergence holds for all choices of $\varphi_1^B, \ldots, \varphi_k^B, \varphi_1^W, \ldots, \varphi_l^W \in C_b(\mathbb{R}^d)$. For any limit point, M_B and M_W are orthogonal martingale random measures satisfying

$$\begin{aligned}
[M_B(\varphi_1), M_B(\varphi_2)]_t &= \int_{\mathbb{R}^d} \varphi_1(y)\varphi_2(y)\Gamma(dy \times [0,t]) \\
[M_W(\varphi_1), M_W(\varphi_2)]_t &= \int_{\mathbb{R}^d} \varphi_1(y)\varphi_2(y)\Gamma(dy \times [0,t]) \\
[M_B(\varphi_1), M_W(\varphi_2)]_t &= 0,
\end{aligned}$$

and X, the limit of X^n, satisfies

$$\begin{aligned}
X_t &= X_0 + \int_{\mathbb{R}^d\times[0,t]} f(X_s, v)\,\Gamma(dv \times ds) + \int_{\mathbb{R}^d\times[0,t]} \sigma(X_s, v)\, M_W(dv \times ds) \\
&\quad + \int_{\mathbb{R}^d\times[0,t]} \bar{\sigma}(X_s, v)\, M_B(dv \times ds), \qquad (2.3)
\end{aligned}$$

where the stochastic integrals are defined as in [18].

Proof. Relative compactness follows from the fact that

$$E[(M_B^n(\varphi, t+h) - M_B^n(\varphi, t))^2|\mathcal{F}_t^n] = E[\int_{\mathbb{R}^d} \varphi(y)^2\Gamma^n(dy \times (t, t+h])|\mathcal{F}_t^n] \le \|\varphi\|^2 h$$

for each $\varphi \in C_b(\mathbb{R}^d)$ and similarly for $\{M_W^n\}$. Along any convergent subsequence, $\{\Gamma^n, M_B^n, M_W^n\}$ satisfies the convergence conditions in Theorem 4.2 of Kurtz and Protter [18] (see Example 12.1 of [18]), and X^n converges to a solution of (2.3) by Theorem 7.4 of Kurtz and Protter [18]. □

Remark 2.3 *Since the time-marginal of Γ in Lemma 2.2 is Lebesgue measure, we can write $\Gamma(C \times [0,t]) = \int_0^t \gamma_s(C)ds$, where γ is a $\mathcal{P}(\mathbb{R}^d)$-valued process. Note that the quadratic covariation matrix for $\int_{\mathbb{R}^d\times[0,t]} \bar{\sigma}(X_s, v)\, M_B(dv \times ds)$ is*

$$\int_{\mathbb{R}^d\times[0,t]} \bar{\sigma}(X_s, v)\bar{\sigma}(X_s, v)^T\Gamma(dv \times ds) = \int_0^t \int_{\mathbb{R}^d} \bar{\sigma}(X_s, v)\bar{\sigma}(X_s, v)^T\gamma_s(dv)ds.$$

If $\bar{\sigma}(x, v)$ is nonsingular for every x and v, then $\bar{a}(x, \gamma) = \int_{\mathbb{R}^d} \bar{\sigma}(x, v)\bar{\sigma}(x, v)^T\gamma(dv)$ is positive definite for every $x \in \mathbb{R}^d$ and $\gamma \in \mathcal{P}(\mathbb{R}^d)$, as is $\bar{\sigma}_0(x, \gamma) = \bar{a}(x, \gamma)^{\frac{1}{2}}$. Similarly, *defining* $\sigma_0(x,\gamma)$ *to be the square root of* $\int_{\mathbb{R}^d}\sigma(x,v)\sigma(x,v)^T\gamma(dv)$ *and setting* $f_0(x,\gamma) = \int_{\mathbb{R}^d} f(x,v)\gamma(dv)$*, there exist independent Brownian motions* $\widetilde{W}$ *and* $\widetilde{B}$ *(perhaps on an enlarged sample space) such that*

$$X_t = X_0 + \int_{\mathbb{R}^d\times[0,t]} f_0\left(X_s,\gamma_s\right) ds + \int_0^t \sigma_0\left(X_s,\gamma_s\right) d\widetilde{W}_s + \int_0^t \bar{\sigma}_0\left(X_s,\gamma_s\right) d\widetilde{B}_s. \tag{2.4}$$

The independence of B *and* (V^n,W) *implies that* B *and*

$$U_t^n = \int_0^t \bar{\sigma}\left(X_s^n,V_s^n\right) dB_s$$

are martingales with respect to the filtration given by $\mathcal{G}_t^n = \mathcal{F}_t^{X^n,B}\vee\sigma(\Gamma^n,W)$*. It follows that* B*,*

$$U_t^B = \int_{\mathbb{R}^d\times[0,t]} \bar{\sigma}\left(X_s,v\right) M_B(dv\times ds),$$

and

$$\widetilde{B}_t = \int_0^t \bar{\sigma}_0^{-1}(X_s,\gamma_s)dU_s$$

are martingales with respect to the filtration given by $\mathcal{G}_t = \mathcal{F}_t^{X,B,U^W}\vee\sigma(\Gamma,W)$ *where*

$$U_t^W = \int_{\mathbb{R}^d\times[0,t]} \sigma\left(X_s,v\right) M_W(dv\times ds).$$

It then is possible to construct $\widetilde{W}$ *so that* $\widetilde{B}$ *is independent of* $(\Gamma,\widetilde{W})$*.*

We also will need to following result on convergence of conditional expectations.

Lemma 2.4 *Let* $\{X_n\}$ *be a* uniformly integrable *sequence of random variables converging in distribution to a random variable* X*, and let* $\{\mathcal{D}_n\}$ *be a sequence of* σ*-fields defined on the probability spaces where* X_n *reside. Let* $\{Y_n\}$ *be a sequence of* S*-valued random variables such that*

$$E\left[X_n|\mathcal{D}_n\right] = G(Y_n),$$

where $G : S\to\mathbb{R}$ *is continuous. Suppose* $(X_n,Y_n)\Rightarrow(X,Y)$*. Then* $E\left[X|Y\right] = G(Y)$*.*

Proof. Since $\{X_n\}$ is uniformly integrable, it follows by Jensen's inequality that $\{G(Y_n)\}$ is uniformly integrable. Then, employing the convergence in distribution and the uniform integrability,

$$E\left[G(Y)g(Y)\right] = \lim_{n\to\infty} E\left[G(Y_n)g(Y_n)\right] = \lim_{n\to\infty} E\left[X_n g(Y_n)\right] = E\left[Xg(Y)\right],$$

for every $g\in C_b(S)$, and the lemma follows. □

To complete the proof of Theorem 1.3, note that by Theorem 1.1, v_t^n charges every open set and

$$V_t^{\alpha,n} = \inf\{x \in \mathbb{R} | v_t^n(-\infty, x] \geq \alpha\} = \sup\{x \in \mathbb{R} | v_t^n(-\infty, x)] < \alpha\}.$$

For each i, the finiteness of $\sup_n E[\sup_{s\leq t}|X_s^{i,n}| + \sup_{s\leq t}|V_s^{t,n}|]$, the linear growth bound on f, σ, and $\bar{\sigma}$, and standard estimates on stochastic integrals imply that the sequence $\{X^{i,n}\}_{n>0}$ is relatively compact (in distribution) in $D_{\mathbb{R}}([0,\infty)$. This relative compactness together with the continuity of the processes ensures relative compactness of $\{X^n\}_{n>0}$ in $D_{\mathbb{R}^\infty}([0,\infty)$. Taking a subsequence, if necessary, we can assume that $\{X^n\}_{n>0}$ converges in distribution to a continuous process $X = (X^i)_{i\geq 0}$. By Lemma 4.4 of [14], v^n converges in distribution to v defined by

$$v_t = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \delta_{X_t^i}. \tag{2.5}$$

More precisely, (X^n, v^n, W) converges in distribution in $D_{\mathbb{R}^\infty\times\mathcal{P}(\mathbb{R})\times\mathbb{R}}[0,\infty)$ to (X, v, W), where X and v satisfy (2.5). Since v^n is $\{\mathcal{F}_t^{W,v^n(0)}\}$-adapted, for $\varphi \in C_b(\mathbb{R})$,

$$v_t^n(\varphi) = E[\varphi(X_t^{i,n})|\mathcal{F}_t^{W,v^n(0)}] = E[\varphi(X_t^{i,n})|\mathcal{F}_t^{W,v^n}].$$

Lemma 2.4 then implies

$$v_t(\varphi) = E[\varphi(X_t^i)|\mathcal{F}_t^{W,v}].$$

Note that we cannot guarantee that v is $\{\mathcal{F}_t^W\}$-adapted.

For each i, X^i will satisfy an equation of the form (2.4), where the $\widetilde{B}^i$ can be taken to be independent. These equations satisfy the conditions of Theorem 1.1, so

$$v_t(\varphi) = E[\varphi(X_t^i)|\mathcal{F}_t^{\widetilde{W},\gamma}]$$

and v_t charges any open set. By Lemma A.3, $V^{\alpha,n}$ converges in distribution to V^α, where

$$V_t^\alpha = \inf\{x \in \mathbb{R} | v_t((-\infty, x]) \geq \alpha\}.$$

In turn, it follows that M_W and M_B satisfy

$$M_B(\varphi, t) = \int_0^t \varphi(V_s^\alpha) dB_s, \quad \varphi \in C_b(\mathbb{R}^d),$$

and

$$M_W^\alpha(\varphi, t) = \int_0^t \varphi(V_s^\alpha) dW_s, \quad C_b(\mathbb{R}^d),$$

that is $\gamma_s = \delta_{V_s^\alpha}$. Consequently, $(X^n, V^{\alpha,n}, v^n)$ converges in distribution to (X, V^α, v) which is a weak solution of (1.2) - (1.4).

Applying Itô's formula to $\phi(X_t^i)$ and averaging the resulting identity as in [19] shows that v satisfies (1.5).

Remark 2.5 *It would be natural to expect a uniqueness result for (1.2) - (1.4), perhaps under the additional assumption that the coefficients were also Lipschitz in the second variable. Unfortunately, quantiles are not well-behaved functions of the corresponding distribution. If* V^α *were replaced by the mean* M *of* v, *then for two solutions* X *and* $\widehat{X}$

$$\begin{aligned} E[|\sigma(X_s^i, M_s) - \sigma(\widehat{X}_s^i, \widehat{M_s})|] &\leq KE[|X_s^i - \widehat{X}_s^i| + |M_s - \widehat{M_s}|] \\ &\leq KE[|X_s^i - \widehat{X}_s^i|] + K \sup_j E|X_s^j - \widehat{X}_s^j|] \\ &\leq 2KE[|X_s^i - \widehat{X}_s^i|], \end{aligned}$$

where the last inequality follows from the exchangeability, and uniqueness for the system would follow by an argument similar to that used in Section 10 of [18]. Unfortunately, there is no similar estimate for quantiles.

We can prove uniqueness for the system given by (1.1). If we define

$$Y_t^i = e^{-\beta t} X_0^i + \int_0^t e^{-\beta(t-s)} \bar{\sigma} dB_s^i,$$

we have

$$X_t^i = Y_t^i + \int_0^t \beta e^{-\beta(t-s)} S_s ds + \int_0^t e^{-\beta(t-s)} dW_s.$$

Consequently, if S_t *is the* α*-quantile of* $\{X_t^i\}$, *then*

$$S_t = U_t^\alpha + \int_0^t \beta e^{-\beta(t-s)} S_s ds + \int_0^t e^{-\beta(t-s)} dW_s, \tag{2.6}$$

where U_t^α *is the* α*-quantile of* $\{Y_t^i\}$ *and is uniquely determined since the* Y_t^i *are. (Note that* U_t^α *will be deterministic if the* X_0^i *are independent.) Clearly, the solution of (2.6) is unique.*

Remark 2.6 *Theorem 1.3 gives existence of a solution of (1.2) - (1.4) that is exchangeable. Suppose that we define*

$$V_t^\alpha = \limsup_{n\to\infty} \inf\{x \in \mathbb{R} : \frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\{X_t^i \leq x\}} \geq \alpha\}$$

and consider the system without an a priori assumption of exchangeability. Then strong uniqueness for the infinite system would imply that V^α *is measurable with respect to* W *and exchangeability of the solution would follow automatically. More generally, if weak uniqueness holds for the system, then any finite permutation of a solution is a solution so all finite permutations have the same distribution, that is, the solution is exchangeable.*

3 A model of prices for multiple assets

As an application of the multidimensional version of Theorem 1.1, we extend the asset price model discussed above to a market with multiple assets. To specify the model, we need to identify an appropriate market clearing condition. Our model essentially sets the prices by solving an assignment game as defined in [25]. The prices can also be interpreted as the result of a multi-item auction [7].

Suppose there are N traders and d assets. Each trader owns at most one unit of one of the assets. If the prices of the assets are $s_1, \ldots, s_d$ and the value that the ith trader places on the kth asset is x_{ik}, then the ith trader will buy the kth asset provided

$$x_{ik} - s_k \geq 0 \vee \max_{l \neq k}(x_{il} - s_l), \tag{3.1}$$

ignoring for the moment the ambiguity that would occur if there were more than one value of k satisfying (3.1). Suppose there are n_k units of the kth asset and $\sum_k n_k < N$. Then the prices should be set so that the assets can be allocated to the traders in such a way that each unit of the kth goes to a trader whose valuations satisfy (3.1) and each trader with valuations satisfying $x_{ik} - s_k > 0 \vee \max_{l \neq k}(x_{il} - s_l)$ receives a unit of asset k. Define

$$A_k^s = \{i : x_{ik} - s_k \geq 0 \vee \max_{l \neq k}(x_{il} - s_l)\}$$

and

$$A_0^s = \{i : x_{ik} \leq s_k, k = 1, \ldots, d\}.$$

Each trader receiving asset k must have index in A_k^s and each trader receiving no asset must have index in A_0^s. Setting $n_0 = N - \sum_{k=1}^d n_k$, the classical marriage theorem states that this allocation can be achieved if and only if for each $I \subset \{0, \ldots, d\}$,

$$\# \cup_{k \in I} A_k^s \geq \sum_{k \in I} n_k. \tag{3.2}$$

Assume that $\frac{n_k}{N} \to a_k$ as $N \to \infty$ and

$$\frac{1}{N} \sum_{i=1}^N \delta_{x_i} \Rightarrow \nu \in \mathcal{P}(\mathbb{R}^d).$$

Now for each $s \in \mathbb{R}^d$, let $A_k^s = \{x \in \mathbb{R}^d : x_k - s_k \geq 0 \vee \max_{l \neq k}(x_l - s_l)\}$, $k = 1, \ldots, d$, and $A_0^s = \{x \in \mathbb{R}^d : x_k \leq s_k, k = 1, \ldots, d\}$. The continuous version of the allocation requirement (3.2) becomes

$$\nu(\cup_{k \in I} A_k^s) \geq \sum_{k \in I} a_k. \tag{3.3}$$

Lemma 3.1 *Let $a_k > 0$, $k = 0, \ldots, d$, satisfy $\sum_{k=0}^{d} a_k = 1$. Then for each $\nu \in \mathcal{P}(\mathbb{R}^d)$, there exists $s \in \mathbb{R}^d$ such that*

$$\int \vee_l (x_l - s_l)^+ \nu(dx) + \sum_{l=1}^{d} a_l s_l = \inf_{s' \in \mathbb{R}^d} \int \vee_l (x_l - s'_l)^+ \nu(dx) + \sum_l a_l s'_l \tag{3.4}$$

and for each $I \subset \{0, \ldots, d\}$, (3.3) holds.

Remark 3.2 *The continuous version of the marriage theorem due to Dudley (see [8], Lemma 1.4) then gives the existence of measures ν_k, $k = 0, \ldots, d$, such that $\nu_k(A_k^s) = a_k$ and $\sum_{k=0}^{d} \nu_k = \nu$. For $k = 1, \ldots, d$, the ν_k determine the allocation of asset k to traders whose valuation satisfies $x_k - s_k \geq 0 \vee \max_{l \neq k}(x_l - s_l)$. The fact that $\nu_0(A_0^s) = a_0 = 1 - \sum_{k=1}^{d} a_k$ ensures that all traders with valuations satisfying $\max_k(x_k - s_k) > 0$ are allocated a unit of the asset.*

Proof. Assume first that ν is absolutely continuous with respect to Lebesgue measure, and let s be any minimizer of (3.4). Differentiation with respect to s_k shows that $\nu(A_k^s) = a_k$, $k = 1, \ldots, d$. The absolute continuity of ν implies $\nu\{x : x_k = s_k, \text{ some } k\} = 0$. Consequently, setting

$$A_k^{s,1} = \{x : x_k - s_k > 0 \vee \max_{l \neq k}(x_l - s_l)\}, \quad k = 1, \ldots, d,$$

$\nu(A_k^{s,1}) = \nu(A_k^s) = a_k$, and since $A_0^s = (\cup_{k=1}^{d} A_k^{s,1})^c$, $\nu(A_0^s) = a_0$. Setting $\nu_k = \nu(\cdot \cap A_k^s)$ gives the desired result.

For general ν, let ρ_ϵ be a mollifier with support in $B_\epsilon(0)$, the ball of radius ϵ around 0, and let ν_ϵ be the probability measure with density $f_\epsilon(x) = \int \rho_\epsilon(x - y)\nu(dy)$. Then there exists s^ϵ minimizing (3.4) with ν replaced by ν_ϵ and $\nu_\epsilon(A_k^{s^\epsilon}) = a_k$, $k = 0, \ldots, d$. As $\epsilon \to 0$, any limit point s of $\{s^\epsilon\}$ will minimize (3.4), and $\nu_\epsilon \Rightarrow \nu$.

For $x \in \mathbb{R}^d$, let $y_l = x_l - s_l^\epsilon + s_l$. For $1 \leq k \leq d$, suppose $x_k - s_k^\epsilon \geq 0 \vee \max_{l \neq k}(x_l - s_l^\epsilon)$. Then

$$y_k - s_k \geq 0 \vee \max_{l \neq k}(y_l - s_l)$$

and $y \in A_k^s$. Similarly, if $x_l \leq s_l^\epsilon$ for $1 \leq l \leq d$, then $y_l \leq s_l$ and $y \in A_0^s$. For any $B \in \mathcal{B}(\mathbb{R}^d)$, let $B^\epsilon = \{y : \inf_{x \in B} |y - x| < \epsilon\}$ and note that $\nu^\epsilon(B) \leq \nu(B^\epsilon)$. Consequently, for any $I \subset \{0, \ldots d\}$,

$$\sum_{k \in I} a_k \leq \nu^\epsilon(\cup_{k \in I} A_k^{s^\epsilon}) \leq \nu\{x : \exists y \in \cup_{k \in I} A_k^s \ni |x - y| \leq \epsilon + \max_l |s_l^\epsilon - s_l|\},$$

and

$$\sum_{k \in I} a_k \leq \liminf_{\epsilon \to 0} \nu^\epsilon(\cup_{k \in I} A_k^{s^\epsilon}) \leq \nu(\cup_{k \in I} A_k)$$

giving (3.3). □

We are interested in an infinite system

$$X_t^i = X_0^i + \int_0^t f\left(X_s^i, S_s\right) ds + \int_0^t \sigma\left(X_s^i, S_s\right) dW_s + \int_0^t \bar{\sigma}\left(X_s^i, S_s\right) dB_s^i,$$

where X^i takes values in $\mathbb{R}^d$ and S_t is the vector of prices determined by the requirement that

$$v_t\{\cup_{k\in I} A_k^{S_t}\} \geq \sum_{k\in I} a_k. \tag{3.5}$$

In other words, for the ith trader, X_t^i gives the valuations at time t of the d assets, and $v_t = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \delta_{X_t^i}$ gives the distribution of valuations by the infinite collection of traders.

Existence of a solution follows essentially as in the quantile example, v_t will have a strictly positive density which ensures that S_t is uniquely determined by v_t, and as before, v satisfies the stochastic partial differential equation

$$\langle \phi, v_t\rangle = \langle \phi, v_0\rangle + \int_0^t \langle L(S_s)\phi\,, v_s\rangle ds + \int_0^t \langle \nabla\phi^T \sigma(\cdot, S_s), v_s\rangle dW_s, \tag{3.6}$$

where

$$L(S)\phi(x) = \frac{1}{2}\sum_{i,j} a_{ij}(x,S)\partial_{x_i}\partial_{x_j}\phi(x) + f(x,S)\cdot\nabla\phi(x)$$

and

$$a(x,S) = \sigma(x,S)\sigma(x,S)^T + \bar{\sigma}(x,S)\bar{\sigma}(x,S)^T.$$

4 Quantile Process

Returning now to the single asset case, we find an equation for the quantile process

$$V_t^\alpha = \inf\{x\in\mathbb{R}\,,\ v_t((-\infty, x]) \geq \alpha\}\,.$$

Recall that we considered an infinite system of (one-dimensional) interacting diffusions

$$X_t^i = X_0^i + \int_0^t f\left(X_s^i, V_s^\alpha\right) ds + \int_0^t \sigma\left(X_s^i, V_s^\alpha\right) dW_s + \int_0^t \bar{\sigma}\left(X_s^i, V_s^\alpha\right) dB_s^i, \tag{4.1}$$

where

$$V_t^\alpha = \inf\left\{x\in\mathbb{R} | v_t\left(-\infty, x\right] \geq \alpha\right\}$$

and

$$v_t = \lim_{n\to\infty} v_t^n \quad \text{where} \quad v_t^n = \frac{1}{n}\sum_{i=1}^n \delta_{X_t^i}. \tag{4.2}$$

To prove the following result we choose a bounded, smooth, strictly positive function $q:\mathbb{R}\to\mathbb{R}$ with bounded first and second derivative such that $\int_{\mathbb{R}} q(x)\,dx = 1$ and

$$\sup_{x\in\mathbb{R}} \frac{q'(x)}{q(x)} < \infty^4. \tag{4.3}$$

Define the functions, $v_t^{n,\epsilon}, v_t^{\epsilon}, F_t^{n,\epsilon}, F_t^{\epsilon} : \mathbb{R}\to\mathbb{R}$ as follows

$$\begin{array}{ll} v_t^{n,\epsilon}(x) = \frac{1}{n}\sum_{i=1}^n q_\epsilon\left(x - X_t^i\right) & F_t^{n,\epsilon}(x) = \int_{-\infty}^x v_t^{n,\epsilon}(y)\,dy \\ v_t^{\epsilon}(x) = \int_{\mathbb{R}} q_\epsilon(x-y)\,v_t(dy) & F_t^{\epsilon}(x) = \int_{-\infty}^x v_t^{\epsilon}(y)\,dy \end{array},$$

where $q_\epsilon:\mathbb{R}\to\mathbb{R},\ q_\epsilon(x) = \frac{1}{\epsilon}q\left(\frac{x}{\epsilon}\right), x\in\mathbb{R}$. Then, the functions $v_t^{n,\epsilon}$ are uniformly bounded smooth functions and, since $\lim_{n\to\infty} v_t^n = v_t$, it follows that $v_t^{n,\epsilon}$ converges pointwise to v_t^{ϵ}. Hence the quantiles $V_t^{\alpha,n,\epsilon}$ of the probability measures with densities $v_t^{n,\epsilon}$ with respect to the Lebesgue measure uniquely defined by the formula

$$F^{n,\epsilon}\left(t, V_t^{\alpha,n,\epsilon}\right) = \alpha$$

converge to the quantiles $V_t^{\alpha,\epsilon}$ of the measure with density v_t^{ϵ} with respect to the Lebesgue measure, $\lim_{n\to\infty} V_t^{\alpha,n,\epsilon} = V_t^{\alpha,\epsilon}$. Moreover, since also the derivatives of the functions $v_t^{n,\epsilon}$ converge to the derivatives of the functions v_t^{ϵ} and are uniformly bounded, it follows that $v_t^{n,\epsilon}$ converges to v_t^{ϵ} *uniformly on compacts*. In particular this implies that $\lim_{n\to\infty} v_t^{n,\epsilon}(V_t^{\alpha,n,\epsilon}) = v_t^{\epsilon}(V_t^{\alpha,\epsilon})$. Similarly, $\lim_{n\to\infty} \frac{dv_t^{n,\epsilon}(x)}{dx}|_{x=V_t^{\alpha,n,\epsilon}} = \frac{dv_t^{\epsilon}(x)}{dx}|_{x=V_t^{\alpha,\epsilon}}$ These two facts will be used in the following proposition.

Proposition 4.1 *Assume Conditions C1, C2, and C3 and that f, σ and $\bar{\sigma}$ are twice continuously differentiable in the first component. Then the quantiles V_t^{α} satisfy the following evolution equation*

$$\begin{aligned} V_t^{\alpha} &= V_s^{\alpha} + \int_s^t f(V_r^{\alpha}, V_r^{\alpha})dr + \int_s^t \sigma(V_r^{\alpha}, V_r^{\alpha})dW_r \\ &\quad - \int_s^t \frac{1}{2v_r(V_r^{\alpha})}\frac{\partial}{\partial x}\left[(\sigma(x, V_r^{\alpha}))^2 v_r(x)\right]\Big|_{x=V_r^{\alpha}} dr. \end{aligned} \tag{4.4}$$

for any $t > s > 0$.

Proof. First, note that, by the definition of the quantiles,

$$\Upsilon^{\alpha,n,\epsilon}\left(V_t^{\alpha,n,\epsilon}, X_t^1, ..., X_t^n\right) = 0,$$

where $\Upsilon^{\alpha,n,\epsilon}:\mathbb{R}^{n+1}\to\mathbb{R}$ is the smooth function

$$\Upsilon^{\alpha,n,\epsilon}(v, x_1, ..., x_n) = \frac{1}{n}\sum_{i=1}^n \int_{-\infty}^v q_\epsilon(y - x_i)\,dy - \alpha.$$

[4] One can choose q such that $q(x) = c_q\exp(-|x|)$ for $|x|\geq 1$, where c_q is the normalization constant.

Since $\frac{\partial \Upsilon^{\alpha,n,\epsilon}}{\partial v}(v,x_1,...,x_n)=\frac{1}{n}\sum_{i=1}^n q_\epsilon(v-x_i)>0$, by the implicit function theorem there exists a countable set of balls $B(x_j,r_j)\in\mathbb{R}^n$ $j\geq 1$ such that $\bigcup_{n\geq 1}B(x_j,r_j)=\mathbb{R}^n$ and a countable set of *smooth* functions $Q^{\alpha,n,\epsilon,j}:B(x_j,r_j)\to\mathbb{R}$ such that

$$V_t^{\alpha,n,\epsilon}=Q^{\alpha,n,\epsilon,j}\left(X_t^1,...,X_t^n\right),\text{ if }\left(X_t^1,...,X_t^n\right)\in B(x_j,r_j).$$

In particular it follows that $V_t^{\alpha,n,\epsilon}$ is a semi-martingale. This fact allows us to deduce the evolution equation for the semimartingales $V_t^{\alpha,n,\epsilon}$. By applying the generalized Itô formula (see, for example, Kunita [16]) we have

$$\begin{aligned}
0 &= d\Upsilon^{\alpha,n,\epsilon}\left(V_t^{\alpha,n,\epsilon},X_t^1,...,X_t^n\right)\\
&= \frac{\partial\Upsilon^{\alpha,n,\epsilon}}{\partial v}\left(V_t^{\alpha,n,\epsilon},X_t^1,...,X_t^n\right)dV_t^{\alpha,n,\epsilon}+\sum_{j=1}^n\frac{\partial\Upsilon^{\alpha,n,\epsilon}}{\partial x_j}\left(V_t^{\alpha,n,\epsilon},X_t^1,...,X_t^n\right)dX_t^j\\
&\quad+\frac{1}{2}\frac{\partial^2\Upsilon^{\alpha,n,\epsilon}}{\partial v^2}\left(V_t^{\alpha,n,\epsilon},X_t^1,...,X_t^n\right)d\langle V^{\alpha,n,\epsilon}\rangle_t+\frac{1}{2}\sum_{j=1}^n\frac{\partial^2\Upsilon^{\alpha,n,\epsilon}}{\partial x_j^2}\left(V_t^{\alpha,n,\epsilon},X_t^1,...,X_t^n\right)d\langle X^j\rangle_t\\
&\quad+\sum_{j=1}^n\frac{\partial\Upsilon^{\alpha,n,\epsilon}}{\partial x_j\partial v}\left(V_t^{\alpha,n,\epsilon},X_t^1,...,X_t^n\right)d\left\langle V^{\alpha,n,\epsilon},X^j\right\rangle_t,
\end{aligned}$$

which implies that

$$\begin{aligned}
0 &= v_t^{n,\epsilon}\left(V_t^{\alpha,n,\epsilon}\right)dV_t^{\alpha,n,\epsilon}-\frac{1}{n}\sum_{j=1}^n f\left(X_t^j,V_t^\alpha\right)q_\epsilon\left(V_t^{\alpha,n,\epsilon}-X_t^j\right)dt\\
&\quad-\frac{1}{n}\sum_{j=1}^n\sigma\left(X_t^j,V_t^\alpha\right)q_\epsilon\left(V_t^{\alpha,n,\epsilon}-X_t^j\right)dW_t-\frac{1}{n}\sum_{j=1}^n\bar\sigma\left(X_t^j,V_t^\alpha\right)q_\epsilon\left(V_t^{\alpha,n,\epsilon}-X_t^j\right)dB_t^j\\
&\quad+\frac{1}{2n}\sum_{j=1}^n\bar\sigma^2\left(X_t^j,V_t^\alpha\right)q_\epsilon'\left(V_t^{\alpha,n,\epsilon}-X_t^j\right)dt+\frac{1}{2n}\sum_{j=1}^n\sigma^2\left(X_t^j,V_t^\alpha\right)q_\epsilon'\left(V_t^{\alpha,n,\epsilon}-X_t^j\right)dt\\
&\quad+\frac{1}{2n}\sum_{j=1}^n q_\epsilon'\left(V_t^{\alpha,n,\epsilon}-X_t^j\right)d\langle V^{\alpha,n,\epsilon}\rangle_t-\frac{1}{n}\sum_{j=1}^n q_\epsilon'\left(V_t^{\alpha,n,\epsilon}-X_t^j\right)\sigma\left(X_s^i,V_s^\alpha\right)d\langle W,V^{\alpha,n,\epsilon}\rangle_t\\
&\quad-\frac{1}{n}\sum_{j=1}^n q_\epsilon'\left(V_t^{\alpha,n,\epsilon}-X_t^j\right)\bar\sigma\left(X_s^i,V_s^\alpha\right)d\left\langle B^j,V^{\alpha,n,\epsilon}\right\rangle_t.
\end{aligned}$$

From this identity it follows that

$$\begin{aligned}
\langle V^{\alpha,n,\epsilon}\rangle_t &= \int_0^t\frac{1}{v_s^{n,\epsilon}\left(V_s^{\alpha,n,\epsilon}\right)^2}\left(\frac{1}{n}\sum_{j=1}^n\sigma\left(X_s^j,V_s^\alpha\right)q_\epsilon\left(V_s^{\alpha,n,\epsilon}-X_s^j\right)\right)^2ds\\
&\quad+\int_0^t\frac{1}{v_s^{n,\epsilon}\left(V_s^{\alpha,n,\epsilon}\right)^2}\left(\frac{1}{n^2}\sum_{j=1}^n\bar\sigma\left(X_s^j,V_s^\alpha\right)^2q_\epsilon\left(V_s^{\alpha,n,\epsilon}-X_s^j\right)^2\right)ds
\end{aligned}$$

$$\begin{aligned}
\langle W, V^{\alpha,n,\epsilon}\rangle_t &= \int_0^t \frac{1}{v_s^{n,\epsilon}\left(V_s^{\alpha,n,\epsilon}\right)} \left(\frac{1}{n}\sum_{j=1}^n \sigma\left(X_s^j, V_s^\alpha\right) q_\epsilon\left(V_s^{\alpha,n,\epsilon} - X_s^j\right)\right) ds \\
\left\langle B^i, V^{\alpha,n,\epsilon}\right\rangle_t &= \int_0^t \frac{1}{v_s^{n,\epsilon}\left(V_s^{\alpha,n,\epsilon}\right)} \frac{1}{n} \bar{\sigma}\left(X_s^j, V_s^\alpha\right) q_\epsilon\left(V_s^{\alpha,n,\epsilon} - X_s^j\right) ds.
\end{aligned}$$

Therefore

$$\begin{aligned}
dV_t^{\alpha,n,\epsilon} &= \frac{1}{nv_t^{n,\epsilon}\left(V_t^{\alpha,n,\epsilon}\right)} \sum_{j=1}^n f\left(X_t^j, V_t^\alpha\right) q_\epsilon\left(V_t^{\alpha,n,\epsilon} - X_t^j\right) dt \\
&+ \frac{1}{nv_t^{n,\epsilon}\left(V_t^{\alpha,n,\epsilon}\right)} \sum_{j=1}^n \sigma\left(X_t^j, V_t^\alpha\right) q_\epsilon\left(V_t^{\alpha,n,\epsilon} - X_t^j\right) dW_t \\
&+ \frac{1}{nv_t^{n,\epsilon}\left(V_t^{\alpha,n,\epsilon}\right)} \sum_{j=1}^n \bar{\sigma}\left(X_t^j, V_t^\alpha\right) q_\epsilon\left(V_t^{\alpha,n,\epsilon} - X_t^j\right) dB_t^j \\
&- \frac{1}{2nv_t^{n,\epsilon}\left(V_t^{\alpha,n,\epsilon}\right)} \sum_{j=1}^n (\bar{\sigma}^2\left(X_t^j, V_t^\alpha\right) + \sigma^2\left(X_t^j, V_t^\alpha\right)) q_\epsilon'\left(V_t^{\alpha,n,\epsilon} - X_t^j\right) dt \\
&- \frac{1}{2nv_t^{n,\epsilon}\left(V_t^{\alpha,n,\epsilon}\right)} \sum_{j=1}^n q_\epsilon'\left(V_t^{\alpha,n,\epsilon} - X_t^j\right) d\langle V^{\alpha,n,\epsilon}\rangle_t \\
&+ \frac{1}{nv_t^{n,\epsilon}\left(V_t^{\alpha,n,\epsilon}\right)} \sum_{j=1}^n q_\epsilon'\left(V_t^{\alpha,n,\epsilon} - X_t^j\right) \sigma\left(X_s^i, V_s^\alpha\right) d\langle W, V^{\alpha,n,\epsilon}\rangle_t \\
&+ \frac{1}{n}\sum_{j=1}^n q_\epsilon'\left(V_t^{\alpha,n,\epsilon} - X_t^j\right) \bar{\sigma}\left(X_s^i, V_s^\alpha\right) d\left\langle B^j, V^{\alpha,n,\epsilon}\right\rangle_t . \qquad (4.5)
\end{aligned}$$

Observe that the term $\frac{1}{nv_t^{n,\epsilon}\left(V_t^{\alpha,n,\epsilon}\right)} \sum_{j=1}^n f\left(X_t^j, V_t^\alpha\right) q_\epsilon\left(V_t^{\alpha,n,\epsilon} - X_t^j\right)$ is bounded by $\|f\|_\infty$, the supremum norm of f, with similar bounds holding for the second and the third term in (4.5) and for the terms appearing in the expression for $\langle V^{\alpha,n,\epsilon}\rangle_t$, $\langle W, V^{\alpha,n,\epsilon}\rangle_t$, $\langle B^i, V^{\alpha,n,\epsilon}\rangle_t$. The term

$$x \to \frac{1}{2nv_t^{n,\epsilon}(x)} \sum_{j=1}^n (\bar{\sigma}^2(x, V_t^\alpha) + \sigma^2(x, V_t^\alpha)) q_\epsilon'\left(x - X_t^j\right)$$

is uniformly bounded by $\frac{1}{\epsilon}(\|\bar{\sigma}\|^2 + \|\sigma\|^2)$ following property (4.3) of the function q. A similar bound can be proved for all the remaining terms in (4.5) are uniformly bounded on compacts as $\inf_n \inf_{r\in[s,t]} v_s^{n,\epsilon}(x)$ is strictly positive on compacts (using the tightness of the sequence v^n) and $\bar{\sigma}$, σ, and q_ϵ' are bounded. Using these bounds, we take the limit in (4.5) as n tends to infinity to obtain that

$$dV_t^{\alpha,\epsilon} = \frac{1}{v_t^\epsilon\left(V_t^{\alpha,\epsilon}\right)} \left(\int_{\mathbb{R}} f\left(x, V_t^\alpha\right) q_\epsilon\left(V_t^{\alpha,\epsilon} - x\right) v_t(dx)\right) dt$$

$$
\begin{aligned}
&+\frac{1}{v_t^\epsilon\left(V_t^{\alpha,\epsilon}\right)}\left(\int_{\mathbb{R}}\sigma\left(x,V_t^\alpha\right)q_\epsilon\left(V_t^{\alpha,\epsilon}-x\right)v_t\left(dx\right)\right)dW_t\\
&-\frac{1}{2v_t^\epsilon\left(V_t^{\alpha,\epsilon}\right)}\left(\int_{\mathbb{R}}(\bar{\sigma}^2\left(x,V_t^\alpha\right)+\sigma^2\left(x,V_t^\alpha\right))q_\epsilon'\left(V_t^{\alpha,\epsilon}-x\right)v_t\left(dx\right)\right)dt\\
&-\frac{1}{2v_t^\epsilon\left(V_t^{\alpha,\epsilon}\right)}\left(\int_{\mathbb{R}}q_\epsilon'\left(V_t^{\alpha,\epsilon}-x\right)v_t\left(dx\right)\right)\\
&\qquad\qquad\times\frac{1}{v_t^\epsilon\left(V_t^{\alpha,\epsilon}\right)^2}\left(\int_{\mathbb{R}}\sigma\left(x,V_t^\alpha\right)q_\epsilon\left(V_t^{\alpha,\epsilon}-x\right)v_t\left(dx\right)\right)^2dt\\
&+\frac{1}{v_t^\epsilon\left(V_t^{\alpha,\epsilon}\right)}\left(\int_{\mathbb{R}}q_\epsilon'\left(V_t^{\alpha,\epsilon}-x\right)\sigma\left(x,V_s^\alpha\right)v_t\left(dx\right)\right)\\
&\qquad\qquad\times\frac{1}{v_t^\epsilon\left(V_t^{\alpha,\epsilon}\right)}\left(\int_{\mathbb{R}}\sigma\left(x,V_t^\alpha\right)q_\epsilon\left(V_t^{\alpha,\epsilon}-x\right)v_t\left(dx\right)\right)dt. \qquad (4.6)
\end{aligned}
$$

Next since $v_t(x)=\lim_{\epsilon\to0}v_t^\epsilon(x)$ as ϵ tends to 0, it follows that $V_t^\alpha=\lim_{\epsilon\to0}V_t^{\alpha,\epsilon}$. Following from Corollary 6.1 and the boundedness of both $v_r(x)$ and $\frac{\partial}{\partial x}\left[v_r(x)\right]$ on sets of the form $[s,t]\times[-k,k]$, we can take the limit in (4.6) as ϵ tends to 0 to obtain that

$$
\begin{aligned}
dV_t^\alpha &= f(V_t^\alpha,V_t^\alpha)dt+\sigma(V_t^\alpha,V_t^\alpha)dW_t-\frac{1}{2v(t,V_t^\alpha)}\frac{\partial}{\partial x}\left[(\bar{\sigma}^2(x,V_t^\alpha)+\sigma^2(x,V_t^\alpha))v_t(x)\right]\Big|_{x=V_t^\alpha}dt\\
&\quad-\frac{1}{2v(t,V_t^\alpha)}\sigma^2(V_t^\alpha,V_t^\alpha)\frac{\partial}{\partial x}\left[v_t(x)\right]\Big|_{x=V_t^\alpha}dt+\frac{\sigma(V_t^\alpha,V_t^\alpha)}{v(t,V_t^\alpha)}\frac{\partial}{\partial x}\left[\sigma(x,V_t^\alpha)v_t(x)\right]\Big|_{x=V_t^\alpha}dt,
\end{aligned}
$$

which gives (4.4). □

Remark 4.2 *See also [22] for the equation (4.4).*

Remark 4.3 *Under additional assumptions on the initial distribution of X (for example if the distribution of X_0 is absolutely continuous with respect to the Lebesgue measure with strictly positive density), one can show that (4.4) holds true also for $s=0$.*

5 Application to nonlinear filtering

Let $(\Omega,\mathcal{F},P)$ be a probability space on which we have defined two independent d-dimensional, respectively m -dimensional standard Brownian motions $B=\{(B_t^i)_{i=1}^d,t\geq0\}$ and $W=\{(W_t^i)_{i=1}^m,t\geq0\}$ Let (X,Y) be the solution of the following stochastic system

$$
\begin{aligned}
X_t &= X_0+\int_0^t f\left(X_s,Y_s\right)ds+\int_0^t\bar{\sigma}\left(X_s,Y_s\right)dW_s+\int_0^t\bar{\sigma}\left(X_s,Y_s\right)dB_s\\
Y_t &= \int_0^t h\left(X_s,Y_s\right)ds+\int_0^t k\left(Y_s\right)dW_s.
\end{aligned}
$$

Let $\mathcal{F}_t^Y$ be σ-field generated by the process Y and π_t be the conditional distribution of X_t given the σ-field generated by the process Y. We show that π_t charges any open set. Moreover, under additional conditions, we show that it is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$ and has a positive density. We have the following

Corollary 5.1 *Assume the following:*

- $f : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$, $h : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^m$, $\sigma : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^d$, *and* $\bar{\sigma} : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d \times \mathbb{R}^d$ *are continuous functions, uniformly Lipschitz in the first argument.*
- $\bar{\sigma}$ *is non-singular,* $k : \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^m$ *is invertible,* k^{-1} *is bounded and* $\sigma k^{-1} h : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ *is a continuous functions, uniformly Lipschitz in the first argument.*
- *The random variable* X_0 *has finite second moment.*

Then there exists a set $\widetilde{\Omega} \in \mathcal{F}$ *of full measure such that for every* $\omega \in \widetilde{\Omega}$ *and* $t > 0$, π_t^ω *charges any nonempty open set.*

If, in addition, the functions f, $\sigma k^{-1} h$, σ *and* $\bar{\sigma}$ *are continuously differentiable in the first component then there exists a set* $\widetilde{\Omega} \in \mathcal{F}$ *of full measure such that for every* $\omega \in \widetilde{\Omega}$ *and* $t > 0$, π_t^ω *is absolutely continuous with respect to the Lebesgue measure with a strictly positive density.*

Proof. Let $Z = \{Z_t, t \geq 0\}$ be defined as

$$\begin{aligned} Z_t &= \exp\Bigg(-\int_0^t \left(k^{-1}(Y_s)\, h(X_s, Y_s)\right)^\top dW_s \\ &\qquad -\frac{1}{2}\int_0^t \left(k^{-1}(Y_s)\, h(X_s, Y_s)\right)^\top \left(k^{-1}(Y_s)\, h(X_s, Y_s)\right) ds \Bigg). \end{aligned}$$

Under the above assumption Z is a martingale. Consider the probability measure $\widetilde{P}$ absolutely continuous with respect to P defined as

$$\left.\frac{d\widetilde{P}}{dP}\right|_{\mathcal{F}_t} = Z_t.$$

Then, by Girsanov's theorem, the process $\widetilde{W} = \left\{\widetilde{W}_t, t \geq 0\right\}$ defined by

$$\widetilde{W}_t = W_t - \int_0^t k^{-1}(Y_s)\, h(X_s, Y_s)\, ds$$

for $t \geq 0$ is a Brownian motion under $\widetilde{P}$ independent of B and, by Kallianpur-Striebel's formula,

$$E\left[\varphi(X_t) | \mathcal{F}_t^Y\right] = \widetilde{E}\left[\varphi(X_t)\, \zeta_t | \mathcal{F}_t^Y\right], \tag{5.1}$$

where $\zeta_t = \frac{Z_t^{-1}}{\widetilde{E}\left[Z_t^{-1}|\mathcal{F}_t^Y\right]}$ and

$$X_t = X_0 + \int_0^t \left(f + \sigma k^{-1} h\right)(X_s, Y_s)\, ds + \int_0^t \sigma(X_s, Y_s)\, d\widetilde{W}_s + \int_0^t \bar{\sigma}(X_s, Y_s)\, dB_s.$$

We note that, under $\widetilde{P}$, Y satisfies the SDE

$$Y_t = \int_0^t k(Y_s)\, d\widetilde{W}_s,$$

hence

$$\widetilde{W}_t = \int_0^t k^{-1}(Y_s)\, dY_s$$

and in particular $\mathcal{F}_t^Y = \mathcal{F}_t^{\widetilde{W},Y}$ for all $t \geq 0$. From (5.1) we obtain that as in (6.3) that

$$\pi_t(\varphi) = \int_{\mathbb{R}^d} E\left[\varphi(X_t(z))\, M_t(z)\, \zeta_t \frac{e^{-\frac{1}{2}z^\top z}}{(2\pi)^{\frac{d}{2}}} |\mathcal{F}_t^Y\right] dz$$

where $M_t(z)$ is the martingale defined in (6.2). The analysis then proceeds in an identical fashion to that in the proofs of Theorems 1.1 and 1.2. □

Remark 5.2 *Note that we cannot apply the results of the Theorems 1.1 and 1.2 under the original measure P as the Brownian motion B is not independent of Y under P.*

Remark 5.3 *Corollary 5.1 can be interpreted as a smoothing result of the most basic kind. Essentially we prove under Lipschitz/differentiability conditions (in the first argument only !) on the coefficients that, whilst π_0 is arbitrary, π_t charges every open set and, respectively, has a positive density with respect to the Lebesgue measure for any $t > 0$. Recently, Krylov proved in [15] that if π_0 has a density that belongs to the Sobolev space $H_p^{1-2/p}(\mathbb{R}^d)$ for all $p \geq 2$, then π_t is $1 - \varepsilon$ Hölder continuous. In addition to the boundedness and the Lipschitz assumptions on the coefficients (imposed both in the x and in the y variable), the results in [15] also require uniform ellipticity of the diffusion matrix.*

6 Proof of the properties of the conditional distributions

Let F be a function $F : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ with the following properties:

- For each $z \in \mathbb{R}^d$, the function $t \to F(t,z)$ is a measurable, locally-bounded function.
- For each $t \in [0,\infty)$, the function $z \to F(t,z)$ is differentiable. $F'(t,z)$ will denote the matrix of partial derivatives

$$\left(F'(t,z)\right)_{ij} = \partial_j F_i(t,z).$$

- For each $z \in \mathbb{R}^d$, the function $t \to \digamma'(t,z)$ is a measurable, locally-bounded function.

Now consider a new probability measure P^z, absolutely continuous with respect to P, defined by

$$\left.\frac{dP^z}{dP}\right|_{\mathcal{F}_t} = \exp\left(-\int_0^t \digamma(s,z)^\top dB_s - \frac{1}{2}\int_0^t |\digamma(s,z)|^2 ds\right),$$

where $\digamma(s,z)^\top$ is the row vector $(\digamma(s,z)_1, \digamma(s,z)_2, \ldots, \digamma(s,z)_d)$. Then, by Girsanov's theorem, the process $B^z = \{B_t^z, t \geq 0\}$

$$B_t^z = B_t + \int_0^t \digamma(s,z)\, ds$$

is a Brownian motion under P^z, independent of W and V. Since (B^z, W, V) has the same law under P^z as (B, W, V) has under P, it follows that $X(z)$ given by

$$\begin{aligned} dX_t(z) &= f(X_t(z), V_t)\, dt + \sigma(X_t(z), V_t)\, dW_t + \bar{\sigma}(X_t(z), V_t)\, dB_t^z \qquad (6.1)\\ &= f(X_t(z), V_t)\, dt + \sigma(X_t(z), V_t)\, dW_t + \bar{\sigma}(X_t(z), V_t)\, dB_t \\ &\quad + \bar{\sigma}(X_t(z), V_t)\, \digamma(t,z)\, dt. \end{aligned}$$

has the same law under P^z as X has under P, and for $\varphi \in B(\mathbb{R}^d)$,

$$\begin{aligned} E\left[\varphi(X_t) | \mathcal{F}_t^{W,V}\right] &= E^z\left[\varphi(X_t(z)) | \mathcal{F}_t^{W,V}\right] \\ &= E\left[\varphi(X_t(z))\, M_t(z) | \mathcal{F}_t^{W,V}\right] \end{aligned}$$

where $M_t(z)$ is defined as

$$M_t(z) = \exp\left(-\int_0^t \digamma(s,z)^\top dB_s - \frac{1}{2}\int_0^t |\digamma(s,z)|^2 ds\right), \quad t \geq 0. \qquad (6.2)$$

In the following, we will use a Fubini argument for the function ι, where

$$(z,\omega) \xrightarrow{\iota} \varphi(X_t(z))\, M_t(z)\, \frac{e^{-\frac{1}{2}|z|^2}}{(2\pi)^{\frac{d}{2}}}$$

is defined on the product space $\mathbb{R}^d \times \Omega$. Consequently, we need to know that ι is $\mathcal{B}(\mathbb{R}^d) \times \mathcal{F}_t^{W,V,B}$-measurable. Measurability is not immediate as $X_t(z)$ is initially defined for each z individually. However, one can prove the existence of a process $\bar{X}_t(z)$ such that for each z, $\bar{X}(z)$ and $X(z)$ are indistinguishable and

$$(z,\omega) \xrightarrow{\bar{\iota}} \bar{X}_t(z)$$

is $\mathcal{B}(\mathbb{R}^d) \times \mathcal{F}_t^{W,V,B}$-measurable. More precisely, we can assume that $\bar{X}$ is optional, that is, the mapping

$$(t,z,\omega) \in [0,\infty) \times \mathbb{R}^d \times \Omega \to \bar{X}_t(z)$$

is measurable with respect to the σ-algebra generated by processes of the form

$$\sum \xi_i f_i(z)\mathbf{1}_{[t_i,t_{i+1})}(t),$$

where $0 = t_0 < t_1 < \cdots$, $f_i \in C(\mathbb{R}^d)$, and ξ_i is $\mathcal{F}_{t_i}^{W,V,B}$-measurable. To avoid further measurability complications, from now on, we will use this version of the solution of (6.1). Hence, if $\varphi : \mathbb{R}^d \times \Omega \to \mathbb{R}$ is a non-negative, $\mathcal{B}(\mathbb{R}^d) \times \mathcal{F}_t^{W,V}$-measurable function, the conditional version of Fubini's theorem (for nonnegative functions) gives

$$\begin{aligned} E\left[\varphi\left(X_t,\cdot\right)|\mathcal{F}_t^{W,V}\right] &= \int_{\mathbb{R}^d} E\left[\varphi\left(X_t\left(z\right),\cdot\right)M_t\left(z\right)|\mathcal{F}_t^{W,V}\right]\frac{e^{-\frac{1}{2}z^\top z}}{(2\pi)^{\frac{d}{2}}}dz \\ &= E\left[\int_{\mathbb{R}^d}\varphi\left(X_t\left(z\right),\cdot\right)M_t\left(z\right)\frac{e^{-\frac{1}{2}z^\top z}}{(2\pi)^{\frac{d}{2}}}dz|\mathcal{F}_t^{W,V}\right]. \end{aligned} \tag{6.3}$$

We treat the one-dimensional and the multi-dimensional cases separately.

6.1 The one-dimensional case

Consider the function

$$F\left(t,z\right) = z, \quad \forall t \geq 0. \tag{6.4}$$

In this case, (6.1) becomes

$$dX_t(z) = f\left(X_t\left(z\right),V_t\right)dt + \sigma\left(X_t\left(z\right),V_t\right)dW_t + \bar{\sigma}\left(X_t\left(z\right),V_t\right)\left(dB_t + zdt\right). \tag{6.5}$$

Since $\bar{\sigma}$ is positive, with probability 1, the function $z \to X_t(z)$ is a strictly increasing, continuous function and $\lim_{z\to-\infty} X_t(z) = -\infty$ and $\lim_{z\to\infty} X_t(z) = \infty$. In particular, $z \to X_t(z)$ is a continuous bijection, so if $(\underline{\beta},\overline{\beta})$ is a (non-empty) open interval, then $X_t^{-1}(\underline{\beta},\overline{\beta})$ is a non-empty open interval. In particular, $X_t^{-1}(\underline{\beta},\overline{\beta})$ has positive Lebesgue measure. Hence, if we choose φ in (6.3) to be the indicator function of an open interval $(\underline{\beta},\overline{\beta})$, then

$$P\left[X_t \in (\underline{\beta},\overline{\beta})|\mathcal{F}_t^{W,V}\right] = \frac{1}{\sqrt{2\pi}}E\left[\int_{X_t^{-1}(\underline{\beta},\overline{\beta})} e^{-zB_t-\frac{z^2(t+1)}{2}}dz|\mathcal{F}_t^{W,V}\right]. \tag{6.6}$$

Since $z \to e^{-zB_t-\frac{z^2(t+1)}{2}}$ is positive on $X_t^{-1}(\underline{\beta},\overline{\beta})$, it follows that $\int_{X_t^{-1}(\underline{\beta},\overline{\beta})} e^{-zB_t-\frac{z^2(t+1)}{2}}dz$ is positive (with probability 1) as is its conditional expectation. *This proves Theorem 1.1* in the case $d = 1$.

Assuming that f, σ and $\bar{\sigma}$ are differentiable, $z \to X_t(z)$ is differentiable with probability 1. Its (positive) derivative is given by

$$J_t\left(z\right) \overset{\text{def}}{=} \frac{dX_t(z)}{dz} = \int_0^t \bar{\sigma}\left(X_s\left(z\right),V_s\right)\exp\left(i_s^t\left(z\right)\right)ds, \tag{6.7}$$

where

$$\begin{aligned} i_s^t(z) &= \int_s^t \left(f'(X_r(z), V_r) - \frac{1}{2}(\sigma'(X_r(z), V_r))^2 - \frac{1}{2}(\bar{\sigma}'(X_r(z), V_r))^2 \right) dr \\ &\quad + \int_s^t \sigma'(X_r(z), V_r) dW_r + \int_s^t \bar{\sigma}'(X_r(z), V_r) dB_r + \int_s^t \bar{\sigma}'(X_r(z), V_r) z dr. \end{aligned}$$

Now, since $z \to X_t(z)$ is a bijection, it is invertible, and we can define

$$\nu_t(y) = \frac{\exp\left\{ -X_t^{-1}(y) B_t - \frac{\left(X_t^{-1}(y)\right)^2 (t+1)}{2} \right\}}{J_t\left(X_t^{-1}(y)\right)}. \tag{6.8}$$

Taking $\varphi = \mathbf{1}_A$, $A \in \mathcal{B}(\mathbb{R})$, in (6.3) and using the change of variable $y = X_t(z)$,

$$P\left[X_t \in A | \mathcal{F}_t^{W,V}\right] = \frac{1}{\sqrt{2\pi}} E\left[\int_A \nu_t(y) dy \middle| \mathcal{F}_t^{W,V} \right] = \frac{1}{\sqrt{2\pi}} \int_A E\left[\nu_t(y) | \mathcal{F}_t^{W,V} \right] dy.$$

Hence, the conditional distribution of X_t given $\mathcal{F}_t^{W,V}$ is absolutely continuous with respect to Lebesgue measure with density

$$\rho_t(y) = \frac{1}{\sqrt{2\pi}} E\left[\nu_t(y) | \mathcal{F}_t^{W,V} \right]. \tag{6.9}$$

Since $\nu_t(y)$ is strictly positive, by Lemma A.7, there exists a version of $\rho_t(y)$ such that with probability one, $\rho_t(y) > 0$ for all $y \in \mathbb{R}$ and $t \geq 0$. *This proves Theorem 1.2* in the case $d = 1$.

Corollary 6.1 *Under Conditions C1, C2, and C3, there exists a random variable $c(s,t,k)$ positive almost surely such that*

$$\inf_{(r,y) \in [s,t] \times [-k,k]} \rho_r(y) \geq c(s,t,k). \tag{6.10}$$

In particular, the set $\widetilde{\Omega} \in \mathcal{F}$ of full measure appearing in the statement of Theorem 1.2 on which π_t^ω is absolutely continuous with respect to Lebesgue measure and the density of π_t^ω with respect to Lebesgue measure is strictly positive can be chosen independent of the time variable $t \in (0, \infty)$.

Proof. Using the independence properties of X_0, B, W, and V, we have

$$E[f(X_0, B) | \mathcal{F}_\infty^{W,V}] = E[f(X_0, B) | V_0],$$

for any reasonable function f. Hence, there exists h_f such that

$$E[f(X_0, B, W_{\cdot \wedge t}, V_{\cdot \wedge t}) | \mathcal{F}_\infty^{W,V}] = h_f(V_0, W_{\cdot \wedge t}, V_{\cdot \wedge t})).$$

Since $\nu_t(y)$ is a function of X_0, B, $W_{\cdot\wedge t}$ and $V_{\cdot\wedge t}$, this implies that

$$\rho_t(y) = \frac{1}{\sqrt{2\pi}} E\left[\nu_t(y) | \mathcal{F}_t^{W,V} \right] = \frac{1}{\sqrt{2\pi}} E\left[\nu_t(y) | \mathcal{F}_\infty^{W,V} \right].$$

Choose m to be an arbitrary positive constant. Since the function $(t,x) \to \min(\nu_t(x), m)$ is bounded, positive and jointly continuous in (t,x) it follows that its conditional expectation

$$\rho_t^m(y) = \frac{1}{\sqrt{2\pi}} E\left[\min(\nu_t(x), m) | \mathcal{F}_\infty^{W,V} \right]$$

has a version which is bounded, positive and jointly continuous in (t,x). Hence, (6.10) holds true with $c(s,t,k) = \inf_{(r,y)\in[s,t]\times[-k,k]} \rho_r^m(y) > 0$. □

Lemma 6.2 *Under Conditions C1, C2, and C3, the density function $y \to \rho_t(y)$ is absolutely continuous. Moreover, it is differentiable almost everywhere and*

$$\frac{d\rho_t}{dy}(y) = \frac{1}{\sqrt{2\pi}} E\left[\frac{d\nu_t}{dy}(y) \middle| \mathcal{F}_t^{W,V} \right]. \tag{6.11}$$

More generally, if f, σ and $\bar{\sigma}$ are m-times continuously differentiable in the first component, then the density function $y \to \rho_t(y)$ is $(m-1)$-times continuously differentiable and m-times differentiable almost everywhere. A similar formula to (6.11) holds for higher derivative of ρ_t as well.

Proof. The function $y \to \nu_t(y)$ is continuously differentiable under Conditions C1, C2, and C3, and

$$\frac{d\nu_t(y)}{dy} = \iota_t^1(x) - \iota_t^2(x), \tag{6.12}$$

where

$$\begin{aligned} \iota_t^1(x) &= -\frac{\exp\left\{ -X_t^{-1}(y) B_t - \frac{\left(X_t^{-1}(y)\right)^2 (t+1)}{2} \right\}}{J_t\left(X_t^{-1}(y)\right)} \frac{B_t + X_t^{-1}(y)(t+1)}{J_t\left(X_t^{-1}(y)\right)} \\ \iota_t^2(x) &= \frac{\exp\left\{ -X_t^{-1}(y) B_t - \frac{\left(X_t^{-1}(y)\right)^2 (t+1)}{2} \right\}}{J_t\left(X_t^{-1}(y)\right)} \frac{\frac{dJ_t}{dx}(X_t^{-1}(y))}{J_t\left(X_t^{-1}(y)\right)^2} \end{aligned}$$

We want to prove that

$$E\left[\int_{\mathbb{R}} \left| \frac{d\nu_t(y)}{dx} \right| dy \right] < \infty.$$

In order to do that, we show that the property holds for both functions on the right hand side of (6.12). We show how this is done for the first function. We have that

$$
\begin{aligned}
E\left[\int_{\mathbb{R}} |\iota_t^1(y)| dy\right] &= E\left[\int_{\mathbb{R}} \frac{\exp\left\{-zB_t - \frac{z^2(t+1)}{2}\right\}}{J_t(z)} |B_t + z(t+1)| dz\right] && (6.13)\\
&\leq \int_{\mathbb{R}} e^{-\frac{z^2(t+1)}{2}} E\left[e^{-pzB_t}\right]^{\frac{1}{p}} E\left[J_t(z)^{-q}\right]^{\frac{1}{q}} E\left[|B_t + z(t+1)|^r\right]^{\frac{1}{r}} dz && (6.14)\\
&\leq \int_{\mathbb{R}} e^{-\frac{z^2((t+1-pt)}{2}} Q_r(|z|)^{\frac{1}{r}} E\left[J_t(z)^{-q}\right]^{\frac{1}{q}} dz, && (6.15)
\end{aligned}
$$

where $p, q, r \in (1, \infty)$ are chosen so that $p < \frac{t+1}{t}$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$ and Q_r is a suitably chosen polynomial so that $E[|B_t + z(t+1)|^r] \leq Q_r(|z|)$ for any $z \in \mathbb{R}$. To get (6.13), we used the change of variable $z = X_t^{-1}(y)$ and applied Hölder's inequality to obtain (6.14). From (6.7) it follows that

$$
J_t \geq tc_{\bar{\sigma}} \exp\left(-tc_{f,\sigma,\bar{\sigma}} - tc'_{\bar{\sigma}}|z| - 2\sup_{s\in[0,t]}|C_s|\right), \tag{6.16}
$$

where C is the martingale

$$
C_s = \int_0^s \sigma'(X_r(z), V_r) dW_r + \int_0^s \bar{\sigma}'(X_r(z), V_r) dB_r, \quad s \in [0, t].
$$

In (6.16) we used the fact that $c_{\bar{\sigma}} \stackrel{\text{def}}{=} \inf_{x,y} \bar{\sigma}(x, y) > 0$ and that

$$
\begin{aligned}
c_{f,\sigma,\bar{\sigma}} &\stackrel{\text{def}}{=} \sup_{x,y} |f'(x, y) - \frac{1}{2}\sigma'(x, y)^2 - \frac{1}{2}\bar{\sigma}'(x, y)^2|\\
c'_{\bar{\sigma}} &\stackrel{\text{def}}{=} \sup_{x,y} |\bar{\sigma}'(x, y)|
\end{aligned}
$$

are finite quantities. This follows from Conditions C1, C2, and C3. Hence, immediately,

$$
E\left[J_t(z)^{-q}\right] \leq ke^{qtc'_{\bar{\sigma}}|z|}, \tag{6.17}
$$

where

$$
k = (tac_{\bar{\sigma}})^{-q} \exp(qtc_{f,\sigma,\bar{\sigma}}) E\left[\exp\left(2q\sup_{s\in[0,t]}|C_s|\right)\right].
$$

Note that k is finite as the running maximum of the martingale C has exponential moments of all orders. From (6.15) and (6.17) we deduce immediately the integrability of ι_t^1. The integrability of ι_t^2 follows in a similar manner as all the terms involved as similar to those appearing in ι_t^1. The only term that is different $\frac{dJ_t}{dz}$. Explicitly $\frac{dJ_t}{dz}$ is given by

$$
\frac{dJ_t}{dz}(z) = \int_0^t \bar{\sigma}(X_s(z), V_s) \exp\left(i_s^t(z)\right)\left(\bar{\sigma}'(X_s(z), V_s) J_s(z) + \frac{di_s^t}{dz}(z)\right)\right) ds,
$$

and one proves in a similar manner that

$$E\left[\left|\frac{dJ_t}{dz}\right|\right] \le k' e^{k''|z|}, \tag{6.18}$$

where k' and k'' are some suitably chosen constants. It follows that

$$\begin{aligned} \rho_t\left(y^1\right) - \rho_t\left(y^2\right) &= \frac{1}{\sqrt{2\pi}} E\left[\nu_t\left(y^1\right) - \nu_t\left(y^2\right) \middle| \mathcal{F}_t^{W,V}\right] \\ &= \frac{1}{\sqrt{2\pi}} E\left[\int_{y_2}^{y^1} \frac{d\nu_t}{dy}(y)\,dy \middle| \mathcal{F}_t^{W,V}\right] \\ &= \int_{y_2}^{y^1} \frac{1}{\sqrt{2\pi}} E\left[\frac{d\nu_t}{dy}(y)dy \middle| \mathcal{F}_t^{W,V}\right] dy \end{aligned} \tag{6.19}$$

and we deduce from the above the absolute continuity of ρ_t and, therefore, its differentiability almost everywhere. We note that the last identity follows by the (conditional) Fubini's theorem as we have proved the integrability of $\frac{d\nu_t}{dy}$ over the product space $\Omega \times \mathbb{R}$. The methodology to show that ρ_t has higher derivatives is similar. Observe first that

$$\begin{aligned} \frac{d^m \nu_t(y)}{dy^m} = & \quad \frac{\exp\left\{-X_t^{-1}(y)\,B_t - \frac{\left(X_t^{-1}(y)\right)^2(t+1)}{2}\right\}}{J_t\left(X_t^{-1}(y)\right)} \\ & \times T(t, B_t, X_t^{-1}(y), \frac{dX_t}{dx}\left(X_t^{-1}(y)\right), \ldots \frac{dX_t^m}{dx^m}\left(X_t^{-1}(y)\right))) \end{aligned}$$

where $T(t, B_t, X_t^{-1}(y), \frac{dX_t}{dx}\left(X_t^{-1}(y)\right), \ldots \frac{dX_t^m}{dx^m}\left(X_t^{-1}(y)\right)))$ is a random variable which has moments of all order controlled by an upper bound of the type (6.18). One then shows the integrability of $\frac{d^m\nu_t(y)}{dy^m}$ over the product space $\Omega \times \mathbb{R}$ which implies the m-times differentiability of ρ_t. □

Lemma 6.3 *If in addition to Conditions C1, C2 and C3, the coefficients f, σ and $\bar{\sigma}$ are bounded, then there exists a constant $a = a(t)$ independent of z and a positive random variable c_t such that almost surely*

$$\rho_t(z) \le c_t e^{-az^2}, \qquad \forall z \in \mathbb{R}. \tag{6.20}$$

Proof. It suffices to show that $E[\sup_{z\in\mathbb{R}} \rho_t(z) e^{az^2}] < \infty$. This inequality is satisfied provided $E[\sup_{y\in\mathbb{R}} \nu_t(y) e^{ay^2}] < \infty$, which, substituting $y = X_t(z)$ (see (6.8)), is satisfied if $E[\sup_{z\in\mathbb{R}} \nu_t((X_t(z))) e^{a(X_t(z))^2}] < \infty$. Moreover the latter is satisfied if

$$E\left[\int_{\mathbb{R}} \left|\frac{d}{dz}\left(\nu_t((X_t(z)) e^{a(X_t(z))^2}\right)\right| dz\right] = E\left[\int_{\mathbb{R}} (q_1(z) + q_2(z)) dz\right] < \infty$$

where

$$q_1(z) = 2a|X_t(z)| \exp\left\{-zB_t - \frac{z^2\,(t+1)}{2}\right\} e^{a(X_t(z))^2}$$

$$q_2(z) = \left|\frac{d}{dz}\left(\nu_t((X_t(z)))\right|e^{a(X_t(z))^2}.$$

Since the coefficients f, σ and $\bar{\sigma}$ are bounded, by a standard argument one can easily show that there exists a positive constant b_t^1 such that for any $0 < b < b_t^1$

$$\sup_{z\in\mathbb{R}} e^{-kbz^2} E[e^{b(X_t(z))^2}] < \infty.$$

where $k = 2\|\bar{\sigma}\|_\infty^2 t^2$. The proof then follows similar to that of Lemma 6.2 . □

6.2 The multidimensional case

For $X(z)$ given by

$$dX_t(z) = f\left(X_t\left(z\right), V_t\right) dt + \sigma\left(X_t\left(z\right), V_t\right) dW_t + \bar{\sigma}\left(X_t\left(z\right), V_t\right) dB_t + F\left(t, z\right) dt, \qquad (6.21)$$

define

$$\left.\frac{dP^z}{dP}\right|_{\mathcal{F}_t} = M_t(z) \equiv \exp\left(-\int_0^t \bar{\sigma}^{-1}(X_s(z), V_s)F\left(s, z\right)^\top dB_s - \frac{1}{2}\int_0^t |\bar{\sigma}^{-1}(X_s(z), V_s)F\left(s, z\right)|^2 ds\right),$$

where $F\left(s, z\right)^\top$ is the row vector $(F\left(s, z\right)_1, F\left(s, z\right)_2, ..., F\left(s, z\right)_d)$. Then $M(z)$ is a martingale under the filtration $\mathcal{G}_t = \mathcal{F}_t^B \vee \sigma(W, V)$ and by Girsanov's theorem, the process $B^z = \{B_t^z, t \geq 0\}$

$$B_t^z = B_t + \int_0^t \bar{\sigma}^{-1}(X_s(z), V_s)F\left(s, z\right) ds$$

is a Brownian motion under P^z with respect to the filtration $\{\mathcal{G}_t\}$. Consequently, (B^z, W, V) has the same law under P^z as (B, W, V) has under P, and it follows that $X(z)$ given by

$$\begin{aligned} dX_t(z) &= f\left(X_t\left(z\right), V_t\right) dt + \sigma\left(X_t\left(z\right), V_t\right) dW_t + \bar{\sigma}\left(X_t\left(z\right), V_t\right) dB_t^z \qquad (6.22) \\ &= f\left(X_t\left(z\right), V_t\right) dt + \sigma\left(X_t\left(z\right), V_t\right) dW_t + \bar{\sigma}\left(X_t\left(z\right), V_t\right) dB_t + F\left(t, z\right) dt \end{aligned}$$

has the same law under P^z as X has under P. As before, for $\varphi \in B(\mathbb{R}^d)$,

$$\begin{aligned} E\left[\varphi\left(X_t\right)|\mathcal{F}_t^{W,V}\right] &= E^z\left[\varphi\left(X_t\left(z\right)\right)|\,\mathcal{F}_t^{W,V}\right] \\ &= E\left[\varphi\left(X_t\left(z\right)\right) M_t\left(z\right)|\mathcal{F}_t^{W,V}\right] \end{aligned}$$

since $E[M_t(z)|\sigma(W, V)] = 1$. For $\alpha \geq n^{-1}$, $n = 1, 2, \ldots$, define

$$F^{\alpha,n}\left(s, z\right) = n\mathbf{1}_{[\alpha-\frac{1}{n},\alpha]}(s)z. \qquad (6.23)$$

Let $X^{\alpha,n}(z) = \{X_t^{\alpha,n}(z), t \geq 0\}$ be the solution of (6.1) with F replaced by $F^{\alpha,n}$. Then $X_\alpha^n(z) \to X_\alpha + z$ almost surely, where the convergence will be uniform for z in compact sets, and

$$\begin{aligned}\pi_\alpha(A) &= P\{X_\alpha \in A | \mathcal{F}_t^{W,V}\} \\ &= \sum_{n=1}^{\infty} 2^{-n} \int_{\mathbb{R}^d} E[\mathbf{1}_{\{X_\alpha^n(z)\in A\}} M_\alpha^n(z) | \mathcal{F}_t^{W,V}]\theta(z)dz,\end{aligned}$$

where θ is a probability density that is strictly positive on $\mathbb{R}$. If A is open then

$$\sum_{n=1}^{\infty} 2^{-n} \mathbf{1}_{\{X_\alpha^n(z)\in A\}} M_\alpha^n(z) > 0$$

on $\{(z,\omega) : X_\alpha(\omega) + z \in A, \lim_{n\to\infty} X_\alpha^n = X_\alpha + z\}$. Since the limit holds almost surely, Theorem 1.1 follows.

Next, let $J_t^{\alpha,n}(z)$ be the Jacobian of $z \to X^{\alpha,n}(z)$

$$\left(J_t^{\alpha,n}(z)\right)_{ij} = \partial_j \left(X_t^{\alpha,n}\right)_i (z).$$

Then $J^{\alpha,n}(z) = \{J_t^{\alpha,n}(z), t \geq 0\}$ is zero for $t \leq \alpha - n^{-1}$, and for $t \geq \alpha - n^{-1}$, $J^{\alpha,n}$ satisfies the following stochastic differential equation

$$\begin{aligned}J_t^{\alpha,n}(z) &= \int_{\alpha-\frac{1}{n}}^{t} f'\left(X_s^{\alpha,n}(z), V_s\right) J_s^{\alpha,n}(z)\, ds + \sum_{i=1}^{d'} \int_{\alpha-\frac{1}{n}}^{t} \sigma_i'\left(X_s^{\alpha,n}(z), V_s\right) J_s^{\alpha,n}(z)\, dW_s^i \\ &\quad + \sum_{i=1}^{d} \int_{\alpha-\frac{1}{n}}^{t} \bar{\sigma}_i'\left(X_s^{\alpha,n}(z), V_s\right) J_s^{\alpha,n}(z)\, dB_s^i + (1 - n(\alpha - t)) I_d, \qquad (6.24)\end{aligned}$$

where $f' : \mathbb{R}^d \times E \to \mathbb{R}^{d\times d}$ is the matrix-valued function defined as

$$\left(f'(x,v)\right)_{ij} \stackrel{\text{def}}{=} \frac{\partial_j f(x,v)_i}{\partial x_j}$$

and σ_i', $i = 1, ..., d'$ $\bar{\sigma}_i'$, $i = 1, ..., d$ are functions defined in the same manner (σ_i, $i = 1, ..., d'$ $\bar{\sigma}_i$, $i = 1, ..., d$ are the column vectors of σ, respectively $\bar{\sigma}$, $\sigma = (\sigma_1, \sigma_2, ..., \sigma_{d'})$, $\bar{\sigma} = (\bar{\sigma}_1, \bar{\sigma}_2, ..., \bar{\sigma}_d)$) and I_d is the d-dimensional identity matrix. Let $\Phi^{\alpha,n}(z) = \{\Phi_t^{\alpha,n}(z), t \geq 0\}$ and $\Upsilon^{\alpha,n}(z) = \{\Upsilon_t^{\alpha,n}(z), t \geq 0\}$ be the solutions of the following matrix stochastic differential equations

$$\begin{aligned}\Phi_t^{\alpha,n}(z) &= I_d + \int_{(\alpha-\frac{1}{n})\wedge t}^{t} f'\left(z, X_s^{\alpha,n}(z), V_s\right) \Phi_s^{\alpha,n}(z)\, ds \\ &\quad + \sum_{i=1}^{d'} \int_{(\alpha-\frac{1}{n})\wedge t}^{t} \sigma_i'\left(X_s^{\alpha,n}(z), V_s\right) \Phi_s^{\alpha,n}(z)\, dW_s^i \\ &\quad + \sum_{i=1}^{d} \int_{(\alpha-\frac{1}{n})\wedge t}^{t} \bar{\sigma}_i'\left(X_s^{\alpha,n}(z), V_s\right) \Phi_s^{\alpha,n}(z)\, dB_s^i,\end{aligned}$$

$$\begin{aligned}
\Upsilon_t^{\alpha,n}(z) &= I_d - \int_{(\alpha-\frac{1}{n})\wedge t}^{t} \Upsilon_s^{\alpha,n}(z)\,\kappa\left(z, X_s^{\alpha,n}(z), V_s\right) ds \\
&\quad - \sum_{i=1}^{d'} \int_{(\alpha-\frac{1}{n})\wedge t}^{t} \Upsilon_s^{\alpha,n}(z)\,\sigma_i'\left(X_s^{\alpha,n}(z), V_s\right) dW_s^i \\
&\quad - \sum_{i=1}^{d} \int_{(\alpha-\frac{1}{n})\wedge t}^{t} \Upsilon_s^{\alpha,n}(z)\,\bar{\sigma}_i'\left(X_s^{\alpha,n}(z), V_s\right) dB_s^i,
\end{aligned}$$

where

$$\kappa\left(z, X_s^{\alpha,n}(z), V_s\right) = f'\left(z, X_s^{\alpha,n}(z), V_s\right) - \sum_{i=1}^{d'} \sigma_i'\left(X_t^{\alpha,n}(z), V_t\right)^2 - \sum_{i=1}^{d} \bar{\sigma}_i'\left(X_t^{\alpha,n}(z), V_t\right)^2.$$

It is easy to check that

$$d\left(\Upsilon_t^{\alpha,n}(z)\,\Phi_t^{\alpha,n}(z)\right) = 0,$$

and since $\Upsilon_0^{\alpha,n}(z)\,\Phi_0^{\alpha,n}(z) = I$, it follows that $\Upsilon_t^{\alpha,n}(z)\,\Phi_t^{\alpha,n}(z) = I$, for all $t \geq 0$, i.e., $\Phi_t^{\alpha,n}(z)$ and $\Upsilon_t^{\alpha,n}(z)$ are non-singular and inverse to each other. Then, for $t \in [\alpha - \frac{1}{n}, \alpha]$ we can write the solution of (6.24) explicitly as

$$\begin{aligned}
J_t^{\alpha,n}(z) &= n\Phi_t^{\alpha,n}(z) \int_{\alpha-\frac{1}{n}}^{t} \Upsilon_s^{\alpha,n}(z)\, ds \\
&= I_d + n\Phi_t^{\alpha,n}(z) \int_{\alpha-\frac{1}{n}}^{t} (\Upsilon_s^{\alpha,n}(z) - \Upsilon_t^{\alpha,n}(z)) ds.
\end{aligned}$$

Since $\Upsilon_s^{\alpha,n}(z)$ is jointly continuous in s and z, we have that, almost surely, for each compact $K \subset \mathbb{R}^d$,

$$\lim_{n\to\infty} \sup_{z\in K} \left| n\Phi_\alpha^{\alpha,n}(z) \int_{\alpha-\frac{1}{n}}^{\alpha} (\Upsilon_s^{\alpha,n}(z) - \Upsilon_t^{\alpha,n}(z)) ds \right| = 0.$$

Hence, almost surely, $\lim_{n\to\infty} \sup_{z\in K} |J_\alpha^{\alpha,n}(z) - I| \to 0.$[5]

Let

$$\Gamma^{\alpha,n} = \{(z,\omega) : \det(J_\alpha^{\alpha,n}(z)) \neq 0\}.$$

Then there exists a partition $\Gamma^{\alpha,n} = \bigcup_k \Gamma_k^{\alpha,n}$ and processes $Y_\alpha^{n,k}(y)$, $y \in \mathbb{R}^d$ such that $X_\alpha^n(Y_\alpha^{n,k}(y)) = y$ for $(Y_\alpha^{n,k}(y), \omega) \in \Gamma_k^{\alpha,n}$ and $Y_\alpha^{n,k}(X_\alpha^n(z)) = z$ for $(z,\omega) \in \Gamma_k^{\alpha,n}$. For each n, we have

$$\begin{aligned}
\pi_\alpha(A) &= P\{X_\alpha \in A | \mathcal{F}_\alpha^{W,V}\} \\
&= E\left[\int_{\mathbb{R}^d} \mathbf{1}_{\{X_\alpha^n(z)\in A\}} M_\alpha^n(z)\theta(z)dz | \mathcal{F}_\alpha^{W,V}\right]
\end{aligned}$$

[5]We also have that $\lim_{n\to\infty} \sup_{z\in K} |X_\alpha^n(z) - (X_\alpha + z)| \to 0$.

$$\begin{aligned} &\geq E\left[\int_{\mathbb{R}^d} \mathbf{1}_{\Gamma^{\alpha,n}}(z,\cdot)\mathbf{1}_{\{X_\alpha^n(z)\in A\}} M_\alpha^n(z)\theta(z)dz|\mathcal{F}_\alpha^{W,V}\right] \\ &= \int_{\mathbb{R}^d}\sum_k E\left[\mathbf{1}_{\Gamma_k^{\alpha,n}}(Y_\alpha^{n,k}(y),\cdot)\mathbf{1}_{\{y\in A\}}\frac{M_\alpha^n(Y_\alpha^{n,k}(y))\theta(Y_\alpha^{n,k}(y)}{\det(J_\alpha^{\alpha,n}(Y_\alpha^{n,k}(y)))}|\mathcal{F}_\alpha^{W,V}\right]dy. \end{aligned}$$

Let

$$r^{n,\alpha}(y) = E\left[\sum_k \mathbf{1}_{\Gamma_k^{\alpha,n}}(Y_\alpha^{n,k}(y),\cdot)\frac{M_\alpha^n(Y_\alpha^{n,k}(y))\theta(Y_\alpha^{n,k}(y))}{\det(J_\alpha^{\alpha,n}(Y_\alpha^{n,k}(y)))}|\mathcal{F}_\alpha^{W,V}\right].$$

There exists a version of $r^{n,\alpha}$ such that

$$\pi_\alpha(A) \geq \int_A r^{n,\alpha}(y)dy$$

for all Borel sets A. Then

$$\pi_\alpha(A) = \int_A \max_n r^{n,\alpha}(y)dy. \tag{6.25}$$

for all Borel sets A. To see this observe first that

$$\pi_\alpha(A) \geq \int_A \max_n r^{n,\alpha}(y)dy.$$

Then, it is enough to show that

$$E\left[\int_{\mathbb{R}^d}\max_n r^{n,\alpha}(y)dy\right] = 1. \tag{6.26}$$

But this expectation is greater than or equal to

$$E\left[\int_{\mathbb{R}^d}\mathbf{1}_{\Gamma^{\alpha,n}}(z,\cdot)M_\alpha^n(z)\theta(z)dz\right] = \int_{\mathbb{R}^d} P^z\{\det(J_\alpha^{\alpha,n}(z)) \neq 0\}\theta(z)dz. \tag{6.27}$$

For each z and $\epsilon > 0$, $\lim_{n\to\infty} P^z\{|J_\alpha^{\alpha,n}(z) - I| < \epsilon\} = 1$ by essentially the same argument as for $z = 0$. Consequently, (6.27) converges to 1 giving (6.26).

To see that the density r^α can be taken to be strictly positive, note that

$$r^\alpha(y) \geq E\left[\sum_{n=1}^\infty 2^{-n}\sum_k \mathbf{1}_{\Gamma_k^{\alpha,n}}(Y_\alpha^{n,k}(y),\cdot)\frac{M_\alpha^n(Y_\alpha^{n,k}(y))\theta(Y_\alpha^{n,k}(y))}{\det(J_\alpha^{\alpha,n}(Y_\alpha^{n,k}(y)))}|\mathcal{F}_\alpha^{W,V}\right]$$

and that for almost every ω and for each y and $\epsilon > 0$, there exists n such that

$$B_\epsilon(y) \subset \{u : \exists z \text{ such that } X_\alpha^n(z,\omega) = u \text{ and } \det(J_\alpha^{\alpha,n}(z,\omega)) \neq 0\}.$$

Consequently, the sum inside the conditional expectation is almost surely strictly positive, and hence, the conditional expectation can be taken to be strictly positive. *This proves Theorem 1.2 for the multidimensional case.*

A Appendix

A.1 Convergence of quantiles

For $0 < \alpha < 1$, and for $\mu \in \mathcal{P}(\mathbb{R})$, define $q_\alpha(\mu) = \inf\{x : \mu(-\infty, x] \geq \alpha\}$. Note that μ is a point of continuity for q_α if and only if $\mu(q_\alpha(\mu), q_\alpha(\mu) + \epsilon) > 0$ and $\mu(q_\alpha(\mu) - \epsilon, q_\alpha(\mu)) > 0$ for every $\epsilon > 0$.

Lemma A.1 *Let $\{Y_n\}$ be a sequence of $\mathcal{P}(\mathbb{R})$-valued random variables such that $Y_n \Rightarrow Y$. Suppose that with probability 1, the measure Y charges every open set. Then $q_\alpha(Y_n) \Rightarrow q_\alpha(Y)$ for each $0 < \alpha < 1$.*

Proof. The lemma follows by the continuous mapping theorem. □

Lemma A.2 *Suppose $z \in D_{\mathcal{P}(\mathbb{R})}[0, \infty)$ and for each $t \geq 0$, $z(t)$ and $z(t-)$ charge every open set. Then if $0 < \alpha < 1$ and $z_n \to z$ in $D_{\mathcal{P}(\mathbb{R})}[0, \infty)$, $q_\alpha(z_n) \to q_\alpha(z)$ in $D_{\mathbb{R}}[0, \infty)$.*

Proof. The lemma follows by Proposition 3.6.5 of Ethier and Kurtz [8] and the continuity properties of q_α. □

The continuous mapping theorem gives the following:

Lemma A.3 *Suppose $\{Z_n\}$ is a sequence of processes in $D_{\mathcal{P}(\mathbb{R})}[0, \infty)$ such that $Z_n \Rightarrow Z$. If, with probability 1, $Z(t)$ and $Z(t-)$ charge every open set for all t, then for $0 < \alpha < 1$, $q_\alpha(Z_n) \Rightarrow q_\alpha(Z)$.*

A.2 Convergence of random measures

The following results are from Kurtz [17]. Let $\mathcal{L}(S)$ be the space of measures μ on $[0, \infty) \times S$ such that $\mu([0, t] \times S) < \infty$ for each $t > 0$, and let $\mathcal{L}_m(S) \subset \mathcal{L}(S)$ be the subspace on which $\mu([0, t] \times S) = t$. For $\mu \in \mathcal{L}(S)$, let μ^t denote the restriction of μ to $[0, t] \times S$. Let ρ_t denote the Prohorov metric on $\mathcal{M}([0, t] \times S)$, and define the metric $\widehat{\rho}$ on $\mathcal{L}(S)$ by

$$\widehat{\rho}(\mu, \nu) = \int_0^\infty e^{-t} 1 \wedge \rho_t(\mu^t, \nu^t) dt,$$

that is, $\{\mu_n\}$ converges in $\widehat{\rho}$ if and only if $\{\mu_n^t\}$ converges weakly for almost every t.

Lemma A.4 *A sequence of $(\mathcal{L}_m(S), \widehat{\rho})$-valued random variables $\{\Gamma_n\}$ is relatively compact if and only if for each $\epsilon > 0$ and each $t > 0$, there exists a compact $K \subset S$ such that $\inf_n E[\Gamma_n([0, t] \times K)] \geq (1 - \epsilon)t$.*

Lemma A.5 *Let* $\{(x_n,\mu_n)\} \subset D_E[0,\infty)\times\mathcal{L}(S)$*, and* $(x_n,\mu_n)\to(x,\mu)$*. Let* $h\in\bar{C}(E\times S)$*. Define*

$$u_n(t)=\int_{[0,t]\times S} h(x_n(s),y)\mu_n(ds\times dy), \quad u(t)=\int_{[0,t]\times S} h(x(s),y)\mu(ds\times dy)$$

$z_n(t)=\mu_n([0,t]\times S)$*, and* $z(t)=\mu([0,t]\times S)$*.*

a) If x *is continuous on* $[0,t]$ *and* $\lim_{n\to\infty} z_n(t)=z(t)$*, then* $\lim_{n\to\infty} u_n(t)=u(t)$*.*

b) If $(x_n,z_n,\mu_n)\to(x,z,\mu)$ *in* $D_{E\times\mathbb{R}}[0,\infty)\times\mathcal{L}(S)$*, then* $(x_n,z_n,u_n,\mu_n)\to(x,z,u,\mu)$ *in* $D_{E\times\mathbb{R}\times\mathbb{R}}[0,\infty)\times\mathcal{L}(S)$*. In particular,* $\lim_{n\to\infty} u_n(t)=u(t)$ *at all points of continuity of* z*.*

c) The continuity assumption on h *can be replaced by the assumption that* h *is continuous a.e.* ν_t *for each* t*, where* $\nu_t\in\mathcal{M}(E\times S)$ *is the measure determined by* $\nu_t(A\times B)=\mu\{(s,y): x(s)\in A, s\le t, y\in B\}$*.*

d) In both (a) and (b), the boundedness assumption on h *can be replaced by the assumption that there exists a nonnegative convex function* ψ *on* $[0,\infty)$ *satisfying* $\lim_{r\to\infty}\psi(r)/r=\infty$ *such that*

$$\sup_n \int_{[0,t]\times S} \psi(|h(x_n(s),y)|)\mu_n(ds\times dy)<\infty \tag{A.1}$$

for each $t>0$*.*

A.3 Measurability and positivity of random functions given by conditional expectations

Lemma A.6 *Let* $(\Omega,\mathcal{F},P)$ *be a complete probability space,* E *a complete, separable metric space, and* $\{\mathcal{F}_x, x\in E\}$ *a collection of complete sub-*σ*-algebras of* $\mathcal{F}$*. Suppose that for each* $A\in\mathcal{F}$*, there exists a* $\mathcal{B}(E)\times\mathcal{F}$ *measurable process* X_A *indexed by* E *such that for each* x*,*

$$P(A|\mathcal{F}_x)=X_A(x) \quad a.s.$$

Then for each bounded, $\mathcal{B}(E)\times\mathcal{F}$*-measurable process* Y *there exists another* $\mathcal{B}(E)\times\mathcal{F}$*-measurable process* $\widehat{Y}$ *such that*

$$E[Y(x)|\mathcal{F}_x]=\widehat{Y}(x) \quad a.s.$$

Proof. If $Y(x)=\mathbf{1}_B(x)\mathbf{1}_A$ for $B\in\mathcal{B}(E)$ and $A\in\mathcal{F}$, then $\widehat{Y}(x)=\mathbf{1}_B(x)X_A(x)$ satisfies the requirements of the lemma. Since $\{B\times A: B\in\mathcal{B}(E), A\in\mathcal{F}\}$ is closed under intersections and generates $\mathcal{B}(E)\times\mathcal{F}$ and the collection of Y for which the conclusion of the lemma holds is closed under bounded monotone increasing limits, the lemma follows by the monotone class theorem for functions. (See Theorem 4.3 in the Appendix of Ethier and Kurtz [8].) □

Lemma A.7 *Suppose that the conclusion of Lemma A.6 holds and that Y is $\mathcal{B}(E) \times \mathcal{F}$-measurable and strictly positive. Then $\widehat{Y}$ can be taken to be strictly positive.*

Proof. Let $A_0 = \{(x,\omega) : Y(x,\omega) \geq 1\}$ and $A_n = \{(x,\omega) : 2^{-n} \leq Y(x,\omega) < 2^{-(n-1)}\}$, $n = 1, 2, \ldots$. Then $\cup_{n=0}^{\infty} A_n = E \times \Omega$, and we can assume that $E[\mathbf{1}_{A_n}|\mathcal{F}_x] \geq 0$ for all (x,ω). Note that

$$1 = \lim_{n\to\infty} \sum_{k=0}^{n} E[\mathbf{1}_{A_k}|\mathcal{F}_x] \quad a.s.$$

for all x. If necessary, we can replace $E[\mathbf{1}_{A_n}|\mathcal{F}_x]$ by

$$1 \wedge \sum_{k=0}^{n} E[\mathbf{1}_{A_k}|\mathcal{F}_x] - 1 \wedge \sum_{k=0}^{n-1} E[\mathbf{1}_{A_k}|\mathcal{F}_x]$$

to ensure $\sum_{k=0}^{\infty} E[\mathbf{1}_{A_k}|\mathcal{F}_x] \leq 1$ and then replace $E[\mathbf{1}_{A_0}|\mathcal{F}_x]$ by

$$1 - \sum_{k=1}^{\infty} E[\mathbf{1}_{A_k}|\mathcal{F}_x]$$

to ensure $\sum_{k=0}^{\infty} E[\mathbf{1}_{A_k}|\mathcal{F}_x] = 1$ for all (x,ω). Then

$$\sum_{n=0}^{\infty} 2^{-n} E[\mathbf{1}_{A_n}|\mathcal{F}_x] \leq \widehat{Y}(x) \quad a.s.$$

and we can replace $\widehat{Y}(x)$ by $\widehat{Y}(x) \vee \sum_{n=0}^{\infty} 2^{-n} E[\mathbf{1}_{A_n}|\mathcal{F}_x]$ to be assured that $\widehat{Y}(x) > 0$ for all (x,ω). □

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