DIFFUSION APPROXIMATION OF TRANSPORT PROCESSES WITH GENERAL REFLECTING BOUNDARY CONDITIONS

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Abstract. Diffusion approximations are obtained for time-homogeneous linear transport models with reflecting boundary conditions. The collision kernel is not required to satisfy any balance condition and the scattering kernel on the boundary is general enough to include all examples of boundary conditions known to the authors (with conservation of the number of particles) and, in addition, to model the Debye sheath. The mathematical approach does not rely on Hilbert expansions, but rather on martingale and stochastic averaging techniques.

Key words. diffusion approximation, boundary conditions, linear transport, parabolic hydrodynamic limit, Debye sheath, stochastic averaging

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1. Introduction. Linear transport models are used in many areas of physics as microscopic models for particle behavior. Such models arise, for instance, in neutron transport (for an example of recent work, see Allaire and Bal (1999)), in electron transport in a plasma (Degond (1998) and Degond, Latocha, Garrigues, and Boeuf (1998)), in the study of semiconductors (Ben Abdallah and Degond (1996)), in medical imaging (Arridge and Hebden (1997)), in the study of the propagation of high frequency waves (Van Rossum and Nieuwenhuizen (1998), Bal and Ryzhik (2000)). In some instances, the particles are confined in a region with boundary and the behavior on the boundary is described by an appropriately defined scattering kernel. Approximating models obtained under various types of scaling are of interest.

The linearity of the model reflects an assumption that interactions among the particles whose behavior is being modeled can be neglected. Consequently, the model can be described either by an equation for the evolution of the distribution of particles over the phase space or by a Markov process modeling the position and the velocity (or the wave vector) of a typical particle. Here we take the latter point of view and consider a position and velocity process determined by a general collision operator \( Q \) in the interior of a domain \( D \), and by a quite general scattering kernel \( \eta_b \) on the boundary of \( D \). Under appropriate scaling, we obtain a diffusion approximation which, in terms of the particle distribution function, gives the parabolic hydrodynamic limit. The collision operator \( Q \) is not required to satisfy any balance condition. The scattering kernel \( \eta_b \) is general enough to include the classical Maxwell and Cercignani-Lampis boundary conditions (see, for example, Cercignani et al. (1994)), all the boundary conditions considered in the above mentioned works (with conservation of the number of particles), and to model the Debye sheath (a phenomenon that occurs in plasmas; see Chen (1974)).

More precisely, we construct a (rescaled) process \( (X^\varepsilon, V^\varepsilon) \), where \( X^\varepsilon \) is the location of the particle, and the velocity is given by \( \dot{X}^\varepsilon = \varepsilon^{-1}V^\varepsilon \). In the interior of the domain \( D \), the process evolves according to a generator of the form

\[
L^\varepsilon f(x, v) = \varepsilon^{-1}(v, \nabla_x f(x, v)) + \varepsilon^{-2}Qf(x, v)
\]

\[
Qf(x, v) = \mu(x, v) \int [f(x, v') - f(x, v)]\eta(x, v, dv'),
\]

and when the process hits a point \( x \in \partial D \) with velocity \( \varepsilon^{-1}v \), \( V^\varepsilon \) jumps instantaneously from \( v \) to a new random value distributed according to the scattering kernel \( \eta_b(x, v, \cdot) \). We suppose that \( D \) is bounded with \( C^2 \) boundary. Except in Section 2, we require that \( V^\varepsilon \) takes values in a closed ball, \( \mathcal{V} \subset \mathbb{R}^d \), centered at the origin. We define \( O = D \times \mathcal{V}, \partial_+ O = \{(x, v) \in \partial D \times \mathcal{V} : \langle v, \nu(x) \rangle \leq 0\} \) and \( \partial_- O = \{(x, v) \in \partial D \times \mathcal{V} : \langle v, \nu(x) \rangle > 0\} \).

We assume that the scattering kernel \( \eta_b(x, v, \cdot), (x, v) \in \partial_- O \), is of the form

\[
\eta_b(x, v, \cdot) \equiv \alpha(x, v)\eta^C_b(x, v, \cdot) + [1 - \alpha(x, v)]\eta^H_b(x, v, \cdot),
\]

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where \( 0 \leq \alpha \leq 1 \), \( \eta^C_0 \) satisfies conditions \((A_C)\) below, and \( \eta^H_0 \) satisfies conditions \((A_H)\). For some models, \( \alpha(x,v) \) can be interpreted as being the accomodation coefficient (in the terminology of Cercignani et. al. (1994), Chapter 8), and we will use this terminology.

The kernel \( \eta^C_0 \) generalizes specular reflection: In fact, we assume:

\((A_C)\) For \((x,v) \in \partial O\),

\[
\eta^C_0(x,v,\cdot) \equiv P(v + \rho(x,v)\zeta_{x,v} \in \cdot),
\]

\[
\rho(x,v) \geq |\langle v, \nu(x) \rangle|, \quad \inf_{(x,v) \in \partial O} \rho(x,v) = 0,
\]

where \( \rho : \partial O \to \mathbb{R}_+ \) is a continuous function and \( \{\zeta_{x,v} : (x,v) \in \partial O\} \) is a uniformly integrable family of \( \mathbb{R}^d \)-valued random variables such that the mapping \((x,v) \in \partial O \to P(\zeta_{x,v} \in \cdot) \in \mathcal{P}(\mathbb{R}^d)\) is continuous, \( \eta^C_0(x,v,\nu) = 1 \), and

\[
(\zeta_{x,v}, \nu(x)) > 1, \quad \text{a.s.} \quad \forall (x,v) \in \partial O.
\]

Specular reflection corresponds to

\[
(1.3) \quad \rho(x,v) = |\langle v, \nu(x) \rangle|,
\]

\[
(1.4) \quad \zeta_{x,v} = 2\nu(x).
\]

Note that if \( \alpha(x,v) \equiv 1 \) and \((1.3)\) holds, then the change in the velocity on the boundary is \( O(|\langle v, \nu(x) \rangle|) \) and, in particular, a particle with low energy in the normal direction does not gain significant energy. For this reason we think of the kernel \( \eta^C_0 \) as describing a cool or smooth boundary.

For \( \eta^H_0 \) we assume instead:

\((A_H)\) The mapping \((x,v) \in \partial O \to \eta^H_0(x,v,\cdot) \in \mathcal{P}(\mathbb{R}^d)\) is continuous and

\[
\eta^H_0(x,v,\{v' : \langle v', \nu(x) \rangle > 0\}) = 1, \quad \forall (x,v) \in \partial O.
\]

In particular \((A_H)\) implies that for \( \epsilon > 0 \) there exists \( \delta_\epsilon > 0 \) such that

\[
(1.5) \quad \inf_{(x,v) \in \partial O} \eta^H_0(x,v,\{v' : \langle v', \nu(x) \rangle \geq \delta_\epsilon\}) \geq 1 - \epsilon.
\]

Therefore under the action of the kernel \( \eta^H_0 \), a particle hitting the boundary with low energy in the normal direction may gain significant energy. For this reason we think of \( \eta^H_0 \) as describing a hot or rough boundary.

We make the following assumptions on the accomodation coefficient \( \alpha \):

\((A_M)\) \( \alpha : \partial O \to [0,1] \) is a continuous function and either

\[
(1.6) \quad \sup_{(x,v) \in \partial O : \rho(x,v) = 0} \alpha(x,v) < 1,
\]

or

\[
(1.7) \quad \theta(x,v) \equiv \frac{1 - \alpha(x,v)}{\rho(x,v)}, \quad (x,v) \in \{(x,v) \in \partial O : \rho(x,v) > 0\}
\]

extends to a continous function on all of \( \partial O \).

Notice that the second alternative in \((A_M)\) is satisfied if, for instance, \( \alpha \) is of the form \( \alpha(x,v) = \alpha_0(\rho(x,v)) \), with \( \alpha_0(0) = 1 \) and \( \alpha_0 \) differentiable at 0. Notice also that, for boundary conditions of the form \((1.2)-(A_C)-(A_H)\), \((1.6)\) is equivalent to the assumption that the whole kernel \( \eta_0 \) satisfies \((1.5)\).

Due to the general form of the boundary conditions, the construction of the process \((X^\epsilon, V^\epsilon)\) is not obvious and is carried out in Section 3.
The parameter $\epsilon$ in (1.1) is related to the mean free path. The rescaling in (1.1) (sometimes referred to as parabolic hydrodynamic rescaling) corresponds to observing the motion on a space scale $\epsilon^{-1}$ times coarser than the microscopic one and over time intervals with length of order $\epsilon^{-2}$. As $\epsilon$ goes to zero, both $X'$ and $V'$ vary faster and faster, but while the speed of $X'$ is of order $\epsilon^{-1}$, changes in the velocity $V'$ occur at rate $\epsilon^{-2}$.

The fundamental idea of the approximation is that, for $\epsilon$ small, $V'$ should reach a local equilibrium with mean zero during time periods in which $X'$ moves very little. For example, in the time interval $[t, t + \epsilon^{3/2}]$, the distance $X'$ can move is order $\epsilon^{1/2}$ while the number of jumps taken by $V'$ is order $\epsilon^{-1/2}$. To study the local behavior of $V'$, for each $x \in D$, we define the operator $Q^x$ by

$$Q^x f(v) = \mu(x,v) \int \left( (f(v') - f(v)) \eta(x,v,dv') \right), \ f \in C_c(\mathbb{R}^d),$$

where, as usual, $C_c(\mathbb{R}^d)$ denotes the space of continuous functions on $\mathbb{R}^d$ with compact support.

Under this rescaling, we expect $X'$ to converge, as $\epsilon$ goes to 0, to a diffusion process. We can obtain the diffusion approximation of $X'$ in the interior of $D$ (Section 2) without the restriction that $V$ be a closed ball and, of course, without the restriction that $D$ be bounded. Therefore, in (A0) through (A2) below, $D$ is any open subset of $\mathbb{R}^d$ and $V$ is any closed subset of $\mathbb{R}^d$, possibly unbounded. Some of the assumptions will be trivially satisfied when $D$ is bounded and $V$ is a closed ball centered at the origin.

(A0) $\mu$ and the mapping $(x,v) \to \eta(x,v,\cdot) \in \mathcal{P}(V)$ are continuous on $\overline{D} \times V$. For each $x \in \overline{D}$, $Q^x$ generates a Feller semigroup on $\mathcal{C}(V)$ (the space of continuous functions vanishing at infinity). For $f \in C_c(\mathbb{R}^d)$ and for any compact set $K \subseteq \overline{D}$, $\sup_{(x,v) \in K \times V} |Q^x f(v)| < \infty$.

(A1) For each $x \in \overline{D}$, $Q^x$ has a unique stationary probability distribution $\pi(x,\cdot)$ satisfying $\int v' \pi(x,dv') = 0$, and the mapping $x \in \overline{D} \to \pi(x,\cdot) \in \mathcal{P}(V)$ is continuous.

(A2) There exists a function $h \in C^{1,1}(\overline{D} \times V)$ satisfying

$$Q^x h(x,v) = -v,$$

and for each compact set $K \subseteq \overline{D} \times V$ and each $\epsilon > 0$, there exists a compact set $K_\epsilon \subseteq V$ such that

$$\sup_{(x,v) \in K} \int_{K_\epsilon} |h(x,v')\eta(x,v,dv')| < \epsilon.$$

Remark 1.1. Let $\{S^x(t)\}$ denote the semigroup generated by $Q^x$, and let $i(v) \equiv v$. Then by (A1), under suitable ergodicity assumptions on $Q^x$, the function

$$h(x,v) \equiv \int_0^\infty S^x(t)i(v)dt$$

is well defined for $(x,v) \in \overline{D} \times V$ and satisfies (1.9).

The following theorem is the main result of Section 2. Since it is stated in $\mathbb{R}^d \times \mathbb{R}^d$ it requires some additional Liapunov type assumptions which are stated in Section 2.

Theorem 1.2. Let $D = \mathbb{R}^d$ and $V = \mathbb{R}^d$. Assume (A0) through (A2) and (A4) through (A6) of Section 2. In addition assume that the functions

$$b_i(x) \equiv \int \langle v', \nabla x h_i(x,v') \rangle \pi(x,dv'), \ i = 1, \cdots, d$$

$$a_{i,j}(x) \equiv 2 \int [h_i(x,v')v_j'] \pi(x,dv') \quad i,j = 1, \cdots, d$$

are continuous and that the matrix $a(x) \equiv ((a_{i,j}(x)))$ is uniformly positive definite. If $X_0 \to X_0$ as $\epsilon \to 0$ and the solution of the martingale problem for the operator

$$\partial f(x) \equiv \sum_i b_i(x) \frac{\partial f}{\partial x_i}(x) + \frac{1}{2} \sum_{i,j} a_{i,j}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x), \ f \in C_c^2(\mathbb{R}^d),$$
with initial distribution given by $X_0$ does not blow up in finite time, then $X^\epsilon$ converges in distribution to the solution of the martingale problem for $\overline{L}$ with initial distribution given by $X_0$.

Because of the diffusive behavior in the interior, when the particle hits the boundary, it will hit the boundary a large number of times before moving away. Consequently, on the boundary, averaging takes place under the stationary probability distribution of a transition probability kernel describing the changes in the hitting velocity between one reflection and the next. The smoothness assumptions on the boundary assure that, locally, it looks flat. Treating $x \in \partial D$ as a parameter, let $(Z^x, V^x)$ denote the spatially homogeneous transport process with generator

$$L^x f(z, v) \equiv \langle v, \nabla_z f(z, v) \rangle + \mu(x, v) \int [f(z, v') - f(z, v)]\eta(x, v, dv'),$$

starting at $(0, v)$, and let $\tau^x$ be the first exit time of $Z^x$ from the closed half space $\{z : \langle z, \nu(x) \rangle \geq 0\}$ (the half space determined by the tangent hyperplane for $D$ at $x$), that is,

$$\tau^x \equiv \inf\{t \geq 0 : \langle Z^x, \nu(x) \rangle < 0\}.$$

Define

$$Tf(x, v) \equiv E[f(Z^x(\tau^x))|V^x(0) = v],$$

and

$$R_\theta f(x, v) \equiv \int_V f(x, v')\eta_\theta(x, v, dv').$$

Then the desired transition kernel is the kernel of the composed operator

$$R_\theta Tf(x, v) \equiv \int_V Tf(x, v')\eta_\theta(x, v, dv').$$

We require the additional hypothesis:

$$(A_3)$$ For each $x \in \partial D$, the transition kernel corresponding to the operator $(1.14)$ has a unique stationary probability distribution $\pi_\theta(x, \cdot)$, and the mapping $x \mapsto \pi_\theta(x, \cdot) \in P(\mathbb{R}^d)$ is continuous.

Assumptions $(A_0)$ through $(A_3)$ allow us to prove our main result, that is, the diffusion approximation in the presence of boundary conditions of the form $(1.2)$ satisfying $(A_C)$, $(A_H)$ and $(A_M)$. It is convenient to consider three cases: The case when $\alpha \equiv 0$, which we call the hot boundary case and we treat in Section 4; the case when $\alpha \equiv 1$, which we call the cool boundary case and we treat in Section 5, and the general case for $\alpha$ satisfying $(A_M)$, which we treat in Section 6. In the cool boundary and general cases we require $D$ to be convex and $Q$ to satisfy a mild nondegeneracy condition which is formulated in $(A_7)$ and $(A_8)$ in Section 5. The precise statements of our results are contained in the following theorems.

**Theorem 1.3.** (Hot boundary) Suppose $\alpha \equiv 0$. Let $D$ be bounded with $C^2$-boundary, and let $V$ be a closed ball centered at the origin. Assume $(A_H)$ and $(A_0)$ through $(A_3)$. In addition assume that the functions defined in $(1.10)$ are continuous and that the matrix $a(x) = (a_{i,j}(x))$ is strictly positive definite for every $x \in \overline{D}$. Suppose that the vector field $\gamma : \partial D \to \mathbb{R}^d$ defined by

$$\gamma(x) \equiv \int_V [R_\theta - I]h(x, v)\pi_\theta(x, dv)$$

is Lipschitz continuous and satisfies

$$\inf_{x \in \partial D} \langle \gamma(x), \nu(x) \rangle > 0.$$

If as $\epsilon \to 0$, $X^\epsilon_0 \to X_0$, then $X^\epsilon$ converges to the unique solution of the submartingale problem on for $(\overline{L}, (\gamma, \nabla))$ with $D(\overline{L}) = C^2(D)$ and initial condition $X_0$.

**Theorem 1.4.** Let $D$ be bounded with $C^2$-boundary and convex, and let $V$ be a closed ball centered at the origin. Assume $(A_C)$, $(A_H)$ and $(A_M)$, $(A_0)$ through $(A_3)$, and $(A_7) - (A_8)$ of Section 5. In addition
assume that the functions defined in (1.10) are continuous and that the matrix $a(x) = ((a_{i,j}(x)))$ is strictly positive definite for every $x \in \overline{D}$. Suppose that the vector field $\gamma$ defined by (1.15) is Lipschitz continuous and satisfies (1.16). If as $\epsilon \to 0$, $X_0^\epsilon \to X_0$, then $X^\epsilon$ converges to the unique solution of the submartingale problem on $(\bar{X}, \langle \gamma, \nabla \rangle)$ with $D(\bar{X}) = C^2(\overline{D})$ and initial condition $X_0$.

**Remark 1.5.** Following Stroock and Varadhan (1971), $X$ is a solution of the submartingale problem for $(\bar{X}, \langle \gamma, \nabla \rangle)$ with $D(\bar{X}) = C^2(\overline{D})$ if and only if $X(t) \in \overline{D}$, $t \geq 0$, and for each $f \in C^2(\overline{D})$ satisfying $\gamma \cdot \nabla f \geq 0$ on $\partial D$,

$$f(X(t)) - \int_0^t \bar{L}(X(s))ds$$

is a $\{\mathcal{F}_t^X\}$-submartingale. Under our assumptions, $X$ is a solution of the submartingale problem if and only if $X(t) \in \overline{D}$, $t \geq 0$, and there exists a continuous, nondecreasing process $\lambda$ such that

$$f(X(t)) - \int_0^t \bar{L}f(X(s))ds - \int_0^t \gamma(X(s)) \cdot \nabla f(X(s))d\lambda(s)$$

is a $\{\mathcal{F}_t^{X,\lambda}\}$-martingale for each $f \in C^2_c(\mathbb{R}^d)$ and $\int_0^t 1_D(X(s))d\lambda(s) = 0$, $t \geq 0$.


Models with boundary conditions are considered in Bensoussan et al. (1979) and in Costantini (1991). In the former work, $V$ is compact and the scattering kernel on the boundary, $\eta_b$, is required to satisfy (1.5) and a form of Doeblin’s condition ((3.6.7) and (3.6.8) in Bensoussan et al. (1979)). In Costantini (1991) the boundary condition is specular reflection, the mean change of the velocity in the interior is approximately linear in the current velocity ((1.5) in Costantini (1991)), and $V$ can be taken to be $\mathbb{R}^d$.

1.1. **Examples.** Although our primary interest is in general conditions under which the diffusion approximation can be rigorously justified, we would like to ensure that these conditions cover examples of physical interest. In the work on specific transport models having boundary conditions, the conditions are usually of the form (1.2) with $\eta_b^\alpha$ the kernel of specular reflection, $\eta_b^H$ satisfying $(A_H)$, and $\alpha$ satisfying (1.6). Here we give several examples from the physics and engineering literature that satisfy the conditions of our theorems. This list is illustrative rather than exhaustive. Cercignani, Illner, and Pulvirenti (1994) provides a general discussion of the derivation of the scattering kernels as well as a number of particular examples.

1.1.1. **Maxwell model.** (Cercignani et al. (1994), page 236) The boundary scattering kernel has the form

$$\eta_b(x, v, dv') \equiv [1 - \alpha] \delta_{v' = 2\langle v, \nu(x)\rangle \nu(x)}(dv') + \alpha C I_{\{v': \langle v', \nu(x)\rangle > 0\}}|\langle v', \nu \rangle|M_w(v')dv'$$

where

$$M_w(v') = \exp \{-\beta_w|v|^2\}, \quad \beta_w \equiv \frac{1}{2RT_w},$$

$$C \equiv \left(\int_{\{v': \langle v', \nu \rangle > 0\}}|\langle v', \nu \rangle|M_w(v')dv'\right)^{-1}, \quad 0 \leq \alpha \leq 1,$$

$R$ is a universal constant and $T_w$ is the temperature of the boundary (wall). Cercignani et al. say that the use of this model with $\alpha = 1$ is justified for low velocity flows. The model with $\alpha = 0$ (pure specular reflection) is frequently used, but Cercignani and Lampis (1971) argue that it is far from being consistent with experimental data. (However, see the discussion of the Debye sheath below.)
1.1.2. Cercignani - Lampis model. (Cercignani and Lampis (1971), Cercignani et al. (1994), page 237) The boundary scattering kernel is given by:

\[ \eta_b(x, v, dv') \equiv \frac{2}{\pi} \left[ \alpha \omega (2 - \alpha \omega) \right]^{-1} \beta \omega v' \]

\[ \times \exp \left\{ -\beta \omega \frac{v''^2 + (1 - \alpha \omega \omega) v''}{\alpha \omega} - \beta \omega \frac{|v'' - (1 - \alpha \omega \omega)^2|}{\alpha \omega (2 - \alpha \omega)} \right\} \]

\[ \times I_0 \left( \frac{2(1 - \alpha \omega)^{1/2} v' \omega}{\alpha \omega} \right) dv' \]

where \( v'' \equiv \langle v', \nu(x) \rangle, \ v_\omega \equiv \langle v, \nu(x) \rangle, \ v''_\omega \equiv v''_\omega \nu(x), \ v_\omega \equiv v - v_\omega \nu(x), \ 0 < \alpha \omega \leq 1, \ 0 < \alpha \omega \leq 1, \ \beta \omega \) is the same as in the Maxwell model, and \( I_0 \) is the modified Bessel function of first kind and zeroth order, defined by

\[ I_0(y) \equiv (2\pi)^{-1} \int_0^{2\pi} \exp(y \cos \phi) d\phi. \]

1.1.3. Electron kinetics. (Morozov and Shubin (1984), Degond (1998), and Degond, Latocha, Gar rigues, and Boeuf (1998)) Following Morozov and Shubin, Degond et. al. model the motion of electrons using a boundary scattering kernel of the form

\[ \eta_b(x, v, dv') \equiv [1 - \alpha(x, v)] \delta_{v - 2(v, v)\nu} (dv') \]

\[ + \alpha(x, v) \left[ 1 - e^2 \beta(x) \right] K(x, v, dv') + \alpha(x, v) e^2 \beta(x) H(x, v, dv'), \]

where \( K \) is an elastic (energy preserving) diffuse kernel. In particular, \( K \) is assumed to satisfy

\[ K(x, v, \{v' : \langle v', \nu \rangle > 0 \}) = 1, \ \forall v : \langle v, \nu \rangle < 0, \]

\[ K(x, v, dv') \equiv K(x, v, dr \times d\omega) = \delta_{\{|v|\}}(dr) \tilde{K}(x, v, \omega) d\omega, \quad \omega \in S_2, \]

where \( \{r, \omega\} \) are the radial and spherical components of \( v' \).

\( H \) is an inelastic kernel related to attachment to the wall and secondary emissions, that is \( H \) allows for the possible creation or loss of particles at the boundary. We do not consider the possibility of creation or loss, so our conditions cover the case \( \beta \equiv 0 \), or if \( \beta > 0 \), the case in which \( H \) models the transformation of a single incoming particle of velocity \( v \) into a single outgoing particle of velocity \( v' \) of possibly different energy.

Degond (1998) comments that the presence of a boundary layer known as the Debye sheath causes low energy particles to reflect specularly while high energy particles will reflect diffusively leading to the assumption that \( \lim_{|v| \to 0} \alpha(x, v) = 0 \) and \( \lim_{|v| \to \infty} \alpha(x, v) = 1 \) or perhaps preferably that \( \lim_{|\langle v, \nu(x) \rangle| \to 0} \alpha(x, v) = 0 \) and \( \lim_{|\langle v, \nu(x) \rangle| \to \infty} \alpha(x, v) = 1 \).

Our asymptotic results are very different from Degond’s. In Degond’s model, \( Q = 0 \), that is, there are no collisions in the interior of the domain. Degond assumes that the boundary of \( D \) consists of two concentric cylinders and that the difference in the radii of the two cylinders is small compared to the radii. The diffusive behavior of the particles then results from the large number of collisions that occur because the particles move rapidly between the two boundaries.

1.2. Constrained martingale problems. The approach followed in this work is new and of independent interest. In all the works mentioned above, with the exception of Costantini (1991), the proofs are based on the asymptotic expansion in \( \epsilon \) (Hilbert expansion) of the operators involved and on analytical techniques. Costantini (1991) employs martingale techniques and the properties of the Skorohod oblique reflection problem.

Here we follow the approach of Kurtz (1990, 1991) and treat the transport process as a solution of a suitable constrained martingale problem. The key point here is the definition of the boundary operator, which we take to be the pure jump Markov operator corresponding to the transition kernel \( \eta_b \), with jump rate \( 1/(\alpha \rho + (1 - \alpha)) \). The transport process is constructed from a solution of the corresponding patchwork martingale problem (see Kurtz (1990)) by a random time change (Section 3).
Solutions of the patchwork martingale problem are slowed down on the boundary. This enables one to easily derive relative compactness, as $\epsilon$ goes to zero, of the family of rescaled solutions of the patchwork martingale problem. The limit is then identified by stochastic averaging techniques. The stochastic averaging argument exploits certain martingale relations that hold if the operator $T$ defined in (1.13) is sufficiently regular. To avoid discussing the regularity of $T$, we approximate the solution of the patchwork martingale problem by pure jump Markov processes, and exploit the corresponding martingale relations for the approximating processes instead. Finally the inverse random time transformation gives the desired convergence of the rescaled transport processes (Sections 4, 5 and 6).

1.3. Notation. For a metric space $E$, $C(E)$ ($C_b(E)$, $C_c(E)$) will denote the space of continuous (bounded continuous, continuous with compact support), real-valued functions on $E$. $C_0(0, \infty)$ will denote the space of continuous, $E$-valued functions on $[0, \infty)$ and $D_E[0, \infty)$, the space of $E$-valued functions that are right continuous and have left hand limits. For a complete, separable metric space $E$ (bounded continuous, continuous with compact support), real-valued functions on $E$ will denote the space of measures on $[0, \infty)$ satisfying $\mu([0, t] \times S) < \infty$, for each $t \in [0, \infty)$, such that $\{\mu_n\} \subset \mathcal{L}(S)$ converges to $\mu \in \mathcal{L}(S)$ if and only if $\int f \, d\mu_n \to \int f \, d\mu$ for every $f \in C_b([0, \infty) \times S)$ with support in $[0, t] \times S$ for some $t$. For closed subsets $E_1, E_2 \subset \mathbb{R}^d$, $C^m,n(E_1 \times E_2, \mathbb{R}^k)$ ($C^m,n_b(E_1 \times E_2, \mathbb{R}^k)$), $m, n \in \mathbb{Z}^+$, $k \in \mathbb{N}$, will denote the space of functions $f : E_1 \times E_2 \to \mathbb{R}^k$ differentiable $m$ times with respect to the first variable and $n$ times with respect to the second one in the interior of $E_1 \times E_2$, with derivatives that can be extended to continuous (bounded continuous) functions on $E_1 \times E_2$; the subscript $b$ and the superscript 0 will be omitted whenever the meaning is clear without them, and if $k = 1$, we will write $C^m,n_b(E_1 \times E_2)$. $\nabla f$ will denote the gradient or the Jacobean matrix if $f$ is vector valued; subscripts $x$ and $v$ will be used occasionally to distinguish between differentiation with respect to the first and second variable. $\| \cdot \|$ will always denote the supremum norm. For $x \in \mathbb{R}^d$, the notation $d(x, \partial D)$ means the distance of $x$ from $\partial D$, that is, $d(x, \partial D) = \inf_{y \in \partial D} d(x, y)$.

2. The Unconstrained Process. In this section we will obtain the diffusion approximation of $X^\epsilon$ in the interior of the domain $D$, or equivalently for $D = \mathbb{R}^d$. Here the velocity state space $V$ can be any closed subset of $\mathbb{R}^d$, possibly unbounded: Therefore processes are stopped at suitable stopping times, which is unnecessary if $V$ is compact, and certain Liapunov type assumptions, which are always satisfied if $V$ is compact, are required.

First of all let us construct rigorously the transport process $(X^\epsilon, V^\epsilon)$. Consider the operator $L^\epsilon$ defined by (1.1), the operator $Q^\epsilon$ defined in (1.8) and an initial datum $(X_0^\epsilon, V_0^\epsilon)$.

**Proposition 2.1.** Assume that for each $k \in \mathbb{N}$, there exist $p^k > 0$, $K^k \geq 0$, and $\psi^k$ on $V$, continuous, such that $\psi^k(v) \geq |v|^p$, $E[\psi^k(V_0^\epsilon)] < \infty$ and $Q^x \psi^k(v) \leq K^k$ for $v \in V$ and $|x| \leq k$. Then there exists a stochastic process $\{(X^\epsilon, V^\epsilon)\}$ such that

$$
(2.1) \quad f(X^\epsilon(t \wedge \beta_k^\epsilon), V^\epsilon(t \wedge \beta_k^\epsilon)) - f(X_0^\epsilon, V_0^\epsilon) - \int_0^{t \wedge \beta_k^\epsilon} L^\epsilon f(X^\epsilon(s), V^\epsilon(s)) \, ds,
$$

where

$$
(2.2) \quad \beta_k^\epsilon = \inf\{t : |X^\epsilon(t)| \geq k\},
$$

is a local martingale for every $f \in C^{0,1}_0(\mathbb{R}^d \times V)$. The stochastic process $\{(X^\epsilon, V^\epsilon)\}$ is uniquely defined up to time $\beta_{\infty}^\epsilon = \lim_{k \to \infty} \beta_k^\epsilon$. In addition, setting

$$
\gamma_n^\epsilon = \inf\{t : |V^\epsilon(t)| \geq n\},
$$

it holds that

$$
(2.3) \quad f(X^\epsilon(t \wedge \beta_k^\epsilon \wedge \gamma_n^\epsilon), V^\epsilon(t \wedge \beta_k^\epsilon \wedge \gamma_n^\epsilon)) - f(X_0^\epsilon, V_0^\epsilon) - \int_0^{t \wedge \beta_k^\epsilon \wedge \gamma_n^\epsilon} L^\epsilon f(X^\epsilon(s), V^\epsilon(s)) \, ds
$$

is a martingale for every $f \in C^{1,0}_0(\mathbb{R}^d \times V)$ and

$$
\lim_{n \to \infty} \beta_k^\epsilon \wedge \gamma_n^\epsilon = \beta_k^\epsilon.
$$
Proof. The proof consists of a fairly standard localization procedure. First one constructs a family \( \{(X_{k,n}^t, V_{k,n}^t)\} \) of Markov processes with generator \( L_{k,n}^t \) of the form (1.1) with \( \mu(x,v) \) and \( \eta(x,v,\cdot) \) replaced by \( \mu_{k,n}(x,v) \) and \( \eta_{k,n}(x,v,\cdot) \) respectively. Then \( f(X_{k,n}^t(t \wedge \gamma_{k,n}^n), V_{k,n}^t(t \wedge \gamma_{k,n}^n)) - \int_0^{t \wedge \gamma_{k,n}^n} L_{k,n}^t (X_{k,n}^t(s), V_{k,n}^t(s))ds \) is a martingale for every \( f \in C_b^{1,0} (\mathbb{R}^d \times \mathcal{V}) \). Next, setting \( \gamma_{k,n}^n := \inf\{t \geq 0 : |V_{k,n}^t(t)| \geq n\} \), one defines the processes \( (X_{k,n}^t, V_{k,n}^t) \) by
\[
(X_{k,n}^t(t), V_{k,n}^t(t)) \equiv (X_{k,n}^t(t), V_{k,n}^t(t)), \quad t \leq \gamma_{k,n}^n.
\]
Then \( f(X_{k,n}^t(t \wedge \gamma_{k,n}^n), V_{k,n}^t(t \wedge \gamma_{k,n}^n)) - \int_0^{t \wedge \gamma_{k,n}^n} L_{k,n}^t (X_{k,n}^t(s), V_{k,n}^t(s))ds \), where \( L_{k}^t \) is the operator of the form (1.1) with \( \mu(x,v) \) and \( \eta(x,v,\cdot) \) replaced by \( \mu_{k}(x,v) \) and \( \eta_{k}(x,v,\cdot) \) respectively, is a martingale, for every \( f \in C_b^{1,0} (\mathbb{R}^d \times \mathcal{V}) \). In particular, for the function \( \psi^{k,a}(\cdot) \equiv \psi^k(\cdot) \wedge a \), where \( \psi^k \) is the function in the assumption of the proposition, it holds, for \( a \geq \sup_{|v| \leq n} \psi^k(v) \),
\[
E \left[ |\psi^{k,a}(V_{k,n}^t(t \wedge \gamma_{k,n}^n))|^2 \right] = E \left[ \int_0^{t \wedge \gamma_{k,n}^n} e^{-2Q(kx^t)} \psi^{k,a}(V_{k,n}^t(s)) ds \right] \\
\leq E \left[ \int_0^{t \wedge \gamma_{k,n}^n} e^{-2Q(kx^t)} \psi^k(V_{k,n}^t(s)) ds \right],
\]
and letting \( a \) go to infinity, we have
\[
P \left( (\gamma_{k,n}^n \leq t) \leq \frac{K}{\eta^{1/2}},
\]
so that \( \gamma_{k,n}^n \to \infty \) in probability. The assertion then follows by setting \( \beta_k \equiv \inf\{t \geq 0 : |X_{k}^t(t)| \geq k\} \) and
\[
(X_{k}^t(t), V_{k}^t(t)) \equiv (X_{k,n}^t(t), V_{k,n}^t(t)), \quad t \leq \beta_k,
\]
and by observing that \( \beta_k = \inf\{t \geq 0 : |X_{k}^t(t)| \geq k\} \wedge \beta_k^n = \inf\{t \geq 0 : |V_{k}^t(t)| \geq n\} \wedge \beta_k^n \).

Our convergence technique relies on the representation of time integrals as integrals with respect to the occupation measure \( \Gamma_{k}^n \) on \([0,\infty) \times \mathcal{V} \) defined by
\[
\Gamma_{k}^n([0,t] \times A) = \int_0^{t \wedge \beta_{k}^n} I_A(V_{k}^t(s)) ds, \quad A \in B(\mathcal{V}).
\]
The advantage of this approach is twofold: On one hand, relative compactness of \( \{\Gamma_{k}^n\} \), for each \( k \), is almost immediate (relative compactness of \( \{X_{k}^t(\cdot \wedge \beta_{k}^n)\} \) is also easy to prove): on the other hand one avoids dealing explicitly with the behavior of \( V_{k}^t \) as \( \epsilon \) goes to 0, which would be delicate because \( V_{k}^t \), roughly speaking, reaches a different local equilibrium at any value \( x \) of the limiting position process.

The outline of the rest of this section is as follows: Lemma 2.2 is a technical lemma providing the moment estimates that allow us to prove relative compactness of \( \{X_{k}^t(\cdot \wedge \beta_{k}^n)\} \) (Lemma 2.3) and of \( \{\Gamma_{k}^n\} \) (Lemma 2.4). The limit points of \( \{\Gamma_{k}^n\} \) are then identified in Lemma 2.4. The argument used for this purpose in Lemma 2.4 is crucial and will be used repeatedly in Sections 4 and 5. The proof of Theorem 1.2 concludes this section.

The following hypotheses assure relative compactness of \( \{X_{k}^t(\cdot \wedge \beta_{k}^n)\} \) and \( \{\Gamma_{k}^n\} \) for each \( k \), as well as some uniform integrability conditions that are needed in the identification of the limit. Note that assumption \( (A_6) \) below is a slightly reinforced version of the assumption of Proposition 2.1, so that under \( (A_4)-(A_6) \) Proposition 2.1 always holds. All the conditions are immediately satisfied if the state space \( \mathcal{V} \) is compact. Recall that \( \nabla_x f \) is the Jacobian matrix of \( f \) with respect to \( x \). Let \( h \) be the function of \((A_2)\).

\((A_4)\) For each \( k \in \mathbb{N} \), there exist \( K_{k}^1, K_{k}^2, q \geq 0 \) such that \( |h(x,v)| \leq |\nabla_x h(x,v)| \leq K_{k}^1 + K_{k}^2 |v|^q \) for \( v \in \mathcal{V} \) and \( |x| \leq k \).
\(A_5\) For each \(k \in \mathbb{N}\), there exist \(p_k^1 > q^k + 1\), \(K_3^k, K_4^k > 0\), and \(\psi_1^k \geq 0\) on \(V\), continuous, such that 
\[Q^v \psi_1^k(v) \leq K_3^k - K_4^k |v|^{p_k^1} \text{ for } v \in V \text{ and } |x| \leq k, E[|\psi_1^k(V_0)|] < \infty.\]

\(A_6\) For each \(k \in \mathbb{N}\), there exist \(p_2^k > 2q^k\), \(K_5^k \geq 0\), and \(\psi_2^k\) on \(V\), continuous, such that 
\[\psi_2^k(v) \geq |v|^{p_2^k}, Q^v \psi_2^k(v) \leq K_5^k \text{ and } E[|\psi_2^k(V_0)|] < \infty, \text{ for } v \in V \text{ and } |x| \leq k.\]

Lemma 2.2. Fix \(k \in \mathbb{N}\). Let \(p_k^1\) and \(q^k\) be as in \((A_4)\) and \((A_5)\). Then there exist stopping times \(\tau^*\) such that for each \(T > 0\),
\[
\lim_{\epsilon \to 0} P\{\tau^* \leq T \wedge \beta_2^k\} = 0,
\]
\[
\lim_{\epsilon \to 0} E[\sup_{0 \leq t \leq T \wedge \beta_2^k} \epsilon|h(X^\epsilon(t), V^\epsilon(t))|] = 0,
\]
\[
\sup_{c > 0} E[\int_0^{T \wedge \beta_2^k} |\nabla_x h(X^\epsilon(s), V^\epsilon(s))V^\epsilon(s)|^{p_k^1/(q^k + 1)}ds] < \infty,
\]
and
\[
\sup_{c > 0} E[\int_0^{T \wedge \beta_2^k} (|V^\epsilon(s)||h(X^\epsilon(s), V^\epsilon(s))|)^{p_k^1/(q^k + 1)}ds] < \infty.
\]

\textbf{Proof.} To simplify notation, assume that the constants and the functions in \((A_4)\), \((A_5)\) and \((A_6)\) are independent of \(k\). (Since \(X^\epsilon\) is continuous, the stopped process \(X^\epsilon(\cdot \wedge \beta_2^k)\) never enters a region where the constants would be different.) Let \(0 < \delta < 1\) satisfy \((1 - \delta)p_2 > 2q\), and define \(\tau^* = \inf\{t : |V^\epsilon(t)|^q > \epsilon^{\delta - 1}\}\).

For \(a > 0\), define \(\psi_2^a = a \wedge \psi_2\). Clearly, for \(n\) large enough, \(\tau^* < \gamma_n\), so that, by \((2.1)\) and \((A_6)\),
\[E[|\psi_2^a(V^\epsilon(T \wedge \beta_2^k \wedge \tau^*))| \leq E[|\psi_2^a(V^\epsilon(0))]| + K_5 T \epsilon^2],\]

By letting \(a\) go to infinity we see that the same inequality holds for \(\psi_2\), and we obtain
\[
e^{(\delta - 1)\frac{p_2}{q}} P\{\tau^* \leq T \wedge \beta_2^k\} \leq E[|V^\epsilon(T \wedge \beta_2^k \wedge \tau^*)|^{p_2}
\leq E[|\psi_2(V^\epsilon(T \wedge \beta_2^k \wedge \tau^*))| \leq E[|\psi_2(V^\epsilon(0))]| + K_5 T \epsilon^2],
\]

which implies that \(\lim_{\epsilon \to 0} P\{\tau^* \leq T \wedge \beta_2^k\} = 0\). Note that, since \(|h(x, v)| \vee |\nabla_x h(x, v)| \leq K_1 + K_2|v|^q\), to prove the lemma it is sufficient to show that
\[
\lim_{\epsilon \to 0} E[\sup_{0 \leq t \leq T \wedge \beta_2^k \wedge \tau^*} \epsilon|V^\epsilon(t)|^q] = 0
\]
and
\[
\sup_{c > 0} E[\int_0^{T \wedge \beta_2^k} |V^\epsilon(s)|^{p_1}ds] < \infty.
\]

Observe that
\[
\sup_{0 \leq t \leq T \wedge \beta_2^k \wedge \tau^*} \epsilon|V^\epsilon(t)|^q \leq \epsilon^\delta + \epsilon|V^\epsilon(T \wedge \beta_2^k \wedge \tau^*)|^q,
\]

and hence
\[
E[\sup_{0 \leq t \leq T \wedge \beta_2^k \wedge \tau^*} \epsilon|V^\epsilon(t)|^q] \leq \epsilon^\delta + \epsilon E[|V^\epsilon(T \wedge \beta_2^k \wedge \tau^*)|^q]
\]
\[
\leq \epsilon^\delta + \epsilon E[|V^\epsilon(T \wedge \beta_2^k \wedge \tau^*)|^{p_2}]^\frac{1}{p_2}
\]
\[
\leq \epsilon^\delta + (E[|\psi_2(V^\epsilon(0))]| + K_5 T \epsilon^2)^\frac{1}{p_2},
\]
which gives (2.10).

For \( a > 0 \), let, as before, \( \psi_1^a = a \wedge \psi_1 \). Then, by (2.1)

\[
\psi_1^a (V^*(t \wedge \gamma^*_k \wedge \beta^*_k)) - \int_0^{t \wedge \gamma^*_k \wedge \beta^*_k} e^{-2Q^*(s)\psi_1^a (V^*(s))} ds
\]

is a martingale. For \( a \geq \sup_{|v| \leq n} \psi_1(v) \), \( \psi_1^a (v) = \psi_1(v) \) for \( |v| \leq n \), and

\[
Q^* \psi_1^a (v) \leq Q^* \psi_1(v) \leq K_3 - K_4 |v|^p, \quad v \leq n, \ |x| \leq k.
\]

Consequently,

\[
0 \leq E[\psi_1^a (V^*(0))] + E\int_0^{T \wedge \gamma^*_k \wedge \beta^*_k} e^{-2Q^*(s)\psi_1^a (V^*(s))} ds
\]

\[
\leq E[\psi_1(V^*(0))] + E\int_0^{T \wedge \gamma^*_k \wedge \beta^*_k} e^{-2(K_3 - K_4 |V^*(s)|^p)} ds,
\]

and hence

\[
E[\int_0^{T \wedge \gamma^*_k \wedge \beta^*_k} K_4 |V^*(s)|^p ds] \leq c^2 E[\psi_1(V^*(0))] + K_3T.
\]

Letting \( n \to \infty \), we get (2.11).

Lemma 2.3. If \( \{X_0^\epsilon, V_0^\epsilon\} \) is relatively compact in \( \mathbb{R}^d \times \mathcal{V} \) as \( \epsilon \) goes to zero, then for each \( k > 0 \), \( \{X^\epsilon(\cdot \wedge \beta^*_k)\} \) is relatively compact in \( C_\mathbb{R}^\mathbb{R}^{0, \infty} \times \mathcal{V} \), as \( \epsilon \) goes to zero.

Proof. By (2.6), it is enough to prove relative compactness for \( \{X^\epsilon(\cdot \wedge \beta^*_k)\} \). Consider the function \( h \) of (A2) and (A4). For \( f \in C_\mathbb{R}^\mathbb{R}_c(\mathbb{R}^d) \), set

\[
f^\epsilon(x, v) = f(x) + \epsilon \langle \nabla f(x), h(x, v) \rangle.
\]

Then

\[
L^\epsilon f^\epsilon(x, v) = \sum_i (\nabla_x h_i(x, v) \frac{\partial f}{\partial x_i}(x)) + \sum_{ij} (h_i(x, v) v_j) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \equiv Lf(x, v),
\]

and, by (2.1),

\[
f(X^\epsilon(t \wedge \gamma^* \wedge \beta^*_k)) + \epsilon \langle \nabla f(X^\epsilon(t \wedge \gamma^* \wedge \beta^*_k)), h(X^\epsilon(t \wedge \gamma^* \wedge \beta^*_k)) \rangle
\]

\[
- \int_0^{t \wedge \gamma^* \wedge \beta^*_k} Lf(X^\epsilon(s), V^\epsilon(s)) ds
\]

is a martingale. The results of Lemma 2.2 and Theorems 9.1, and 9.4, Chapter 3, of Ethier and Kurtz (1986) (see also Remark 9.5(b)) imply the conclusion of the lemma.

Lemma 2.4. For each \( k > 0 \), the family \( \{r_k\} \) is relatively compact in \( \mathcal{L}(\mathcal{V}) \) as \( \epsilon \) goes to zero, and \( \{\beta_k\} \) is relatively compact in \( [0, \infty] \) as \( \epsilon \) goes to zero. For every limit point \( (X_k, \Gamma_k, \tilde{\beta}_k) \) of \( (X^\epsilon(\cdot \wedge \beta^*_k), \Gamma^\epsilon, \beta^*_k) \), \( \Gamma_k \) almost surely has the form

\[
\Gamma_k([0, t] \times A) = \int_0^{t \wedge \tilde{\beta}_k} \pi(X_k(s), A) ds, \quad A \in \mathcal{B}(\mathcal{V})
\]

where \( \pi \) is defined in (A1).

Remark 2.5. Note that \( \tilde{\beta}_k \) may not equal \( \beta_k \equiv \inf \{t : |X_k(t)| \geq k\} \), but it will satisfy \( \tilde{\beta}_k \geq \beta_k \). Note also that \( X_k(\cdot) = X_k(\cdot \wedge \tilde{\beta}_k) \).

Proof. By (2.11), \( \{r_k\} \) is relatively compact. (See Lemma 1.3 of Kurtz (1992).) In addition, since \( \Gamma_k([0, t] \times \mathcal{V}) \equiv t \wedge \tilde{\beta}_k \), every limit point \( \Gamma_k \) has the form

\[
\Gamma_k([0, t] \times A) = \int_0^{t \wedge \tilde{\beta}_k} \gamma(s, A) ds, \quad A \in \mathcal{B}(\mathcal{V}),
\]
for some process $\gamma$ with values in $\mathcal{P}(\mathcal{V})$ (see, for example, Kurtz (1992)).

For $f \in C_c(\mathcal{V})$, let

$$M_f^\epsilon(t) = f(V^\epsilon(t \wedge \tau^{\epsilon} \wedge \beta_k^{\epsilon})) - f(V_0) - \epsilon^{-2} \int_0^{t \wedge \tau^{\epsilon} \wedge \beta_k^{\epsilon}} Q^X(s) f(V^\epsilon(s)) ds$$

Then, by (2.1), $M_f^\epsilon(\cdot)$ is a martingale. Multiplying by $\epsilon^2$, $\{\epsilon^2 M_f^\epsilon(\cdot)\}$ is relatively compact in $D_R[0, \infty)$.

Considering convergent subsequences if necessary, we see that $\epsilon^2 M_f^\epsilon(\cdot)$ and

$$\int_0^{\wedge \tau^{\epsilon} \wedge \beta_k^{\epsilon}} Q^X(s) f(V^\epsilon(s)) ds,$$

and hence, by (2.6),

$$\int_0^{\wedge \beta_k} Q^X(s) f(V^\epsilon(s)) ds,$$

converge in distribution to the same process. Consequently, any limit point must be a martingale and must be continuous and of bounded variation, and hence the limit must be identically zero. (See, for instance, Corollary II.6.1, Protter (1990)). Therefore, for every limit point $(X_k, \Gamma_k, \hat{\beta}_k)$ and for every $f \in C_c(\mathcal{V})$,

$$\int_{[0.1] \times \mathcal{V}} Q^{X_{kn}(s)} f(v) \Gamma_k(ds \times dv) = \int_{[0.1] \times \mathcal{V}} Q^{X_{kn}(s)} f(v) \gamma(s, dv) ds = 0$$

so that, almost surely for almost every $s < \hat{\beta}_k$,

$$\int_{\mathcal{V}} Q^{X_{kn}(s)} f(v) \gamma(s, dv) = 0$$

for all $f \in C_c(\mathcal{V})$. In addition, $\gamma(s, \mathcal{V}) = 1$, for almost every $s < \hat{\beta}_k$, almost surely. Therefore, by (A1), $\gamma(s, \cdot)$ must agree with $\pi(X_k(s), \cdot)$, for almost every $s < \hat{\beta}_k$, almost surely. (See Proposition 9.2, Chapter 4, Ethier and Kurtz (1986).) □

**Proof of Theorem 1.2.** Every subsequence of $\epsilon$’s has a further subsequence $\{\epsilon_n\}$ such that for each $k$,

$$\{(X^{\epsilon_n}(\cdot \wedge \beta_k^{\epsilon_n}), \Gamma_k^{\epsilon_n}, \beta_k^{\epsilon_n})\}$$

converges in distribution in $D_R[0, \infty) \times \mathcal{L}(\mathcal{V}) \times [0, \infty]$ to a limit $(X_k, \Gamma_k, \hat{\beta}_k)$. For every $f \in C^2_c(\mathbb{R}^d)$, by the same argument used in the proof of Lemma 2.3 the stochastic process (2.14) is a martingale. It follows from Theorem 1.6 in Kurtz (1992), Lemma 2.2, and Lemma 2.4 that

$$f(X_k(t)) - \int_{[0,t \wedge \beta_k] \times \mathcal{V}} L f(X_k(s), v) \Gamma_k(ds \times dv)$$

$$= f(X_k(t)) - \int_0^{t \wedge \beta_k} L f(X_k(s)) ds$$

is a martingale for every $f \in C^2_c(\mathbb{R}^d)$.

By Remark 2.5, $\lim_{\epsilon \to \infty} \beta_k = \infty$, and the theorem follows. □

### 3. Construction of the reflecting process.

We now consider the case in which $X^\epsilon$ is constrained in a domain $D$ and reflects on the boundary according to a transition function $\eta_b(x, v, \cdot)$ of the form (1.2). The behavior of $(X^\epsilon, V^\epsilon)$ in $D$ is still defined by the operator $L^\epsilon$ in (1.1).

We assume that there is a function $\varphi \in C^2_b(\mathbb{R}^d)$ such that

$$D = \{x : \varphi(x) > 0\}, \quad \partial D = \{x : \varphi(x) = 0\},$$

$$\partial D = \{x : \varphi(x) = 0\},$$
(3.2) \[ \inf_{x \in \partial D} |\nabla \varphi(x)| > 0. \]

For simplicity we will take \( D \) bounded. This restriction can be relaxed using the localization techniques of Section 2. We also assume that the velocity process \( V^\prime \) assumes values in a closed ball \( V \) centered at the origin. It should be possible to eliminate this hypothesis under conditions similar to \( (A_4) \) - \( (A_6) \), but we have not been able to find a simple argument to accomplish this goal. We suppose that \( (A_0) \) holds. Therefore, since \( \overline{D} \) and \( V \) are compact, we have

(3.3) \[ \sup_{(x,v) \in \overline{D} \times V} \mu(x,v) = \overline{\mu} < \infty. \]

Recall that \( O = D \times V, \partial_x O = \{(x,v) \in \partial D \times V : \langle v, \nu(x) \rangle \leq 0 \} \), and \( \partial_n O = \{(x,v) \in \partial D \times V : \langle v, \nu(x) \rangle > 0 \} \). As far as the boundary condition is concerned, we assume that \( (A_C), (A_H), \) and \( (A_M) \) are satisfied. Recall that

(3.4) \[ R_0 f(x,v) \equiv \int_V f(x,v') \eta_0(x,v,dv'). \]

We assume that \((X^0_0,V^0_0) \in O \cup \partial_n O\).

The goal of this section is to construct a transport process \((X^\epsilon,V^\epsilon)\) with the desired behavior both in \( D \) and on \( \partial D \). The outline of the section is as follows: First a ‘natural’ system of stochastic equations ((3.6)) is introduced and it is proved that if (1.6) holds or if there is pure specular reflection, the system has a unique solution, well defined for all times (Theorem 3.1). Next (Proposition 3.4 and Theorem 3.6) it is shown that any solution of (3.6) that is well defined for all times is a solution of a suitable constrained martingale problem (see Kurtz (1990) and Definition 3.3 below). More generally, a solution of the constrained martingale problem can be obtained, by a specific random time change, from any solution of the corresponding patchwork martingale problem (see Kurtz (1990) and Definition 3.5). A solution of the patchwork martingale problem, \((Y^\epsilon,U^\epsilon)\), is constructed as a limit point of certain pure jump Markov processes (Theorem 3.8). The corresponding solution of the constrained martingale problem, \((X^\epsilon,V^\epsilon)\), is the desired reflecting transport process. The process \((Y^\epsilon,U^\epsilon)\) will have more than just an auxiliary role: In fact, in the next three sections we will first prove convergence, as the scaling parameter \( \epsilon \) goes to 0, of \((Y^\epsilon)\), and from this convergence, we will derive convergence of \((X^\epsilon)\). One may construct a stochastic process \((X^\epsilon,V^\epsilon)\) evolving in the interior of \( D \) according to the operator \( L^\prime \), and on \( \partial D \) according to the transition kernel \( \eta_0 \), by iterating the following procedure: follow the unconstrained process until it first hits \( \partial D \), then switch velocity according to \( \eta_0 \) and start following the unconstrained process again. Unfortunately, this approach may not yield a process well defined for all times, because one cannot rule out that, with positive probability, the boundary hitting times have a finite accumulation point; also, if \( D \) is not convex, the process may not stay confined in \( \overline{D} \) because the boundary may be hit at a point \( x \) with a velocity \( v \) such that \( \rho(x,v) = 0 \). However, it is shown, in Bensoussan et al. (1979) when (1.6) holds, and in Costantini (1991) for pure specular reflection, that, under mild assumptions, neither of these happens.

More precisely, let \( F(x,v,\theta) \) and \( F_0(x,v,\theta) \) satisfy

(3.5) \[ \int_0^1 I_A(F(x,v,\theta))d\theta = \frac{\mu(x,v)}{\overline{\mu}} \eta(x,v,A) + \left(1 - \frac{\mu(x,v)}{\overline{\mu}}\right) I_A(v) \]

\[ \int_0^1 I_A(F_0(x,v,\theta))d\theta = \eta_0(x,v,A). \]

Following Blackwell and Dubins (1983), one can assume that \( \lim_{(x',v') \to (x,v)} F(x',v',\theta) = F(x,v,\theta) \) for almost every \( \theta \), and similarly for \( F_0 \). Let \( \Xi^\epsilon \) be a Poisson process of parameter \( \epsilon^{-2}\overline{\mu}, \{\Theta_n\} \) and \( \{\Theta_n^k\} \) be sequences of i.i.d. \([0,1]\) uniform random variables, such that \((X^0_0,V^0_0), \Xi^\epsilon, \{\Theta_n\} \) and \( \{\Theta_n^k\} \) are independent. Finally let \( m \) denote Lebesgue measure (indifferently on \( V, \overline{D} \) or \( O \)).
Theorem 3.1. Assume either \((1.6)\), or \(\alpha \equiv 1\), \((1.3)\) and \((1.4)\). In the latter case assume also that the law of \((X_0^\epsilon, V_0^\epsilon)\) is absolutely continuous with respect to \(m\) and that, for every nonnegative \(p \in L^1(m)\),

\[
\int_V \eta(x,v,:)p(v)m(dv)
\]

is absolutely continuous with respect to \(m\), for \(m\)-almost all \(x \in \bar{D}\). Then there exists a unique solution \((X^\epsilon, V^\epsilon)\), well defined for all \(t \geq 0\) with probability 1, to the following system of stochastic equations:

\[
\begin{align*}
X^\epsilon(t) &= X_0^\epsilon + \int_0^t \epsilon^{-1} V^\epsilon(s) ds \\
V^\epsilon(t) &= V_0^\epsilon + \int_0^t [F(X^\epsilon(s), V^\epsilon(s), \Theta^\epsilon_{\chi_{s}(s)+1}) - V^\epsilon(s)] d\Xi^\epsilon(s) \\
&+ \int_0^t [F_b(X^\epsilon(s), V^\epsilon(s), \Theta^\epsilon_{\chi_{s}(s)+1}) - V^\epsilon(s)] dN^\epsilon(s), \\
N^\epsilon(t) &= \sum_{s \leq t} I_{\partial D}(X^\epsilon(s)).
\end{align*}
\]

In particular

\[
N^\epsilon(t) < \infty, \ \forall t \geq 0,
\]

with probability 1.

Proof. For pure specular reflection, the assertion is proved in Theorem 2.2 in Costantini (1991).

In the case when \((1.6)\) holds, the assertion is proved in Bensoussan et al. (1979). The following is an alternative proof. In order to simplify notation, let us omit \(\epsilon\). It is clear that one can define a (unique) solution of \((3.6)\) for \(t < \tau_\infty\), where

\[
\tau_\infty \equiv \lim_{k \to \infty} \tau_k,
\]

\[
\tau_0 \equiv 0, \ \tau_{k+1} \equiv \inf\{t > \tau_k : X(t) \in \partial D\}.
\]

Let us show that \((1.6)\) implies that \(\tau_\infty = \infty\) with probability 1. In fact, since \(|V(t)|\) is bounded above and the curvature of the boundary is bounded above, there exist constants \(c_1\) and \(c_2\), depending on the maximum speed and the maximum curvature, such that

\[
P\{\tau_{k+1} - \tau_k \geq c_1(V(\tau_k), \nu(X(\tau_k))) \mid \mathcal{F}_{\tau_k}^X,V\} \geq e^{-c_2\mu}.
\]

On the other hand, as pointed out in the Introduction, \((1.6)\) implies that \((1.5)\) is satisfied by the whole kernel \(\eta_b\). Therefore there exists \(\delta > 0\) such that

\[
P\{V(\tau_k), \nu(X(\tau_k)) \geq \delta \mid \tau_1, \ldots, \tau_k\} \geq \delta,
\]

which implies

\[
P\{\tau_{k+1} - \tau_k \geq c_1 \delta \mid \tau_1, \ldots, \tau_k\} \geq e^{-c_2\mu} \delta,
\]

and hence

\[
P\{\sum_k (\tau_{k+1} - \tau_k) < \infty\} = 0.
\]

\(\square\)

Remark 3.2. If the domain \(D\) is convex, then there is always a unique solution of \((3.6)\) for \(t < \tau_\infty\) (defined by \((3.7)\)). If, for some solution of \((3.6)\) \(\tau_\infty = \infty\), then \((3.6)\) has a unique solution.
We will avoid the issue of accumulation of boundary hitting times by taking a more abstract point of view. Notice that any solution of (3.6) that is well defined for all times, satisfies a martingale relation, namely

\begin{equation}
(3.8) \quad f(X^*(t),V^*(t)) - f(X_0^*,V_0^*) - \int_0^t L^* f(X^*(s),V^*(s))ds - \int_0^t [R_b - I]f(X^*(s),V^*(s-))dN^*(s)
\end{equation}

is an \(\{F^*,V^*\}\)-(local)-martingale for every \(f \in C^{1,1}(\mathcal{O})\), and \(dN^*\) can be viewed as a counting measure. This observation suggests constructing the reflecting transport process as a solution of a constrained martingale problem. We recall the definition of a constrained martingale problem given in Kurtz (1990), slightly modified to suit our present purposes.

**Definition 3.3.** Let \(E\) be a compact subset of \(\mathbb{R}^d\), \(\{E_0, E_1\}\) be a partition of \(E\) into two Borel sets, \(\mathcal{D}\) be a dense subspace of \(C(E)\), and let \(\mathcal{E}_0 : \mathcal{D} \to C(\overline{E_0})\) and \(\mathcal{E}_1 : \mathcal{D} \to C(\overline{E_1})\) be two dissipative linear operators that map the function identically equal to 1 into 0. A stochastic process \(\mathcal{X}\) is a solution of the constrained martingale problem for \((E_0, E_0, E_1, E_1)\), with initial condition \(X_0\), if there is a filtration \(\{\mathcal{F}_t\}\) such that \(\mathcal{X}\) is an \(\{\mathcal{F}_t\}\) adapted process with sample paths in \(D\mathcal{E}[0,\infty)\) and there exists an \(\{\mathcal{F}_t\}\) adapted random measure \(\Lambda\) on \([0,\infty) \times \overline{E_1}\) such that

\[ f(\mathcal{X}(t)) - f(\mathcal{X}_0) - \int_0^t \mathcal{E}_0 f(\mathcal{X}(s))ds - \int_{[0,t] \times \overline{E_1}} \mathcal{E}_1 f(x) \Lambda(ds \times dx) \]

is an \(\{\mathcal{F}_t\}\) (local) martingale for every \(f \in \mathcal{D}\).

In order to construct the reflecting transport process as a solution of a constrained martingale problem, we need to identify the sets \(E_0\) and \(E_1\), and the operators \(\mathcal{E}_0\) and \(\mathcal{E}_1\). In particular, in view of the following sections, it is crucial that the boundary operator \(\mathcal{E}_1\) be defined properly. We introduce the function \(\mu_b : \partial_- O \to [0,\infty]\)

\[ \mu_b(x,v) \begin{cases} \equiv [\alpha(x,v)\rho(x,v) + 1 - \alpha(x,v)]^{-1}, & \text{for } \alpha(x,v) < 1, \text{ or } \rho(x,v) > 0, \\ \equiv \infty, & \text{for } \alpha(x,v) = 1, \rho(x,v) = 0. \end{cases} \]

For \(\mu_b(x,v) < \infty\), we let the boundary operator have the form

\begin{equation}
(3.9) \quad B^* f(x,v) \equiv \epsilon^{-1} B f(x,v) \quad B f(x,v) \equiv \mu_b(x,v)(R_b - I) f(x,v), \quad f \in C^{1,1}(\mathcal{O}),
\end{equation}

where \(R_b\) is defined in (3.4). \((A_C)\), \((A_H)\) and \((A_M)\) imply that,

\begin{equation}
(3.10) \quad c_b \equiv \sup_{(x,v) \in \partial_- O, \mu_b(x,v) < \infty} \mu_b(x,v) \int |v' - v| \eta_b(x,v, dv') < \infty,
\end{equation}

and that, for \(f \in C^{1,1}(\mathcal{O})\), \(B f\) extends to a bounded, continuous function on all of \(\partial_- O\) and we have

\begin{equation}
(3.11) \quad \sup_{(x,v) \in \partial_- O} |B f(x,v)| \leq c_b \|\nabla f\|.
\end{equation}

For \(\rho(x,v) = 0\) (in particular for \(\mu_b(x,v) = \infty\)), \(B f(x,v)\) is given by

\begin{equation}
(3.12) \quad B f(x,v) = \beta(x,v)[E[k_{x,v}], \nabla_v f(x,v)] + [1 - \beta(x,v)] \int [f(x,v') - f(x,v)] \eta^H_b(x,v, dv')
\end{equation}

where \(\beta\) is defined as

\begin{equation}
(3.13) \quad \beta(x,v) \equiv \frac{\alpha(x,v)\rho(x,v)}{\alpha(x,v)\rho(x,v) + 1 - \alpha(x,v)}, \quad (x,v) \in \partial_- O, \rho(x,v) > 0,
\end{equation}

\(\eta\) and \(\eta^H\) being the corresponding versions of \(\eta\) and \(\eta^H\) defined in (3.5).
extended to a continuous function on all of $\partial \cdot O$ (cf. Condition $A_M$). $C^{1,1}(\overline{O})$ can be taken as the common

domain $\mathcal{D}$ of $L^\prime$ and $B^\prime$. Note that $\overline{O}$ is the closure of $O \cup \partial_+ O$ and that $\partial_+ O$ is already closed.

With the above definitions, we have:

**PROPOSITION 3.4.** If (3.6) has a solution, $(X^\prime, V^\prime)$, well defined for all times, then $(X^\prime, V^\prime)$ is a solution of the constrained martingale problem for $(L^\prime, O \cup \partial_+ O, B^\prime, \partial_\cdot O)$ with initial condition $(X^0_0, V^0_0)$.

**Proof.** By setting

$$A^\prime([0, t] \times A) = \epsilon \int_0^t \mu_b(X(s), V(s-))^{-1} I_A(X(s), V(s-)) dN^\prime(s),$$

one can see that Definition 3.3 is verified.

We will now show that the constrained martingale problem for $(L^\prime, O \cup \partial_+ O, B^\prime, \partial_\cdot O)$ always has solutions. Following Kurtz (1990), solutions of the constrained martingale problem for $(L^\prime, O \cup \partial_+ O, B^\prime, \partial_\cdot O)$ will be obtained from solutions of the corresponding patchwork martingale problem. The definition of patchwork martingale problem, slightly modified to suit our present purposes, is recalled below.

**DEFINITION 3.5.** Let $E, E_0, E_1, \mathcal{E}_0$ and $\mathcal{E}_1$ be as in Definition 3.3. A triple $(Y, \lambda_0, \lambda_1)$ is a solution of the patchwork martingale problem for $(E_0, E_1, \mathcal{E}_0, \mathcal{E}_1)$ with initial condition $\mathcal{Y}_0$ if there is a filtration $\{\mathcal{G}_t\}$ such that $\mathcal{Y}$ is a $\{\mathcal{G}_t\}$ adapted process with sample paths in $D_E[0, \infty)$ and $\lambda_0$ and $\lambda_1$ are $\{\mathcal{G}_t\}$ adapted, nonnegative, nondecreasing processes satisfying

$$\lambda_0(t) + \lambda_1(t) = t, \quad t \geq 0,$$

such that

$$f(\mathcal{Y}(t)) - f(\mathcal{Y}_0) - \int_0^t \mathcal{E}_0 f(\mathcal{Y}(s)) d\lambda_0(s) - \int_0^t \mathcal{E}_1 f(\mathcal{Y}(s)) d\lambda_1(s)$$

is a $\{\mathcal{G}_t\}$ martingale for each $f \in \mathcal{D}$,

$$\int_0^t I_{\mathcal{F}^\prime}(\mathcal{Y}(s)) d\lambda_0(s) = \lambda_0(t),$$

and

$$\int_0^t I_{\mathcal{F}^\prime}(\mathcal{Y}(s)) d\lambda_1(s) = \lambda_1(t),$$

for $t \geq 0$.

Solutions of the patchwork martingale problem evolve according to the slowed down “clocks” $\lambda_0$ and $\lambda_1$. It is shown in Kurtz (1990) that, if a given solution of the patchwork martingale problem satisfies a Liapunov type condition, then, by looking at the solution at time $\gamma^\prime(t), \ t \geq 0$, where $\gamma^\prime$ is a generalized inverse of $\lambda_0^\prime$, one obtains a solution of the constrained martingale problem. The following theorem states that this is the case for any solution of the patchwork martingale problem for $(L^\prime, O \cup \partial_+ O, B^\prime, \partial_\cdot O)$.

**THEOREM 3.6.** Let $(Y^\prime, U^\prime, \lambda_0^\prime, \lambda_1^\prime)$ be a solution of the patchwork martingale problem for $(L^\prime, O \cup \partial_+ O, B^\prime, \partial_\cdot O)$ with initial condition $(X^0_0, V^0_0)$. Define

$$\gamma^\prime = \inf \{ s : \lambda_0^\prime(s) > t \},$$

and

$$X^\prime = Y^\prime \circ \gamma^\prime, \quad V^\prime = U^\prime \circ \gamma^\prime.$$

Then $(X^\prime, V^\prime)$ is a solution of the constrained martingale problem for $(L^\prime, O \cup \partial_+ O, B^\prime, \partial_\cdot O)$ with initial condition $(X^0_0, V^0_0)$. 
Proof. Let \( \psi : \mathbb{R} \to \mathbb{R}_+ \) be a nonincreasing, infinitely differentiable function that is identically equal to 1 on \( (-\infty, 0] \) and identically equal to 0 on \([1, +\infty)\). In addition let \( \mathcal{V} = B\mathcal{R}_0(0) \) and let \( r_0 \) be such that \( \inf_{d(x, \partial D) \leq r_0} |\nabla \varphi(x)| > 0 \). Consider

\[
\psi_b(x,v) \equiv \left( \frac{\nabla \varphi(x)}{|\nabla \varphi(x)|} , v \right) \psi(\frac{d(x,\partial D)}{r_0}) \psi(|v| - R_0).
\]

Then Lemma 1.9 and Proposition 2.2 in Kurtz (1990) apply with \( f \equiv \psi_b \). \( \square \)

Still following Kurtz (1990), we will now construct solutions to the patchwork martingale problem for \((L', O \cup \partial_+ O, B', \partial_- O)\). We will construct our solutions as limits of pure jump Markov processes. To this end we need functions \( \mu^{\epsilon,n}_b : \partial_- O \to [0, \infty) \) that satisfy, for each \( \epsilon > 0 \),

\[
\|\mu^{\epsilon,n}_b\| \equiv \sup_{(x,v) \in \partial_- O} \mu^{\epsilon,n}_b(x,v) < \infty \tag{3.20}
\]

\[\mu^{\epsilon,n}_b(x,v) \leq \mu_b(x,v), \quad n \in \mathbb{N}, \quad \lim_{n \to \infty} \frac{\|\mu^{\epsilon,n}_b\|}{n} = 0,\]

and kernels \( \eta^{\epsilon,n}_b : \partial_- O \to \mathcal{P}(\mathcal{V}) \) that satisfy

\[
\eta^{\epsilon,n}_b(x,v, \{v' : \langle v', \nu(x) \rangle > 0\}) = 1, \quad n \in \mathbb{N}, \tag{3.21}
\]

and, for \( f \in C^{1,1}(\overline{O}) \),

\[
\lim_{n \to \infty} \sup_{x,v \in \partial_- O} |R^{\epsilon,n}_b f(x,v) - R_b f(x,v)| = 0 \tag{3.22}
\]

\[
\sup_{x,v \in \partial_- O} |\mu^{\epsilon,n}_b(x,v)(R^{\epsilon,n}_b - I) f(x,v) - B f(x,v)| = 0,
\]

where \( R^{\epsilon,n}_b f(x,v) \equiv \int f(x,v')\eta^{\epsilon,n}_b(x,v, dv') \), and \( R_b \) and \( B \) are given by (3.4) and (3.9) respectively.

In order to satisfy (3.20)-(3.22), clearly we could take \( \mu^{\epsilon,n}_b \) and \( \eta^{\epsilon,n}_b \) independent of \( \epsilon \). However, in Sections 4, 5 and 6 we will need to consider pure jump Markov processes constructed with functions \( \mu^{\epsilon,n}_b \) and kernels \( \eta^{\epsilon,n}_b \) that satisfy also

\[
\lim_{\epsilon \to 0, n \to \infty} \frac{\|\mu^{\epsilon,n}_b\|}{n} = 0, \tag{3.23}
\]

and

\[
\lim_{\epsilon \to 0, n \to \infty} \sup_{x,v \in \partial_- O} |R^{\epsilon,n}_b f(x,v) - R_b f(x,v)| = 0 \tag{3.24}
\]

\[
\sup_{x,v \in \partial_- O} |\mu^{\epsilon,n}_b(x,v)(R^{\epsilon,n}_b - I) f(x,v) - B f(x,v)| = 0.
\]

Existence of functions and kernels with the desired properties is given by the following Lemma 3.7.

**Lemma 3.7.** Assume \((A_C)\), \((A_H)\) and \((A_M)\). Then there exist functions \( \mu^{\epsilon,n}_b, n \in \mathbb{N} \), and kernels \( \eta^{\epsilon,n}_b, n \in \mathbb{N} \), satisfying (3.20)-(3.22) and (3.23)-(3.24).

**Proof.** We shall prove the assertion first for a purely hot boundary \((\alpha \equiv 0, \eta_b = \eta^{\epsilon,n}_b\) satisfying \((A_H)\)) and for a purely cool boundary \((\alpha \equiv 1, \eta_b = \eta^{\epsilon,n}_b\) satisfying \((A_C)\)) first, and then for the general case.

For a hot boundary one can take directly

\[
\mu^{\epsilon,n}_b \equiv \mu_b \equiv 1, \quad \eta^{\epsilon,n}_b \equiv \eta_b. \tag{3.25}
\]

For a cool boundary, let \( \mathcal{V} = B\mathcal{R}_0(0) \) and let \( \Pi_V \) denote the normal projection on \( \mathcal{V} \). Define

\[
\mu^{\epsilon,n}_b(x,v) \equiv \left( \rho(x,v) \sqrt{\frac{1}{\epsilon n}} \right)^{-1}
\]

\[
\eta^{\epsilon,n}_b(x,v, \cdot) \equiv P \left( V^{\epsilon,n}_{x,v} \in \cdot \right),
\]

\[
V^{\epsilon,n}_{x,v} \equiv \Pi_V(\tilde{V}^{\epsilon,n}_{x,v}), \quad \tilde{V}^{\epsilon,n}_{x,v} \equiv v + \left( \rho(x,v) \sqrt{\frac{1}{\epsilon n}} \right) \zeta_{x,v}.
\]
Then (3.21), (3.20) and (3.23) are clearly satisfied. The arguments to show that (3.22) and (3.24) are verified are exactly the same, so we will prove only (3.24). To this end, it is enough to prove that, for every triple of sequences \( \{\epsilon_k\}, \{n_k\}, \{(x^k, v^k)\} \), such that \( \epsilon_k \to 0 \), \( n_k \to \infty \), \( \epsilon_k n_k \to \infty \), \( \{(x^k, v^k)\} \subseteq \partial O \), \( (x^k, v^k) \to (x, v) \), \( \rho(x, v) = 0 \), it holds

\[
\lim_{k \to \infty} |R_b^{x, n_k} f(x^k, v^k) - R_b f(x^k, v^k)| = 0,
\]

\[
\lim_{k \to \infty} \left| \mu_b^{\epsilon_k, n_k} (x^k, v^k) (R_b^{x, n_k} - I) f(x^k, v^k) - B f(x^k, v^k) \right| = 0.
\]

This follows easily if we prove that

\[
\lim_{k \to \infty} \mu_b^{\epsilon_k, n_k} (x^k, v^k) E \left[ |V_{x, v}^{\epsilon_k, n_k} - \tilde{V}_{x, v}^{\epsilon_k, n_k}| \right] = 0.
\]

Denote \( \mu_b^{\epsilon_k, n_k} \equiv \mu_b^{\epsilon_k, n_k} \), \( V_{x, v}^{\epsilon_k, n_k} \equiv V_{x, v}^{\epsilon_k, n_k} \), \( \tilde{V}_{x, v}^{\epsilon_k, n_k} \equiv \tilde{V}_{x, v}^{\epsilon_k, n_k} \). First, notice that the family of random variables

\[
\left\{ \mu_b(x^k, v^k) \left| V_{x, v}^{\epsilon_k, n_k} - \tilde{V}_{x, v}^{\epsilon_k, n_k} \right| \right\}
\]

is uniformly integrable because

\[
\mu_b(x^k, v^k) \left| V_{x, v}^{\epsilon_k, n_k} - \tilde{V}_{x, v}^{\epsilon_k, n_k} \right| = I \left\{ \left| \tilde{V}_{x, v}^{\epsilon_k, n_k} > R_0 \right\} \mu_b(x^k, v^k) \left( \tilde{V}_{x, v}^{\epsilon_k, n_k} - R_0 \right) \leq |\zeta_{x,v,v}|.
\]

In addition, we have

\[
\mu_b(x^k, v^k) \left| V_{x, v}^{\epsilon_k, n_k} - \tilde{V}_{x, v}^{\epsilon_k, n_k} \right| = I \left\{ \left| V_{x, v}^{\epsilon_k, n_k} > R_0 \right\} \frac{1}{\sqrt{\epsilon_k n_k}} \frac{\tilde{V}_{x, v}^{\epsilon_k, n_k}}{\tilde{V}_{x, v}^{\epsilon_k, n_k}} - R_0 \right\}^2 + R_0 \leq \frac{1}{2R_0} I \left\{ \left| V_{x, v}^{\epsilon_k, n_k} > R_0 \right\} \frac{1}{\sqrt{\epsilon_k n_k}} \left( \frac{1}{\sqrt{\epsilon_k n_k}} \right)^2 + 2|\zeta_{x,v,v}|^2 + 2|\zeta_{x,v,v}|^2 \right)\]

We know that \( \zeta_{x,v,v} \) converges in law to \( \zeta_{x,v} \). Therefore, if \( |v| < R_0 \), then \( I \left\{ \left| V_{x, v}^{\epsilon_k, n_k} > R_0 \right\} \right\} \) converges in law to \( 0 \) as does the right side of (3.27). If \( |v| = R_0 \), let us prove that it must be, almost surely, \( \langle \zeta_{x,v,v}, v \rangle \leq 0 \). To this end, let

\[
\tilde{v}^k = \left( v - \frac{1}{n_k} \rho(x) \right) \frac{R_0}{\sqrt{R_0^2 + 1/n_k}}.
\]

Then

\[
|\tilde{v}^k| = R_0, \quad \rho(x, \tilde{v}^k) > 0, \quad \tilde{v}^k \to v.
\]

Therefore

\[
\langle \zeta_{x,v,v}, v \rangle = \lim_{n \to \infty} \langle \zeta_{x,v,v}, \tilde{v}^k \rangle = \lim_{n \to \infty} \frac{(V_{x,v}^{\tilde{v}^k} - |\tilde{v}^k|^2)}{\rho(x, \tilde{v}^k)} \leq 0,
\]

and the right side of (3.27) must converge to zero.

Finally, we turn to the general case. If (1.6) holds, define

\[
\alpha^{x,n}(x, v) \equiv \alpha(x, v).
\]

If (1.7) holds, notice that the function

\[
\theta(x, v) \equiv \frac{1 - \alpha(x, v)}{\rho(x, v)}, \quad (x, v) \in \partial O : \quad \rho(x, v) > 0,
\]

\[
\alpha^{x,n}(x, v) \equiv \alpha(x, v).
\]
can be extended continuously to \( \partial_- O \), and define

\[
\alpha^{\epsilon,n}(x,v) \equiv 1 - \left( \rho(x,v) \vee \frac{1}{\sqrt{\epsilon n}} \right) \theta(x,v) \right) \vee 0.
\]

(3.30)

With these definitions, set

\[
\mu^{\epsilon,n}_b(x,v) \equiv \left[ \alpha^{\epsilon,n}(x,v) \rho(x,v) \right] + 1 - \alpha^{\epsilon,n}(x,v) \right) \theta(x,v) \right] - 1
\]

\[
\eta^{\epsilon,n}_b(x,v,) \equiv \alpha^{\epsilon,n}(x,v) \mu^{\epsilon,n}_b(x,v) + 1 - \alpha^{\epsilon,n}(x,v) \right) \eta^C(x,v,).\]

where \( \eta^{C,\epsilon,n}_b \) is defined by (3.26) with \( \eta_b = \eta^C_b \). We denote

\[
R_b f(x,v) \equiv \int f(x,v') \eta_b(x,v,dv')
\]

and

\[
R^{\epsilon,n}_b f(x,v) \equiv \int f(x,v') \eta^{\epsilon,n}_b(x,v,dv')
\]

as before, and, in addition,

\[
R^H_b f(x,v) \equiv \int f(x,v') \eta^H_b(x,v,dv')
\]

\[
R^C_b f(x,v) \equiv \int f(x,v') \eta^C_b(x,v,dv')
\]

\[
R^{C,\epsilon,n}_b f(x,v) \equiv \int f(x,v') \eta^{C,\epsilon,n}_b(x,v,dv')
\]

Then (3.21) is satisfied because

\[
\min \{ \alpha^{\epsilon,n}(x,v), 1 - \alpha^{\epsilon,n}(x,v) \} > 0, \quad \forall (x,v) \in \partial_- O.
\]

(3.20) and (3.23) are satisfied because, for \( \epsilon n \geq 1 \), for all \((x,v) \in \partial_- O\),

\[
\rho(x,v) \leq 1 \Rightarrow \alpha(x,v) \geq 0 \quad \forall (x,v) \in \partial_- O.
\]

\[
\rho(x,v) > 1 \Rightarrow \alpha(x,v) \geq 1 \geq \frac{\rho(x,v)}{\|\rho\|}.
\]

so that for \( \epsilon n \geq 1 \),

\[
\mu^{\epsilon,n}_b(x,v) \leq (\|\rho\| \vee 1) \left( \rho(x,v) \vee \frac{1}{\sqrt{\epsilon n}} \right) ^{-1}, \quad (x,v) \in \partial_- O.
\]

(3.31)

Finally, in order to verify (3.22) and (3.24), notice that we have

\[
|R^{\epsilon,n}_b f(x,v) - R_b f(x,v)| \leq \alpha^{\epsilon,n}(x,v) \left| R^{C,\epsilon,n}_b f(x,v) - R^C_b f(x,v) \right| + 2\|f\| \|\alpha^{\epsilon,n}(x,v) - \alpha(x,v)\|.
\]

Setting

\[
\beta^{\epsilon,n}(x,v) \equiv \frac{\alpha^{\epsilon,n}(x,v)}{\alpha^{\epsilon,n}(x,v) \left( \rho(x,v) \vee \frac{1}{\sqrt{\epsilon n}} \right) ^{-1} + 1 - \alpha^{\epsilon,n}(x,v)}.
\]
we have also, taking into account (3.11),
\[|p_\epsilon^n(x,v)(R_\epsilon^n - I)f(x,v) - Bf(x,v)|\]
\[\leq \beta^{\epsilon,n}(x,v) \left( (\rho(x,v) + \frac{1}{\sqrt{\epsilon_n}}) \left( \left(R_\epsilon^nC^{\epsilon,n} - I\right)f(x,v) - (\rho(x,v))^{-1}(R_\epsilon^n - I)f(x,v)\right)\right)\]
\[+ c_\delta \|\nabla f\| \|\beta^{\epsilon,n}(x,v) - \beta(x,v)\|.
\]

The assertion then follows by the fact that, if either (1.6) or (1.7) holds,
\[\lim_{\epsilon,n \to \infty} \sup_{(x,v) \in \partial \partial O} |\alpha^{\epsilon,n}(x,v) - \alpha(x,v)| = 0, \quad \lim_{\epsilon,n \to \infty} \sup_{(x,v) \in \partial \partial O} |\beta^{\epsilon,n}(x,v) - \beta(x,v)| = 0, \quad \forall \epsilon > 0,
\]
and
\[\lim_{\epsilon,n \to \infty} \sup_{(x,v) \in \partial \partial O} |\alpha^{\epsilon,n}(x,v) - \alpha(x,v)| = 0, \quad \lim_{\epsilon,n \to \infty} \sup_{(x,v) \in \partial \partial O} |\beta^{\epsilon,n}(x,v) - \beta(x,v)| = 0.
\]

For \(x \text{ and } v\) such that \(x - tv \in \bar{D}\) for some \(t \geq 0\), let
\[p(x,v) \equiv x - t_0v, \quad \text{where } t_0 = \inf\{t \geq 0 : x - tv \in \bar{D}\}.
\]
(In particular, for \(x \in \bar{D}, p(x,v) = x\).) Let
\[O \equiv \{(x,v) : p(x,v) \text{ is defined and } |p(x,v) - x| \leq |v|\}.
\]
The set \(O\) is compact. The function \(p\) is continuous only if \(D\) is strictly convex, but this has no relevance for our purposes. Let
\[\hat{O} \equiv O - O \cup \partial_+ O,
\]
and for \((x,v) \in \hat{O},\) define
\[B^{\epsilon,n}f(x,v) = \epsilon^{-1} p_\epsilon^{\epsilon,n}(p(x,v),v)[R_\epsilon^n f(p(x,v),v) - f(x,v)]\]
Note that for \((x,v) \in \hat{O}, (p(x,v),v) \in \partial_+ O. \) For \((x,v) \in \partial \hat{O}\) define
\[L^{\epsilon,n}f(x,v) = \epsilon^{-1} n((f(x + \frac{1}{n}v,v) - f(x,v)) + \epsilon^{-2} Qf(x,v).
\]
Finally, define
\[C^{\epsilon,n}f(x,v) = I_{O \cup \partial_+ O} (x,v)L^{\epsilon,n}f(x,v) + I_{\hat{O}}(x,v)B^{\epsilon,n}f(x,v).
\]
Since \(C^{\epsilon,n}\) is a bounded operator on the space of bounded measurable functions on \(O\), for any initial distribution the martingale problem for \(C^{\epsilon,n}\) has a unique solution.

**Theorem 3.8.** Let \((Y^{\epsilon,n},U^{\epsilon,n})\) be the solution of the martingale problem for \(C^{\epsilon,n}\) with initial condition \((X_0^{\epsilon,n},V_0^{\epsilon,n})\) in \(O\). Define
\[\lambda_0^{\epsilon,n}(t) = \int_0^t I_{O \cup \partial_+ O}(Y^{\epsilon,n}(s),U^{\epsilon,n}(s)) ds, \quad \lambda_1^{\epsilon,n}(t) = \int_0^t I_{\hat{O}}(Y^{\epsilon,n}(s),U^{\epsilon,n}(s)) ds.
\]
Then, for every \(\epsilon > 0\), \(\{(Y^{\epsilon,n},U^{\epsilon,n},\lambda_0^{\epsilon,n},\lambda_1^{\epsilon,n})\}\) is relatively compact as \(n\) goes to infinity, and any limit point \((Y^e,U^e,\lambda_0^e,\lambda_1^e)\) is a solution of the patchwork martingale problem for \((L^e,O \cup \partial_+ O,B^e,\partial_+ O)\) with initial condition \((X_0^e,V_0^e)\).

**Proof.** To simplify notation, let us omit the superscript \(\epsilon\). The proof is essentially the same as for Lemma 1.1 of Kurtz (1990). First, note that \(Y^n(t) \notin \bar{D}\) only if it has just jumped from inside \(\bar{D}\) by \(\frac{1}{n}U^n(t)\) and hence, for all possible values of \(Y^n(t),\)
\[|p(Y^n(t),U^n(t)) - Y^n(t)| \leq \frac{1}{n}|U^n(t)|.
\]
It follows that
\[
\begin{align*}
|I_O(Y^n(t),U^n(t))B^n f(Y^n(t),U^n(t))| \\
\leq \epsilon^{-1}(c_b \|\nabla_v f\| + \sup_{(x,v)\in \partial_-O} |\mu^n_b(x,v) (R^n_b - I)f(x,v) - Bf(x,v)| + \frac{\|\mu^n_b\|}{n} \|\nabla_x f\| \sup_{v \in \mathcal{V}} |v|),
\end{align*}
\]
where \(c_b\) is defined in (3.10). This estimate along with the fact that \(\sup_n \|L^n f\| < \infty\), for \(f \in C^{1,1}_{\delta}(\mathbb{R}^d \times \mathbb{R}^d)\), gives the relative compactness of \(\langle Y^n, U^n \rangle\) by Theorems 3.9.1 and 3.9.4 of Ethier and Kurtz (1986). Then by (3.38), \(\{\langle Y^n, U^n, p(Y^n, U^n) \rangle\}\) is relatively compact and all limit points are of the form \((Y, U, \lambda)\). For every \(f \in C^{1,1}_{\delta}(\mathbb{R}^d \times \mathbb{R}^d)\), we have
\[
f(Y^n(t),U^n(t)) - f(X_0, V_0) - \int_0^t L^n f(Y^n(s),U^n(s))d\lambda^n_b(s) \\
- \int_0^t B^n f(Y^n(s),U^n(s))d\lambda^n_b(s) \\
= f(Y^n(t),U^n(t)) - f(X_0, V_0) - \int_0^t Lf(Y^n(s),U^n(s))d\lambda^n_b(s) \\
- \int_0^t Bf(p(Y^n(s),U^n(s)),U^n(s))d\lambda^n_b(s) + \delta^n(t),
\]
where \(\delta^n(t)\) is bounded by \(t\) times
\[
\|L^n f - Lf\| + \epsilon^{-1}(\sup_{(x,v)\in \partial_-O} |\mu^n_b(x,v) (R^n_b - I)f(x,v) - Bf(x,v)| + \frac{\|\mu^n_b\|}{n} \|\nabla_x f\| \sup_{v \in \mathcal{V}} |v|).
\]
It follows that for any limit point \((Y, U, \lambda_0, \lambda_1)\) of \(\{\langle Y^n, U^n, \lambda^n_0, \lambda^n_1 \rangle\}\),
\[
f(Y(t),U(t)) - f(X(0), V(0)) - \int_0^t Lf(Y(s),U(s))d\lambda_0(s) \\
- \int_0^t Bf(Y(s),U(s))d\lambda_1(s)
\]
is a martingale and hence \((Y, U, \lambda_0, \lambda_1)\) is a solution of the patchwork martingale problem for \((L, O \cup \partial_+ O, B, \partial_- O)\).

\[\Box\]

4. Hot boundary reflection. Throughout this section, we will use the same notation as in Section 3.

In Section 3, we constructed the reflecting transport process \((X^\epsilon, V^\epsilon)\) that evolves according to the operator \(L^\epsilon\) defined in (1.1), when \(X^\epsilon\) is in the interior of the domain \(D\), and reflects according to a transition function \(\eta_\epsilon(x, v, \cdot)\) of the form (1.2), when \(X^\epsilon\) hits the boundary. \((X^\epsilon, V^\epsilon)\) is a solution of the constrained martingale problem for \((L^\epsilon, O \cup \partial_+ O, B^\epsilon, \partial_- O)\), where \(B^\epsilon\) is defined in (3.9), and is obtained, by the random time change (3.18)-(3.19), from a process \((Y^\epsilon, U^\epsilon, \lambda^\epsilon_0, \lambda^\epsilon_1)\), which is a solution of the patchwork martingale problem for \((L^\epsilon, O \cup \partial_+ O, B^\epsilon, \partial_- O)\) and a limit point of the processes \((Y^{\epsilon,n}, U^{\epsilon,n}, \lambda^{\epsilon,n}_0, \lambda^{\epsilon,n}_1)\) defined in Theorem 3.8. In this section we prove convergence of \(X^\epsilon\), as the scaling parameter \(\epsilon\) goes to 0, to a reflecting diffusion process, for a hot boundary, i.e. when \(\alpha \equiv 0\) and hence \(\eta_\epsilon \equiv \eta_0^H\) satisfies \((AH)\) (Theorem 1.3). We assume that \((A_0)\) to \((A_3)\) hold.

As anticipated in the Introduction, we will derive the convergence of \(X^\epsilon\) from the convergence of \((X^\epsilon, \lambda^\epsilon_0, \lambda^\epsilon_1)\) to the solution of a limiting patchwork martingale problem (Theorem 4.6). In fact \((Y^\epsilon, \lambda^\epsilon_0, \lambda^\epsilon_1)\) is more easily controllable because \(Y^\epsilon\) is slowed down on the boundary and \(\lambda^\epsilon_0, \lambda^\epsilon_1\) are Lipschitz continuous.

We follow an analogous approach to Section 2: We introduce the occupation measures \(\Gamma^\epsilon_0\) and \(\Gamma^\epsilon_1\) defined by
\[
\Gamma^\epsilon_0([0, t] \times A) = \int_0^t I_A(U^\epsilon(s))d\lambda^\epsilon_0(s), \quad \Gamma^\epsilon_1([0, t] \times A) = \int_0^t I_A(U^\epsilon(s))d\lambda^\epsilon_1(s),
\]

\[(4.1)\]
for every Borel set $A \subseteq \mathcal{V}$ and $t \geq 0$.

Relative compactness of $\{(Y^\epsilon, \Gamma_0^\epsilon, \Gamma_1^\epsilon)\}$ is proved in essentially the same way as relative compactness of $\{(X^\epsilon, \Gamma^\epsilon)\}$ in Section 2 (Lemmas 4.1 and 4.2). The limit points of $\{\Gamma_0^\epsilon\}$ are also identified by the same argument used to identify the limit points of $\{\Gamma^\epsilon\}$ in Lemma 2.4 (Lemma 4.2). Instead, most of the work in this section lies in identifying the limit points of $\{\Gamma_1^\epsilon\}$ (Lemma 4.5). In order to do this, it turns out to be convenient to view any limit point of $\{(Y^\epsilon, \Gamma_0^\epsilon, \Gamma_1^\epsilon)\}$ as a limit point, as $\epsilon$ goes to 0 and $n\epsilon$ goes to infinity, of $(Y^{\epsilon,n}, \Gamma_0^{\epsilon,n}, \Gamma_1^{\epsilon,n})$ (where $\Gamma_0^{\epsilon,n}$ and $\Gamma_1^{\epsilon,n}$ are defined by (4.6). Then a similar argument to the one of Lemma 2.4 can be used, but first it must be proved that there exists a limiting distribution of the exit velocities $U^{\epsilon,n}(\tau^{\epsilon,n})$ (Lemmas 4.3 and 4.4).

**Lemma 4.1.** If $\{X_0^\epsilon\}$ is relatively compact as $\epsilon \to 0$, then $\{Y^\epsilon\}$ is relatively compact in $C_D[0, \infty)$ as $\epsilon \to 0$.

**Proof.** Using the assumption that $U^\epsilon$ takes values in a compact set, the proof is essentially the same as the proof of Lemma 2.3 For $f \in C^2_{cb}(\mathbb{R}^2)$ let

$$f^\epsilon(x, v) = f(x) + \epsilon(h(x, v), \nabla f(x)),$$

where $h$ is the function introduced in $(H_2)$. Then we have

$$L^\epsilon f^\epsilon = Lf, \quad B^\epsilon f^\epsilon = (Bh, \nabla f),$$

where $L$ and $B$ are defined in 2.13 and 3.9 respectively. Therefore

$$f(Y^\epsilon(t)) + \epsilon \langle h(Y^\epsilon(t), U^\epsilon(t)), \nabla f(Y^\epsilon(t)) \rangle - \int_0^t Lf(Y^\epsilon(s), U^\epsilon(s))d\lambda_0^\epsilon(s) + \int_0^t (Bh(Y^\epsilon(s), U^\epsilon(s)), \nabla f(Y^\epsilon(s)))d\lambda_1^\epsilon(s)$$

is a martingale, and the assertion follows from Theorems 3.9.1 and 3.9.4 in Ethier and Kurtz (1986).

**Lemma 4.2.** The families $\{\Gamma_0^\epsilon\}$ and $\{\Gamma_1^\epsilon\}$ are relatively compact in $\mathcal{L}(\mathcal{V})$ as $\epsilon \to 0$.

For every limit point $(Y, \Gamma_0, \Gamma_1, \lambda_0, \lambda_1)$, as $\epsilon \to 0$, of $\{(Y^\epsilon, \Gamma_0^\epsilon, \Gamma_1^\epsilon, \lambda_0^\epsilon, \lambda_1^\epsilon)\}$, $\Gamma_0$ has the form

$$\Gamma_0([0, t] \times A) = \int_0^t \pi(Y(s), A)d\lambda_0(s),$$

where $\pi$ is defined in $(H_1)$.

**Proof.** $\{\Gamma_0^\epsilon\}$ and $\{\Gamma_1^\epsilon\}$ are relatively compact by the compactness of $\mathcal{V}$ and the fact that for each $t$, they are uniformly bounded on $[0, t] \times \mathcal{V}$. The proof of the representation for $\Gamma_0$ is exactly the same as in Lemma 2.4.

As far as $\Gamma_1$ is concerned, we are going to prove that

$$\Gamma_1([0, t] \times A) = \int_0^t \pi_b(Y(s), A)d\lambda_1(s)$$

where $\pi_b$ is defined in $(H_3)$, by showing that, almost surely, for every $t_0 > 0$,

$$\int_{[0, t_0] \times \mathcal{V}} [R_{t_0} T - I]f(Y(s), v)\Gamma_1(ds \times dv) = 0,$$

where the operator $Tf$ is defined by (1.13). In order to do this, we would like to claim that

$$T^\epsilon f(Y^\epsilon(t), U^\epsilon(t)) - T^\epsilon f(Y^\epsilon(0), U^\epsilon(0)) - \epsilon^{-1} \int_0^t B T^\epsilon f(Y^\epsilon(s), U^\epsilon(s))d\lambda_1^\epsilon(s),$$

where

$$T^\epsilon f(x, v) = E[f(Y^\epsilon(\tau^\epsilon), U^\epsilon(\tau^\epsilon))|Y^\epsilon(0) = x, U^\epsilon(0) = v]$$
\[ \tau^e = \inf \{ t > 0 : \lambda^e_t(t) > 0 \}, \]

is a martingale since, formally at least, \( L^T f(x, v) = 0 \) for \((x, v) \in O \cup \partial_v O \). But \( T f \) need not be differentiable and hence, may not be in the domain of \( L^e \). To avoid this technicality, we approximate \((Y, \Gamma_0, \Gamma_1, \lambda_0, \lambda_1)\) directly by the stochastic processes \( \{(Y^{\epsilon_n}, \Gamma_0^{\epsilon_n}, \Gamma_1^{\epsilon_n}, \lambda_0^{\epsilon_n}, \lambda_1^{\epsilon_n})\} \), where \((Y^{\epsilon_n}, U^{\epsilon_n}, \lambda_0^{\epsilon_n}, \lambda_1^{\epsilon_n})\) is the stochastic process of Theorem 3.8 and \( \Gamma_i^{\epsilon_n}, \ i = 0, 1 \), is given by

\[ \Gamma_i^{\epsilon_n} = \int_0^t I_A(U^{\epsilon_n}(s)) d\lambda_i^{\epsilon_n}(s), \ i = 0, 1. \]

By Theorem 3.8, for each \( \epsilon > 0 \), \( \{(Y^{\epsilon_n}, \Gamma_0^{\epsilon_n}, \Gamma_1^{\epsilon_n}, \lambda_0^{\epsilon_n}, \lambda_1^{\epsilon_n})\} \) is the limit, as \( n \to \infty \), of \( \{(Y^{\epsilon_n}, \Gamma_0^{\epsilon_n}, \Gamma_1^{\epsilon_n}, \lambda_0^{\epsilon_n}, \lambda_1^{\epsilon_n})\} \). As weak convergence is metrizable by the Prohorov metric, one can always find sequences \( \{\epsilon_k\} \) and \( \{n_k\} \) such that \( \epsilon_k \to 0 \) and \( n_k \to \infty \), and

\[ (Y^{\epsilon_k,n_k}, \Gamma_0^{\epsilon_k,n_k}, \Gamma_1^{\epsilon_k,n_k}, \lambda_0^{\epsilon_k,n_k}, \lambda_1^{\epsilon_k,n_k}) \Rightarrow (Y, \Gamma_0, \Gamma_1, \lambda_0, \lambda_1), \]

de fining that \((Y, \Gamma_0, \Gamma_1, \lambda_0, \lambda_1)\) is a limit point of \( \{(Y^{\epsilon_n}, \Gamma_0^{\epsilon_n}, \Gamma_1^{\epsilon_n}, \lambda_0^{\epsilon_n}, \lambda_1^{\epsilon_n})\} \) as \( n \to \infty \), \( \epsilon \to 0 \) and \( n \epsilon \to \infty \).

For \((x, v) \in O \cup \partial_v O \), let \((Z^n, V^n)\) be a realization of the solution to the martingale problem for the operator \( L^n \) defined in (1.12), with initial condition \((0, v)\). Let \((\tilde{X}^{\epsilon_n}, \tilde{V}^{\epsilon_n})\) be a realization of the solution to the martingale problem for the operator \( L^{\epsilon_n} \) defined in (3.36) with initial condition \((x^{\epsilon_n}, v^{\epsilon_n})\), \((x^{\epsilon_n}, v^{\epsilon_n}) \to (x, v)\) as \( \epsilon \to 0 \), \( n \to \infty \) and \( n \epsilon \to \infty \).

Define

\[ (Z^{\epsilon_n}(t), \tilde{V}^{\epsilon_n}(t)) \equiv (\epsilon^{-1}(\tilde{X}^{\epsilon_n}(\epsilon^2 t) - x^{\epsilon_n}), \tilde{V}^{\epsilon_n}(\epsilon^2 t)). \]

**Lemma 4.3.** If \( \epsilon \to 0 \), \( n \to \infty \) and \( n \epsilon \to \infty \), \((Z^{\epsilon_n}, \tilde{V}^{\epsilon_n}) \Rightarrow (Z^n, V^n)\).

**Proof.** The family \( \{(Z^{\epsilon_n}, V^{\epsilon_n})\} \) satisfies the compact containment condition and, for every \( f \in C_b^1(R^d \times R^d) \),

\[ f(Z^{\epsilon_n}(t), \tilde{V}^{\epsilon_n}(t)) = \int_0^t \tilde{L}^{\epsilon_n} f(Z^{\epsilon_n}(s), \tilde{V}^{\epsilon_n}(s)) ds, \]

where

\[ \tilde{L}^{\epsilon_n} f(z, v) = \epsilon n[f(z + \frac{1}{\epsilon n} v, v) - f(z, v)] + \mu(x^{\epsilon_n} + \epsilon z, v) \int [f(z, v') - f(z, v)] \eta(x^{\epsilon_n} + \epsilon z, v, dv'), \]

is a martingale. Since \( \sup_{\epsilon_n} \| \tilde{L}^{\epsilon_n} f \| < \infty \), Theorems 9.4 and 9.1 in Chapter 3 of Ethier and Kurtz (1986) yield that \( \{(Z^{\epsilon_n}, \tilde{V}^{\epsilon_n})\} \) is relatively compact. Moreover, every limit point as \( \epsilon \to 0 \), \( n \to \infty \) and \( n \epsilon \to \infty \) is a solution of the martingale problem for \( L^n \) with initial condition \((0, v)\), hence equals in law \((Z^n, V^n)\), because the martingale problem for \( L^n \) is well posed. \( \square \)

Let \((x, v) \in \partial_v O \) and define

\[ H^n \equiv \{ z : \langle z, v(x) \rangle > 0 \}, \]

\[ \tau_0^n \equiv \inf \{ t > 0 : Z^n(t) \notin H^n \}, \quad \tau_1^n \equiv \inf \{ t > 0 : Z^n(t) \notin \overline{T} \}, \]

\[ \tau_0^{\epsilon,n} \equiv \inf \{ t > 0 : \tilde{X}^{\epsilon,n}(t) \notin D \}, \quad \tau_1^{\epsilon,n} \equiv \inf \{ t > 0 : \tilde{X}^{\epsilon,n}(t) \notin \overline{D} \}, \]

In addition, let

\[ \tau^{\epsilon,n} \equiv \inf \{ t > 0 : (Y^{\epsilon,n}(t), U^{\epsilon,n}(t)) \notin O \cup \partial_v O \}. \]

Then (Lemma 5.16, chapter 4 of Ethier and Kurtz (1986)) \((Y^{\epsilon,n}, U^{\epsilon,n})\) agrees with \((\tilde{X}^{\epsilon,n}, \tilde{V}^{\epsilon,n})\) for \( t \leq \tau^{\epsilon,n} \), and hence \( \tau_0^{\epsilon,n} \leq \tau^{\epsilon,n} \leq \tau_1^{\epsilon,n} \).
LEMMA 4.4. Suppose that \( a(x) \), defined in (1.10), satisfies

\[
\nu^T(x)a(x)\nu(x) > 0.
\]

Then \( \tau^* \) is \( \tau^*_1 \) and \( V^T(\tau^*_0) \cdot \nu(x) < 0 \) almost surely. If, in addition, \( \epsilon \to 0 \), \( n \to \infty \), \( n \epsilon_n \to \infty \), \( -\epsilon \partial D \to 0 \), then \( (Y^{\epsilon,n}(x,v),U^{\epsilon,n}(x,v)) = (\hat{X}^{\epsilon,n},\hat{V}^{\epsilon,n}) \Rightarrow (x,V^T(\tau^*_0)) \).

**Proof.** By Theorem 1.2, \( \epsilon Z^{\epsilon}(\epsilon^{-2}) \) converges in law to the diffusion process with initial condition 0 and generator

\[
\mathcal{L}^n f(z) = \sum_i b_i(x) \frac{\partial f}{\partial z_i}(z) + \frac{1}{2} \sum_{i,j} a_{i,j}(x) \frac{\partial^2 f}{\partial z_i \partial z_j}(z),
\]

where \( b(x) \) and \( a(x) \) are defined in (1.10). The assumption that \( \nu^T(x)a(x)\nu(x) > 0 \) implies that the exit time from \( \overline{H}^n \) of the diffusion defined by (4.8) is almost surely zero, and therefore the exit time from \( \overline{H}^n \) of \( \epsilon Z^{\epsilon}(\epsilon^{-2}) \), call it \( \tau^* \), satisfies \( \tau^* \to 0 \). But \( \tau^* = \epsilon^2 \tau^*_1 \) and hence \( \tau^*_0 \) and \( \tau^*_1 \) are almost surely finite. Since \( V^\epsilon \) will change only finitely often in any finite time interval and the probability that \( V^\epsilon \) changes exactly at time \( \tau^*_0 \) is zero, we must have \( (Y^{\epsilon,n}(\tau^*_0),U(v(x))) < 0 \) and \( \tau^*_0 = \tau^*_1 \) almost surely.

Note that \( \epsilon^{-2}\tau^*_0 \) is the exit time of \( Z^{\epsilon,n} \) from the domain \( D^{\epsilon,n} = \{ z : x^{\epsilon,n} + \epsilon z \in D \} \). The boundary of \( D^{\epsilon,n} \) converges to the boundary of \( H^n \) uniformly in any ball centered at \( x \), that is

\[
\sup_{z \in \partial D^{\epsilon,n}(x)} \| d(z,\partial H^n) + \sup_{z \in \partial D^{\epsilon,n}(x)} d(z,\partial D^{\epsilon,n}) \to 0.
\]

As \( \tau^*_0 = \tau^*_1 \) almost surely, we have \( \epsilon^{-2}\tau^*_0 \to \tau^*_0 \) and \( \epsilon^{-2}\tau^*_1 \to \tau^*_0 \), which together yield \( \epsilon^{-2}\tau^* \to \tau^*_0 \). Since the probability of \( V^\epsilon \) jumping exactly at time \( \tau^*_0 \) is zero, \( (Y^{\epsilon,n}(\tau^*_0),U^{\epsilon,n}(\tau^*_0)) = (x^\epsilon + \epsilon Z^{\epsilon,n}(\epsilon^{-2}\tau^*),\hat{V}^{\epsilon,n}(\epsilon^{-2}\tau^*)) \) converges in law to \( (x,V^T(\tau^*_0)) \).

For \( f \in B(O) \), define

\[
T^{\epsilon,n} f(x,v) = E[f(Y^{\epsilon,n}(\tau^*_0),U^{\epsilon,n}(\tau^*_0))|Y^{\epsilon,n}(0) = x, U^{\epsilon,n}(0) = v]
\]

and recall that, for \( f \in C(\hat{O}) \),

\[
T f(x,v) = E[f(x,V^T(\tau^*))|V^T(0) = v].
\]

**LEMMA 4.5.** For every limit point \( (Y,\Gamma,\lambda_0,\lambda_1) \), as \( \epsilon \to 0 \), of \( \{ (Y,\Gamma,\lambda_0,\lambda_1) \} \), \( \Gamma_1 \) has the form (4.4).

**Proof.** As discussed immediately before Lemma 4.3, every limit point \( (Y,\Gamma,\lambda_0,\lambda_1) \), as \( \epsilon \to 0 \), of \( \{ (Y,\Gamma,\lambda_0,\lambda_1) \} \) is also a limit point, as \( \epsilon \to 0 \), \( n \to \infty \) and \( n \epsilon_n \to \infty \), of \( \{ (Y^{\epsilon,n},\Gamma^{\epsilon,n},\lambda_0^{\epsilon,n},\lambda_1^{\epsilon,n}) \} \).

For every \( f \in C_0(\mathbb{R}^d \times \mathbb{R}^d) \),

\[
e^{T^{\epsilon,n}} f(Y^{\epsilon,n}(t),U^{\epsilon,n}(t)) - e^{T^{\epsilon,n}} f(Y^{\epsilon,n}(0),U^{\epsilon,n}(0))
- \int_0^t e^{T^{\epsilon,n}} f(Y^{\epsilon,n}(s),U^{\epsilon,n}(s))d\lambda^{\epsilon,n}_s(s)
= e^{T^{\epsilon,n}} f(Y^{\epsilon,n}(t),U^{\epsilon,n}(t)) - e^{T^{\epsilon,n}} f(Y^{\epsilon,n}(0),U^{\epsilon,n}(0))
- \int_0^t (R_0^{T^{\epsilon,n}} f(p(Y^{\epsilon,n}(s),U^{\epsilon,n}(s)),U^{\epsilon,n}(s)) - f(Y^{\epsilon,n}(s),U^{\epsilon,n}(s)))d\lambda^{\epsilon,n}_s(s)
\]

is a martingale, since \( L^{\epsilon,n} e^{T^{\epsilon,n}} f(x,v) = 0 \) for \( (x,v) \in O \cup \partial O \) (\( L^{\epsilon,n} \) is the operator defined by (3.36), which describes the evolution of \( (Y^{\epsilon,n},U^{\epsilon,n}) \) in \( O \)).

Considering a subsequence if necessary, we can assume that \( \{ (Y^{\epsilon,n},\Gamma^{\epsilon,n},\lambda_0^{\epsilon,n},\lambda_1^{\epsilon,n}) \} \to (Y,\Gamma,\lambda_0,\lambda_1) \) as \( n \to \infty \), \( \epsilon \to 0 \) and \( n \epsilon_n \to \infty \). By Lemma 4.4, for each \( \alpha > 0 \), \( T^{\epsilon,n} f \) converges to \( Tf \) uniformly over \( \{ (x,v) : x \in \partial D, \langle x,v(x) \rangle \geq \alpha \} \). On the other hand, by (A2), \( \{ \eta \beta(x,v,\cdot) \} \), \( (x,v) \in \partial \hat{O} \) is relatively compact as a family of measures on \( \partial \hat{O} \). This observation together with the fact that \( T^{\epsilon,n} f \) is uniformly
bounded by $\|f\|$, implies that $R_{0}T_{\epsilon,n}f$ converges to $R_{0}Tf$ uniformly over $\partial_{-}O$. Therefore, by the same argument as in Lemma 2.4,

$$\lim_{n \to \infty, \epsilon \to 0} \int_{[0,1] \times \mathbb{R}^{d}} (R_{0}Tf(p(Y_{\epsilon,x}^{n}(s), v), v) - f(Y_{\epsilon,x}^{n}(s), v)) \Gamma_{1}^{n}(ds \times dv) = 0,$$

almost surely. By observing that $Tf$ and, hence, $R_{0}Tf$ are continuous, one obtains (4.5). Then the assertion follows by Proposition 4.9.2 in Ethier and Kurtz (1986), as in Lemma 2.4. □

We are now ready to prove the main results of this section.

**Theorem 4.6.**

Suppose $\nu(x)f^{T}a(x)\nu(x) > 0$ for all $x \in \partial D$. Let $\mathcal{L}$ be the operator defined by (1.11), with domain $C^{2}(\mathcal{D})$, and let $\gamma$ be the vector field on $\partial D$ defined by (1.15). Then, if $\{X_{\epsilon}\}$ is relatively compact, any limit point, as $\epsilon \to 0$, of $\{(Y_{\epsilon,x}^{n}, \lambda_{0}^{n}, \lambda_{1}^{n})\}$ is a solution of the patchwork martingale problem for $(\mathcal{L}, \mathcal{D}, \langle \gamma, \nabla \rangle, \partial D)$.

Suppose $a(x)$ is strictly positive definite for every $x \in \mathcal{D}$, and $\gamma$ is Lipschitz continuous and satisfies (1.16), i.e.

$$\inf_{x \in \partial D} \langle \gamma(x), \nu(x) \rangle > 0.$$

Then, for any initial condition $X_{0}$, $X_{0} \in \mathcal{D}$ almost surely, there is a unique solution, $(Y, \lambda_{0}, \lambda_{1})$ to the patchwork martingale problem for $(\mathcal{L}, \mathcal{D}, \langle \gamma, \nabla \rangle, \partial D)$ and $\lambda_{0}$ is strictly increasing and diverging to infinity, almost surely. If $X_{0}^{\epsilon} \to X_{0}$ in law as $\epsilon \to 0$, then $Y^{\epsilon} \to Y$ in law.

**Proof.** As in the proof of Theorem 2.3, for $f \in C^{2}(\mathcal{D})$, let

$$f^{\epsilon}(x, v) = f(x) + \epsilon \partial f(x)h(x, v).$$

Then, with $L$ defined by (2.13),

$$f(Y^{\epsilon}(t) + \epsilon \partial f(Y^{\epsilon}(t))h(Y^{\epsilon}(t), U^{\epsilon}(t)) - \int_{[0,t] \times V} Lf(Y^{\epsilon}(s), v) \Gamma_{0}^{\epsilon}(ds \times dv)$$

$$- \int_{[0,t] \times V} \partial f(Y^{\epsilon}(t)) Bh(Y^{\epsilon}(s), v) \Gamma_{1}^{\epsilon}(ds \times dv)$$

is a martingale. Let $(Y, \lambda_{0}, \lambda_{1})$ be a limit point of $(Y^{\epsilon}, \lambda_{0}^{\epsilon}, \lambda_{1}^{\epsilon})$. Then $(Y, \Gamma_{0}, \Gamma_{1}, \lambda_{0}, \lambda_{1})$, where $\Gamma_{0}$ and $\Gamma_{1}$ are given by (4.3) and (4.4), is a limit point of $(Y^{\epsilon}, \Gamma_{0}^{\epsilon}, \Gamma_{1}^{\epsilon}, \lambda_{0}^{\epsilon}, \lambda_{1}^{\epsilon})$.

$$f(Y(t)) - \int_{[0,t] \times V} Lf(Y(s), v) \Gamma_{0}(ds \times dv)$$

$$- \int_{[0,t] \times V} \partial f(Y(t)) Bh(Y(s), v) \Gamma_{1}(ds \times dv)$$

is a martingale and the assertion follows by Lemmas 4.2 and 4.5.

To prove the second part of the theorem, apply Lemma 1.8 in Kurtz (1990) with

$$\varphi_{n}(x) = 1 - \exp \left\{ \frac{1}{n\varphi(x)} - 1 \right\} I_{(-\infty, 1]}(n\varphi(x)),$$

where $\varphi$ is the function introduced in 3.1, and apply Lemma 1.9 in Kurtz (1990) with $f = \varphi_{1}$, to show that for any solution of the patchwork martingale problem, $\lambda_{0}$ is strictly increasing and tends to infinity as $t$ goes to infinity. Thus $X$ defined by

$$X(t) = Y(\lambda_{0}^{-1}(t))$$
is a solution of the submartingale problem for \( \langle L, \langle \gamma, \nabla \rangle \rangle \). Since the submartingale problem for \( \langle L, \langle \gamma, \nabla \rangle \rangle \) is well posed (Stroock and Varadhan (1971)), \((Y, \lambda_0, \lambda_1)\) is also uniquely determined.

**Proof of Theorem 1.3.** Let \( T(0, \infty) \) be the collection of right continuous, nondecreasing functions \( r : [0, \infty) \rightarrow [0, \infty) \) such that \( r(0) = 0 \) and \( \lim_{t \rightarrow \infty} r(t) = \infty \), topologized by the Skorohod topology. Define \( r^{-1}(t) \equiv \inf \{ s : r(s) > t \} \). Then the mapping \((x, r) \in D_{\sqrt{\epsilon}}[0, \infty) \times T(0, \infty) \rightarrow x \circ r^{-1} \) is continuous at a point \((x_0, r_0)\) if \( r_0 \) is strictly increasing. Let \( \check{Y}^\epsilon \) be defined as in Theorem 3.8. \((Y^\epsilon, \lambda^0_\epsilon)\) converges in law to \((Y, \lambda_0)\) by Theorem 4.6. Since \( \lambda_0 \) is strictly increasing, the continuous mapping theorem gives

\[
X^\epsilon = Y^\epsilon \circ (\lambda^0_\epsilon)^{-1} \Rightarrow Y \circ \lambda^{-1}_0 = X.
\]

and \( X \) is the unique solution of the submartingale problem for \( \langle L, \langle \gamma, \nabla \rangle \rangle \). \( \square \)

### 5. Cool boundary reflection.

In this section we prove the diffusion approximation for a cool boundary, i.e. when \( \alpha \equiv 0 \) and hence \( \eta_{\epsilon} \equiv \eta^C_{\epsilon} \) satisfies \((A_C)\). In contrast to hot boundary reflection, a low energy particle reflecting from a cool boundary will not gain significant energy. An important special case of cool boundary reflection is specular reflection (characterized by \((1.3)\) and \((1.4)\)).

We use the notation and assumptions of Section 3: In particular, we assume that \( D \) is bounded and that \( \mathcal{Y} \) is a closed ball. In addition, we require \( D \) to be convex. As in Section 4, we suppose that \((A_0)\) through \((A_2)\) hold. We make the following non-degeneracy assumptions:

\[
\begin{align*}
(A_7) & \inf_{(x,v) \in \partial D} \mu(x,v) \equiv \mu > 0, \\
(A_8) & \eta(x,v, \{v' : \langle v', \nu(x) \rangle = 0\}) = 0, \quad x \in \partial D_1, \quad (v, \nu(x)) = 0.
\end{align*}
\]

We are interested in the behavior, as \( \epsilon \rightarrow 0 \), of a solution \( X^\epsilon \) of the constrained martingale problem for \((L^\epsilon, O \cup \partial_1 O, B^\epsilon, \partial_1 O)\) (in particular a solution of \((3.6)\)), with initial condition \((X^\epsilon_0, V^\epsilon_0)\) supported in \( O \cup \partial_1 O \). As in Section 4, we first prove convergence of \( Y^\epsilon \), where \((Y^\epsilon, U^\epsilon, \lambda^0_\epsilon, \lambda^1_\epsilon)\) is a solution of the pathwork martingale problem for \((L^\epsilon, O \cup \partial_1 O, B^\epsilon, \partial_1 O)\). \((Y^\epsilon, U^\epsilon, \lambda^0_\epsilon, \lambda^1_\epsilon)\) can be approximated by the pure jump Markov processes \((Y^\epsilon, U^\epsilon, \lambda^0_\epsilon, \lambda^1_\epsilon)\) and their occupation times \((\lambda^0_\epsilon, \lambda^1_\epsilon)\) (Lemma 3.8). The generator of \((Y^\epsilon, U^\epsilon, \lambda^0_\epsilon, \lambda^1_\epsilon)\) is given by \((3.32)-(3.37)\), that is

\[
C^{\epsilon, n} f(x, v) \equiv I_{\partial_1 O} f(x, v) + 2 \eta^C_{\epsilon} \left[ R^\epsilon_{\nu} f(x, v) - f(x) \right],
\]

where \( \partial_1 O \equiv \partial D \) is a closed ball. Note that \( p(x, v) = x \) for \( x \in D \).

Our convergence technique relies on the fact that we replace integrals with respect to \( d\lambda^0_\epsilon \) and \( d\lambda^1_\epsilon \) by integrals with respect to the occupation measures \( \Gamma^0_\epsilon \) and \( \Gamma^1_\epsilon \) defined by \((4.1)\), i.e.

\[
\begin{align*}
\Gamma^0_\epsilon([0, t] \times A) & = \int_0^t I_A(U^\epsilon(s))d\lambda^0_\epsilon(s), \quad \Gamma^1_\epsilon([0, t] \times A) = \int_0^t I_A(U^\epsilon(s))d\lambda^1_\epsilon(s),
\end{align*}
\]

for every Borel set \( A \subset \mathbb{R}^d \) and \( t \geq 0 \).

As pointed out in Section 4 right before Lemma 4.3, any limit point of \((Y^\epsilon, \Gamma^0_\epsilon, \Gamma^1_\epsilon, \lambda^0_\epsilon, \lambda^1_\epsilon)\) as \( \epsilon \rightarrow 0 \) is also a limit point of \((Y^\epsilon, \Gamma^0_n, \Gamma^1_n, \lambda^0_n, \lambda^1_n)\), where \( \Gamma^i_n, i = 0, 1 \) are defined by \((4.6)\), as \( \epsilon \rightarrow 0, n \rightarrow \infty, en \rightarrow \infty \).
The proofs of the following lemmas are exactly the same as for Lemmas 4.1 and 4.2.

**Lemma 5.1.** If \( \{X_0^\epsilon\} \) is relatively compact as \( \epsilon \to 0 \), then \( \{Y^\epsilon\} \) is relatively compact in \( C[0, \infty) \) as \( \epsilon \to 0 \).

**Lemma 5.2.** The families \( \{\Gamma_0^\epsilon\} \) and \( \{\Gamma_1^\epsilon\} \) are relatively compact in \( L(V) \) as \( \epsilon \to 0 \). For every limit point \( (Y, \Gamma_0, \Gamma_1, \lambda_0, \lambda_1) \), as \( \epsilon \to 0 \), of \( \{(Y^\epsilon, \Gamma_0^\epsilon, \Gamma_1^\epsilon, \lambda_0^\epsilon, \lambda_1^\epsilon)\} \), \( \Gamma_0 \) has the form
\[
\Gamma_0([0, t] \times A) = \int_0^t \pi(Y(s), A) d\lambda_0(s),
\]
where \( \pi \) is defined in \( (A_1) \).

The main difficulty of this section is to characterize \( \Gamma_1 \) as
\[
\Gamma_1([0, t] \times A) = \int_0^t \kappa(s) \int_A \rho(Y(s), v) \pi_\epsilon(Y(s), dv) d\lambda_1(s), \ A \in \mathcal{B}(R^2),
\]
where \( \kappa \) is a strictly positive stochastic process and \( \pi_b(y, \cdot) \) is the invariant probability measure for the Markov process on \( \partial_+ O \) with generator
\[
[R_b T - I]f(y, \cdot).
\]

As with the corresponding result in Section 4, we will obtain this characterization by showing that, almost surely, for every \( t_0 > 0 \),
\[
\int_{[0, t_0] \times V} \frac{[R_b T - I]f(Y(s), v)}{\rho(Y(s), v)} \Gamma_1(ds \times dv) = 0,
\]
for a large enough class of functions \( f \). Here, however, we must take into account the fact that \( \rho \) may vanish on part of \( \partial_+ O \). In particular, we need to verify that \( (Y, \Gamma_1) \) satisfies
\[
\Gamma_1\{(s, v) : Y(s) \in \partial D, \rho(Y(s), v) = 0\} = 0,
\]
and
\[
\int_{[0, t_0] \times V} \frac{1}{\rho(Y(s), v)} \Gamma_1(ds \times dv) < \infty.
\]

The key tool is the fact that, for any bounded measurable function \( f \),
\[
T^{\epsilon, n} f(Y^{\epsilon, n}(t), U^{\epsilon, n}(t)) - T^{\epsilon, n} f(X_0^\epsilon, V_0^\epsilon) - \int_{[0, t] \times V} B^{\epsilon, n} T^{\epsilon, n} f(Y^{\epsilon, n}(s), v) \Gamma_1^{\epsilon, n}(ds \times dv)
\]
is a martingale because \( L^{\epsilon, n} T^{\epsilon, n} f \equiv 0 \) (recall that \( T^{\epsilon, n} f(x, v) \equiv E_{x,v}[f(Y^{\epsilon, n}(\tau^{\epsilon, n}), U^{\epsilon, n}(\tau^{\epsilon, n}))] \) and \( \tau^{\epsilon, n} = \inf\{t > 0 : (Y^{\epsilon, n}(t), U^{\epsilon, n}(t)) \notin \hat{O}\} \) ). In particular
\[
E \left[ \int_{[0, t_0] \times V} \epsilon B^{\epsilon, n} T^{\epsilon, n} f(Y^{\epsilon, n}(s), v) \Gamma_1^{\epsilon, n}(ds \times dv) \right] = \epsilon E \left[ T^{\epsilon, n} f(Y^{\epsilon, n}(t), U^{\epsilon, n}(t)) - T^{\epsilon, n} f(X_0^\epsilon, V_0^\epsilon) \right].
\]
Notice also that, for \( (x, v) \in \hat{O}, \)
\[
T^{\epsilon, n} f(x, v) = f(x, v).
\]

Since \( \Gamma_1^{\epsilon, n}([0, t] \times V) \) is bounded by \( t \), to verify (5.3) and (5.4), we only need to be concerned about times when \( (Y^{\epsilon, n}(t), U^{\epsilon, n}(t)) \in \hat{O} \) and \( \rho(p(Y^{\epsilon, n}(t), U^{\epsilon, n}(t)), U^{\epsilon, n}(t)) \) is small, hence times when the particle hits the boundary with a small normal component. Therefore, information about the hitting distribution is critical.
Note that Lemma 4.4 is actually a statement about the unconstrained processes, so that its conclusions still hold. Let

$$\sigma_{1}^{\epsilon,n} \equiv \inf \{ t : U_{\epsilon,n}(t) \neq U_{\epsilon,n}(t^{-}) \}.$$  \hspace{1cm} (5.7)

The following lemma tells us more about the normal component of the velocity when the particle hits the boundary.

**Lemma 5.3.**

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \sup_{(x,v) \in \partial_{+}O} P_{x,v} \left( |(U_{\epsilon,n}(\tau_{\epsilon,n}), \nu(\sigma_{1}^{\epsilon,n}, U_{\epsilon,n}(\tau_{\epsilon,n})))| \leq \delta \sigma_{1}^{\epsilon,n} \right) = 0.$$  \hspace{1cm} (5.8)

**Proof.** Let

$$A_{\delta}^{\epsilon,n} \equiv \left\{ |(U_{\epsilon,n}(\tau_{\epsilon,n}), \nu(\sigma_{1}^{\epsilon,n}, U_{\epsilon,n}(\tau_{\epsilon,n})))| \leq \delta \right\}.$$  

It is enough to prove that for all sequences \( \{\epsilon_{k}\}, \{n_{k}\}, \{(x^{k}, v^{k})\} \subseteq \partial_{+}O, \epsilon_{k} \to 0, n_{k} \to \infty, \epsilon_{k} n_{k} \to \infty, (x^{k}, v^{k}) \to (x, v) \), we have

$$\lim_{\delta \to 0} \limsup_{k \to \infty} P_{x^{k}, v^{k}} \left( A_{\delta}^{\epsilon_{k}, n_{k}} |\tau_{\epsilon_{k}, n_{k}}^{x_{k}, y_{k}} > \sigma_{1}^{\epsilon_{k}, n_{k}} \right) = 0.$$  \hspace{1cm} (5.9)

For \((x,v) \in O\), let

$$\tau(x,v) \equiv \inf \{ t > 0 : x + tv \notin D \}.$$  \hspace{1cm} (5.10)

Since \( D \) is convex, \( \tau(x,v) = 0 \) for \((x,v) \in \bar{O}\). In addition \( \tau \) is continuous on \( O \cup \partial_{+}O \) and, if \( D \) is strictly convex, on \( O \), but this has no relevance for our purposes.

We have

$$P_{x,v} \left( \tau_{\epsilon,n} > \sigma_{1}^{\epsilon,n} \right) = P_{x,v} \left( \sum_{k=1}^{n \tau(x,v)} \frac{\epsilon}{n} \Delta_{k} > \sigma_{1}^{\epsilon,n} \right) \geq P_{x,v} \left( \sum_{k=1}^{n \tau(x,v)} \frac{\epsilon}{n} \Delta_{k} > \frac{\Delta_{0}}{\mu e^{-\frac{n}{\epsilon}}} \right),$$

where \( \{\Delta_{k}\}_{k \geq 1} \) are i.i.d. unit exponentials independent of \( \sigma_{1}^{\epsilon,n}, \Delta_{0} \) is a unit exponential independent of \( \{\Delta_{k}\}_{k \geq 1} \), and \([t] = \min \{k \in \mathbb{N} : k \geq t\} \). Calculating the right hand side, we obtain

$$P_{x,v} \left( \tau_{\epsilon,n} > \sigma_{1}^{\epsilon,n} \right) \geq 1 - \exp \left\{ - [n \tau(x,v)] \ln \left( \frac{\mu}{\epsilon n} + 1 \right) \right\} \geq 1 - e^{-\tau(x,v) \mu/(2\epsilon)},$$  \hspace{1cm} (5.11)

where the last inequality holds if \( \epsilon n \geq \mu \).

To simplify notation, let \( \tau_{k} = \tau_{\epsilon_{k}, n_{k}} \) and \( \sigma_{1}^{k} = \sigma_{1}^{\epsilon_{k}, n_{k}}. \) If \((v, \nu(x)) > 0\), then \( \tau(x^{k}, v^{k}) \to \tau(x,v) > 0 \), so that, by (5.11), \( P_{x^{k}, v^{k}} \left( \tau_{k} > \sigma_{1}^{k} \right) \to 1 \) and (5.9) follows by Lemmas 4.3 and 4.4.

If \((v, \nu(x)) = 0\), define \( \beta_{k} = \frac{1}{\sqrt{\epsilon_{k} v^{k} + \nu(x^{k})}} (\epsilon_{k} |v^{k}| + (v^{k}, \nu(x^{k}))) + |x - x^{k}| + |v - v^{k}|. \) Then \( \beta_{k} \to 0. \) Let \( N_{k} \) be the number of jumps of \( Y_{\epsilon_{k}, n_{k}} \) up to and including \( \sigma_{1}^{k}. \) Then, since \( Y_{k}(\sigma_{1}^{k}) - x^{k} = \frac{N_{k}}{n_{k}} v^{k} \) and, by convexity, \( d(Y_{k}(\sigma_{1}^{k}), \partial D) \leq (Y_{k}(\sigma_{1}^{k}) - x^{k}, \nu(x^{k})) \)

$$\left\{ N_{k} \leq \frac{\epsilon_{k} n_{k} v^{k}}{\sqrt{\epsilon_{k} v^{k} + \nu(x^{k})}} \right\} \subseteq \left\{ |Y_{k}(\sigma_{1}^{k}) - x| \leq \beta_{k}, d(Y_{k}(\sigma_{1}^{k}), \partial D) \leq \epsilon_{k} \beta_{k} \right\}$$

and

$$\left\{ \tau_{k} > \sigma_{1}^{k} \right\} = \left\{ \tau(x^{k}, v^{k}) > \frac{N_{k}}{n_{k}} > 0 \right\}.$$
Setting $A^k_\delta = A^\epsilon_{\epsilon,n,k}$, we have

$$P_{x^k,v^k}(A^k_\delta | \sigma^k > \sigma^1)$$

$$\leq P_{x^k,v^k}(N^k > \frac{\epsilon k_n k}{\sqrt{\epsilon k_v (\nu(x^k))}} | \sigma^k > \sigma^1)$$

$$+ P_{x^k,v^k}(A^k_\delta \cap \{ |Y^k(\sigma^1) - x| \leq \beta_k, d(Y^k(\sigma^1), \partial D) \leq \epsilon_k \beta_k \} | \sigma^k > \sigma^1)$$

$$\leq P_{x^k,v^k} \left( \frac{1}{\sqrt{\epsilon k_v (\nu(x^k))}} < \frac{N^k}{n_k c_k} < \frac{\tau(x^k,v^k)}{\epsilon_k} \right) I \left\{ (\epsilon \sqrt{\nu(x^k)})^{-1/2} \leq (\epsilon \sqrt{\nu(x^k)})^{-1/2} \tau(x^k,v^k) \right\}$$

$$P_{x^k,v^k} \left( 0 < \frac{N^k}{n_k c_k} < \frac{\tau(x^k,v^k)}{\epsilon_k} \right)$$

$$+ E_{x^k,v^k} \left[ P_{Y^k(\sigma^1),U^k(\sigma^1)}(A^k_\delta) I \{ |U^k(\sigma^1) - x| \geq \mu_k \} \right] / P_{x^k,v^k}(\sigma^1 > \sigma^1)$$

$$+ E_{x^k,v^k} \left[ \eta(Y^k(\sigma^1), v^k, \{ u' : |\nu(x^k)| < \mu_b \} \right] / P_{x^k,v^k}(\sigma^1 > \sigma^1)$$

$$\leq P_{x^k,v^k} \left( \frac{1}{\sqrt{\epsilon k_v (\nu(x^k))}} < \frac{N^k}{n_k c_k} < \frac{\tau(x^k,v^k)}{\epsilon_k} \right) I \left\{ (\epsilon \sqrt{\nu(x^k)})^{-1/2} \leq (\epsilon \sqrt{\nu(x^k)})^{-1/2} \tau(x^k,v^k) \right\}$$

$$+ \sup_{|z-x| \leq \beta_k} \frac{1}{d(z,\partial D) \leq \epsilon_k \beta_k} P_{z,v}(A^k)$$

$$+ \sup_{|z-x| \leq \beta_k} \frac{1}{d(z,\partial D) \leq \epsilon_k \beta_k} P_{z,v}(A^k)$$

The second and third summands in the right hand side vanish by Lemmas 4.3 and 4.4 and by $(A_8)$. As far as the first summand is concerned, $N^k$ has the same law as $N(n_k \epsilon_k^{-1} \sigma^1)$, $N$ being a unit Poisson process independent of $\sigma^1_k$, so that \( \left\{ \frac{N^k}{n_k c_k} \right\} \) is relatively compact and the numerator goes to zero. On the other hand, the numerator of the subsequence of $\left\{ k \right\}$ such that the indicator function is 1, the denominator goes to 1. \( \square \)

We want to use (5.6) to obtain the estimates needed to verify (5.3) and (5.4). To this end, for $\delta > 0$, define

$$\psi_3(x,v) \equiv I_{\partial O}(x,v) I_{(\delta,\infty)}(\rho(p(x,v),v)), \quad \text{for} \quad (x,v) \in \mathcal{O},$$

and let $q(\delta) > 0$ be nondecreasing and satisfy limits as $\delta \to 0$ $q(\delta) = 0$ and

$$(5.13) \quad q(\delta) \geq \limsup_{\epsilon \to 0, \epsilon n \to \infty} \sup_{(x,v) \in \mathcal{O}} P_{x,v} \left( |U^{\epsilon,n}(\tau^{\epsilon,n}), \nu(p(U^{\epsilon,n}(\tau^{\epsilon,n}),U^{\epsilon,n}(\tau^{\epsilon,n})))| \leq \delta |\tau^{\epsilon,n} > \sigma^{1,n} \right).$$

LEMMA 5.4. Let $\delta_0$ satisfy $\delta_0 < 1$, $q(\delta_0) < 1$, and let $\psi_{\delta_0}$ be defined as above. Let $0 < \delta < 1$. Then for $\epsilon \leq \epsilon_0 = \epsilon_0(\delta)$ and $\epsilon n \geq \nu_0 = \nu_0(\delta)$, for all $(x,v) \in \overset{\circ}{O}$,

$$\psi_{\delta_0}(x,v) \equiv I_{\partial O}(x,v) I_{(\delta,\infty)}(\rho(p(x,v),v)), \quad \text{for} \quad (x,v) \in \overset{\circ}{O},$$

$$\frac{1}{\epsilon_2(1-q(\delta_0))} \left( \epsilon B^{\epsilon,n} T^{\epsilon,n} \psi_{\delta_0}(x,v) + \frac{1}{\delta_0} \right).$$

Proof. By (5.11), for $\epsilon$ small enough and $\epsilon n$ large enough, for $(x,v) \in \partial_+ O$,

$$T^{\epsilon,n} \psi_{\delta_0}(x,v) \geq \left[ 1 - q(\delta_0) \right] P_{x,v} \left( \tau^{\epsilon,n} > \sigma^{1,n} \right)$$

$$\geq \left[ 1 - q(\delta_0) \right] \left[ 1 - e^{-\tau(x,v)/2} \right].$$

For $(x,v) \in \partial_+ O$, let $V^{\epsilon,n}_{x,v}$ denote the velocity after a reflection (i.e., $V^{\epsilon,n}_{x,v} \equiv \Pi V + [\rho(x,v) \nu + \frac{1}{\sqrt{\epsilon n}}] \zeta_{x,v}$). Then, for $(x,v) \in \overset{\circ}{O}$,

$$\epsilon B^{\epsilon,n} T^{\epsilon,n} \psi_{\delta_0}(x,v) = \mu_{\epsilon,n}(p(x,v),v) \left( E \left[ T^{\epsilon,n} \psi_{\delta_0}(p(x,v),V^{\epsilon,n}_{x,v}(x,v)) \right] - \psi_{\delta_0}(x,v) \right)$$
\[
\begin{align*}
&\geq \mu_b^{\epsilon,n}(p(x,v),v)E\left[T^{\epsilon,n}\psi_{\delta_0}(p(x,v),V_{p(x,v)}^{\epsilon,n})\right] - \frac{1}{\delta_0} \\
&\geq \mu_b^{\epsilon,n}(p(x,v),v)[1-q(\delta_0)]E\left[P_{p(x,v),V_{p(x,v)}^{\epsilon,n}}(\tau^{\epsilon,n} > \sigma_1^{\epsilon,n})\right] - \frac{1}{\delta_0} \\
&\geq [1-q(\delta_0)]\mu_b^{\epsilon,n}(p(x,v),v)E\left[1-e^{-\tau(p(x,v),V_{p(x,v)}^{\epsilon,n})\psi/(2\epsilon)}\right] - \frac{1}{\delta_0}.
\end{align*}
\]

Next note that, for \((x,v) \in \partial_+ O\),

\[
\tau(x,v) \geq c_1(v,\nu(x))
\]

for some positive constant \(c_1\) depending only on \(D\) and \(V\). Therefore, for every \((x,v) \in \hat{O}\),

\[
\frac{1}{[1-q(\delta_0)]}\left[e^{\epsilon_0-n-T^{\epsilon,n}\psi_{\delta_0}(x,v)} + \frac{1}{\delta_0}\right] \\
\geq \mu_b^{\epsilon,n}(p(x,v),v)E\left[\left(e^{-1}(V_{p(x,v)}^{\epsilon,n},\nu(p(x,v)))\right) \land 1\right] \\
= c_2 E\left[\left(e^{-1}\mu_b^{\epsilon,n}(p(x,v),v)(V_{p(x,v)}^{\epsilon,n},\nu(p(x,v)))\right) \land \frac{1}{\rho(p(x,v),v) \lor (\epsilon n)^{-1/2}}\right].
\]

Now, for \((x,v) \in \partial_- O\),

\[
\mu_b^{\epsilon,n}(x,v)(V_{x,v}^{\epsilon,n},\nu(x)) \geq (\langle \xi_{x,v},\nu(x) \rangle - 1) - \mu_b^{\epsilon,n}(x,v)|\tilde{V}_{x,v}^{\epsilon,n} - V_{x,v}^{\epsilon,n}|,
\]

where \(\tilde{V}_{x,v}^{\epsilon,n} \equiv v + [\rho(x,v) \lor \frac{1}{\epsilon n}]\langle \xi_{x,v} \rangle \). By \((AC)\), for every \(0 < \delta < 1\) there exists \(c_3 > 0\) such that, for \(\epsilon\) small enough and \(\epsilon n\) large enough,

\[
\inf_{(x,v) \in \partial_- O} P(\langle \xi_{x,v},\nu(x) \rangle - 1 \geq c_3) \geq 1 - \delta/2.
\]

On the other hand, by \((3.26)\),

\[
\limsup_{\epsilon \to 0, n \to \infty} \sup_{(x,v) \in \partial_- O} \mu_b^{\epsilon,n}(x,v)E\left[|\tilde{V}_{x,v}^{\epsilon,n} - V_{x,v}^{\epsilon,n}|\right] = 0,
\]

which implies that, for \(\epsilon\) small enough and \(\epsilon n\) large enough,

\[
\inf_{(x,v) \in \partial_- O} P\left(\mu_b^{\epsilon,n}(x,v)|\tilde{V}_{x,v}^{\epsilon,n} - V_{x,v}^{\epsilon,n}| \leq \frac{c_3}{2}\right) \geq 1 - \frac{\delta}{2}.
\]

Thus, for \(\epsilon\) small enough and \(\epsilon n\) large enough, for all \((x,v) \in \hat{O}\),

\[
\frac{1}{[1-q(\delta_0)]}\left[e^{\epsilon_0-n-T^{\epsilon,n}\psi_{\delta_0}(x,v)} + \frac{1}{\delta_0}\right] \\
\geq c_2(1-\delta)\left(e^{-1}\frac{c_3}{2} \land \frac{1}{\rho(p(x,v),v) \lor (\epsilon n)^{-1/2}}\right) \\
\geq c_2(1-\delta)\left(\frac{1}{\rho(p(x,v),v) \lor \delta}\right).
\]

\(\square\)
Lemma (5.4) will enable us to prove (5.3) and (5.4) but it is not yet enough to prove (5.1). For this we will need a uniform integrability condition that is guaranteed by the following lemma.

**Lemma 5.5.**

Let $\psi_\delta$ be the function defined by (5.12). Then

$$
\lim_{\delta \to 0} \sup_{t, n, \omega} \left\{ \int_{\{ (t, v) : t \leq t_0, \rho(p(x^n(t), v)) \leq \delta \}} |\epsilon B^{c_n} T^{c_n} \psi_\delta(Y^{c_n}(t), v)| \Gamma_1^{c_n}(dt \times dv) \right\} = 0.
$$

**Proof.** First note that, for $\delta \leq \delta_0$ and $(x, v) \in \hat{O}$ such that $\rho(p(x, v)) \leq \delta$, we have $T^{c_n} \psi_\delta(x, v) = \psi_\delta(x, v) = 0$ and

$$
\epsilon B^{c_n} T^{c_n} \psi_\delta(x, v) = \mu_b^{c_n}((p(x, v), v) E\left[T^{c_n} \psi_\delta(x, V^{c_n}_{p(x, v), v})\right]
\geq \mu_b^{c_n}((p(x, v), v)[1 - q(\delta)]E\left[P_{p(x, v), V^{c_n}_{p(x, v), v}}(\tau^{c_n} > \sigma_1^{c_n})\right]
\geq 0.
$$

By (5.6) we have

$$
E\left[\int_{[0, t_0] \times V} \epsilon B^{c_n} T^{c_n} \psi_\delta(Y^{c_n}(t), U^{c_n}(t)) \Gamma_1^{c_n}(dt \times dv)\right] \leq \epsilon,
$$

and hence

$$
\lim_{\delta \to 0} \sup_{t, n, \omega} \left\{ \int_{\{ (t, v) : t \leq t_0, \rho(p(x^n(t), v)) \leq \delta \}} \epsilon B^{c_n} T^{c_n} \psi_\delta(Y^{c_n}(t), v) \Gamma_1^{c_n}(dt \times dv) \right\}
\leq \lim_{\delta \to 0} \sup_{t, n, \omega} \left\{ \int_{\{ (t, v) : t \leq t_0, \rho(p(x^n(t), v)) \leq \delta \}} \epsilon B^{c_n} T^{c_n} \psi_\delta(Y^{c_n}(t), v) \Gamma_1^{c_n}(dt \times dv) \right\}.
$$

On the other hand, for $(x, v) \in \hat{O}$ such that $\rho(p(x, v)) > \delta$, letting $V^{c_n}_{p(x, v), v}$ be the new velocity after a reflection, i.e. $V^{c_n}_{p(x, v), v} \equiv \Pi_V(v + [\rho(p(x, v)), v]) \vee \frac{1}{\sqrt{1 - \rho^2}} |p_{p(x, v), v}|$, we have

$$
\epsilon B^{c_n} T^{c_n} \psi_\delta(x, v)
= \mu_b^{c_n}(p(x, v), v) E\left[\psi_\delta(Y^{c_n}(\tau^{c_n}), U^{c_n}(\tau^{c_n}))\right] - \psi_\delta(x, v)
\geq [1 - q(\delta)] I_{\tau^{c_n} > \sigma_1^{c_n}}(\psi_\delta(Y^{c_n}(\tau^{c_n}), U^{c_n}(\tau^{c_n})) - \psi_\delta(x, v))
\geq [1 - q(\delta)] I_{\tau^{c_n} > \sigma_1^{c_n}}(\psi_\delta(Y^{c_n}(\tau^{c_n}), U^{c_n}(\tau^{c_n})) - \psi_\delta(x, v))
\geq -q(\delta) \mu_b^{c_n}(p(x, v), v) E\left[P_{p(x, v), V^{c_n}_{p(x, v), v}}(\tau^{c_n} \leq \sigma_1^{c_n})\right],
\geq -q(\delta) \mu_b^{c_n}(p(x, v), v) E\left[P_{p(x, v), V^{c_n}_{p(x, v), v}}(\tau^{c_n} > \sigma_1^{c_n})\right]
\geq -q(\delta) \mu_b^{c_n}(p(x, v), v) E\left[P_{p(x, v), V^{c_n}_{p(x, v), v}}(\tau^{c_n} \leq \sigma_1^{c_n})\right],
$$

and hence, by (5.11) and (5.14),

$$
\epsilon B^{c_n} T^{c_n} \psi_\delta(x, v) \geq \frac{-q(\delta)}{c_2(1 - \delta)[1 - q(\delta)]} \left[ \epsilon B^{c_n} T^{c_n} \psi_\delta(x, v) + \frac{1}{\delta_0} \right]
\geq \frac{-q(\delta)}{c_2(1 - \delta)[1 - q(\delta)]} \left[ \epsilon B^{c_n} T^{c_n} \psi_\delta(x, v) + \frac{1}{\delta_0} \right]
\geq \frac{-q(\delta)}{c_2(1 - \delta)[1 - q(\delta)]} \left[ \epsilon B^{c_n} T^{c_n} \psi_\delta(x, v) + \frac{1}{\delta_0} \right]
\geq \frac{1}{\delta} E\left[e^{-\mu(2c)^{-1} \tau(p(x, v), V^{c_n}_{p(x, v), v})}\right].
$$
The second summand in the right hand side vanishes as \( \epsilon \) goes to 0 and \( cn \) goes to infinity. Therefore, observing that by (5.14) \( \epsilon B^{\epsilon,n}T^{\epsilon,n}\psi_{\delta_0}(x,v) + \frac{1}{\delta_0} \geq 0 \) everywhere on \( \hat{O} \), and setting

\[
G_1(\epsilon, n, \delta, t_0) = \{(t, v) : t \leq t_0, \rho(p(Y^{\epsilon,n}(t), v), v) \leq \delta\}
\]

and

\[
G_2(\epsilon, n, \delta, t_0) = \{(t, v) : t \leq t_0, \rho(p(Y^{\epsilon,n}(t), v), v) > \delta\},
\]

we have

\[
\lim_{\epsilon \to 0, n \to \infty} \lim_{t \to \infty} E \left[ \int_{G_1(\epsilon, n, \delta, t_0)} \epsilon B^{\epsilon,n}T^{\epsilon,n}\psi_{\delta}(Y^{\epsilon,n}(t), v)\Gamma_1^{\epsilon,n}(dt \times dv) \right]
\]

\[
\leq \lim_{\epsilon \to 0, n \to \infty} \lim_{t \to \infty} \frac{\epsilon - E \left[ \int_{G_2(\epsilon, n, \delta, t_0)} \epsilon B^{\epsilon,n}T^{\epsilon,n}\psi_{\delta}(Y^{\epsilon,n}(t), v)\Gamma_1^{\epsilon,n}(dt \times dv) \right]}{\epsilon - E \left[ \int_{G_2(\epsilon, n, \delta, t_0)} \epsilon B^{\epsilon,n}T^{\epsilon,n}\psi_{\delta}(Y^{\epsilon,n}(t), v)\Gamma_1^{\epsilon,n}(dt \times dv) \right]}
\]

which yields (5.18).

**Lemma 5.6.** Let \( (Y, \Gamma_1) \) be a limit point, as \( \epsilon \) goes to zero, of \( \{(Y^{\epsilon}, \Gamma_1^{\epsilon})\} \). Then

\[
\Gamma_1^{\epsilon}\{(s, v) : (Y^{\epsilon}(s), v) \in \partial_\epsilon O, \rho(Y(s), v) = 0\} = 0,
\]

almost surely, and for each \( t_0 > 0 \),

\[
E \left[ \int_{[0,t_0] \times \mathcal{V}} \frac{1}{\rho(Y(s), v)} \Gamma_1^{\epsilon}(ds \times dv) \right] < \infty.
\]

Let \( \{(\epsilon_k, n_k)\}, \epsilon_k \to 0, n_k \to \infty, \epsilon_k n_k \to \infty, \) be such that \( \{(Y^{\epsilon_k,n_k}, \Gamma_1^{\epsilon_k,n_k})\} \) converges to \( (Y, \Gamma_1) \) (recall that \( (Y, \Gamma_1) \) is also a limit point of \( \{(Y^{\epsilon,n}, \Gamma_1^{\epsilon,n})\} \) as \( \epsilon \to 0, n \to \infty, cn \to \infty \)). Then for each function \( f \in C^{1,1}_b(\mathbb{R}^d \times \mathbb{R}^d) \) such that \( f \) vanishes on \( \{(x, v) \in \hat{O}, \rho(p(x, v), v) < \delta_f\} \) for some \( \delta_f > 0 \),

\[
\lim_{k \to \infty} \int_{[0,t] \times \mathcal{V}} \epsilon_k B^{\epsilon_k,n_k}T^{\epsilon_k,n_k} f(Y^{\epsilon_k,n_k}(s), v)\Gamma_1^{\epsilon_k,n_k}(ds \times dv)
\]

\[
= \int_{[0,t] \times \mathcal{V}} \frac{[R_0 T - I]f(Y(s), v)}{\rho(Y(s), v)} \Gamma_1(ds \times dv)
\]

weakly as a stochastic process.

**Proof.** Let \( G_1 \) and \( G_2 \) be defined as in (5.21) and (5.22). By Lemma 5.4, we have, for every \( 0 < \delta < 1 \),

\[
\lim_{\delta \to 0} \lim_{\epsilon \to 0, n \to \infty} \lim_{t \to \infty} E \left[ \int_{G_1(\epsilon, n, \delta, t_0)} \frac{\delta}{\rho(p(Y^{\epsilon,n}(t), v), v) \vee \delta} \Gamma_1^{\epsilon,n}(dt \times dv) \right]
\]

\[
= \lim_{\delta \to 0} \lim_{\epsilon \to 0, n \to \infty} \lim_{t \to \infty} E \left[ \int_{G_1(\epsilon, n, \delta, t_0)} \frac{\delta}{\rho(p(Y^{\epsilon,n}(t), v), v) \vee \delta} \Gamma_1^{\epsilon,n}(dt \times dv) \right]
\]
\[
\frac{\delta}{c_2(1 - \delta)[1 - q(\delta_0)]} \limsup_{\epsilon \to 0, \epsilon^2 \to \infty} E \left[ \int_{G_1(\epsilon, n, \delta, \delta_0)} [\epsilon B^{n,n} T^{\epsilon,n} \psi_1(Y^{\epsilon,n}(t), v) + \frac{1}{\delta_0}] \Gamma^{\epsilon,n}_1(dt \times dv) \right] \leq \frac{\delta}{c_2(1 - \delta)[1 - q(\delta_0)]} \limsup_{\epsilon \to 0, \epsilon^2 \to \infty} E \left[ \int_{[0, t_0] \times \mathbb{V}} [\epsilon B^{n,n} T^{\epsilon,n} \psi_1(Y^{\epsilon,n}(t), v) + \frac{1}{\delta_0}] \Gamma^{\epsilon,n}_1(dt \times dv) \right]
\]
\[
\leq \frac{\delta}{c_2(1 - \delta)[1 - q(\delta_0)]} \limsup_{\epsilon \to 0, \epsilon^2 \to \infty} [\epsilon + \frac{1}{\delta_0}] = 0,
\]
which yields
\[
E[\Gamma_1((t, v) : t \leq t_0, \rho(Y(t), v) = 0)] \leq \lim_{\delta \to 0} \limsup_{\epsilon \to 0, \epsilon^2 \to \infty} E[\Gamma^{\epsilon,n}_1((t, v) : t \leq t_0, \rho(p(Y^{\epsilon,n}(t), v) < \delta))] = 0.
\]
Therefore we have
\[
E \left[ \int_{[0, t_0] \times \mathbb{V}} \frac{1}{\rho(Y(t), v)} \Gamma_1(dt \times dv) \right] = \lim_{\delta \to 0} E \left[ \int_{(t,v) : t \leq t_0, \rho(Y(t), v) \geq 2\delta} \frac{1}{\rho(Y(t), v)} \Gamma_1(dt \times dv) \right] \leq \lim_{\delta \to 0} E \left[ \liminf_{\epsilon \to 0, \epsilon^2 \to \infty} \int_{G_2(\epsilon, n, \delta, \delta_0)} \frac{1}{\rho(p(Y^{\epsilon,n}(t), v))} \Gamma^{\epsilon,n}_1(dt \times dv) \right] \leq \lim_{\delta \to 0} \liminf_{\epsilon \to 0, \epsilon^2 \to \infty} E \left[ \int_{G_2(\epsilon, n, \delta, \delta_0)} \frac{1}{\rho(p(Y^{\epsilon,n}(t), v))} \Gamma^{\epsilon,n}_1(dt \times dv) \right] \leq \frac{1}{c_2(1 - \delta)[1 - q(\delta_0)]} \liminf_{\epsilon \to 0, \epsilon^2 \to \infty} E \left[ \int_{G_2(\epsilon, n, \delta, \delta_0)} \epsilon B^{n,n} T^{\epsilon,n} \psi_1(Y^{\epsilon,n}(t), v) + \frac{1}{\delta_0} \Gamma^{\epsilon,n}_1(dt \times dv) \right] \leq \frac{1}{c_2(1 - \delta)[1 - q(\delta_0)]} \delta_0 < \infty.
\]

For the second part of the theorem, fix a sequence \(\{(\epsilon_k, n_k)\}\), such that \(\epsilon_k \to 0, n_k \to \infty, \epsilon_k n_k \to \infty\) and \(\{(Y^{\epsilon_k,n_k}, \Gamma^{\epsilon_k,n_k}_1) \Rightarrow (Y, \Gamma_1)\}\). By the Skorohod representation theorem we can assume that \(\{(Y^{\epsilon_k,n_k}, \Gamma^{\epsilon_k,n_k}_1) \to (Y, \Gamma_1)\}\) uniformly over compact time intervals, almost surely. Let \(Y^{\epsilon_k,n_k} = Y^{k}, U^{\epsilon_k,n_k} = U^{k}, \mu^{\epsilon_k,n_k}_0 = \mu^{k}_0, B^{\epsilon_k,n_k} = B^{k}, T^{\epsilon_k,n_k} = T^{k}\), and \(\Gamma^{\epsilon_k,n_k} = \Gamma^{k}_1\). For all but countably many \(\delta, 0 < \delta < 1\), taking into account that \(\|p(Y^{k}(s), U^{k}(s)) - Y^{k}(s)\| \leq \frac{\|p^{(1)}(s)\|}{n_k}\), we have
\[
\lim_{k \to \infty} \int_{\{(s,v) : s \leq t, \rho(p(Y^{k}(s), v), v) \geq \delta/2, \rho(Y^{k}(s), v) \geq \delta\}} \epsilon_k B^{k} T^{k} f(Y^{k}(s), v) \Gamma^{k}_1(ds \times dv) = \int_{\{(s,v) : 0 \leq s \leq t, \rho(Y^{k}(s), v) \geq \delta\}} \left[ R_s T - \int f(Y^{k}(s), v) \right] \Gamma^{k}_1(ds \times dv),
\]
\[
\frac{R_s T - \int f(Y^{k}(s), v) \Gamma^{k}_1(ds \times dv)}{\rho(Y^{k}(s), v)} \Gamma_1(ds \times dv),
\]
uniformly over compact time intervals, almost surely.

In addition, by (5.24), for any \( t_0 > 0 \),
\[
\lim_{\delta \to 0} E \left[ \int_{\{(t,v): t \leq t_0, \rho(Y(t),v) < \delta\}} \frac{|R_\delta T - I|}{\rho(Y(t),v)} f(Y(t),v) \Gamma_1(dt \times dv) \right] = 0.
\]
Finally, denoting by \( V^{R_\delta}_{x,v} = V^{R_\delta}_{x,v,x} \) the new velocity after a reflection, we have, for \( \delta \leq \delta f, \delta < 1 \), and \((x,v) \in \hat{O}\) such that \( \rho(p(x,v),v) \leq \delta \),
\[
|\epsilon_k B^k T^k f(x,v)| = |\mu_k^\epsilon(p(x,v),v) E[T^k f(x,V^\prime_{x,v})]|
\leq \mu_k^\epsilon(p(x,v),v) E[T^k f(x,V^\prime_{x,v})]
\leq \|f\| \mu_k^\epsilon(p(x,v),v) E[T^k \psi_\delta(x,V^\prime_{x,v})]
= \|f\| |\epsilon_k B^k T^k \psi_\delta(x,v)|,
\]
and hence, by Lemma 5.5
\[
\lim_{\delta \to 0} \limsup_{k \to \infty} E\left[ \int_{\mathcal{G}_k(x,k,t_0,\delta)} |\epsilon_k B^k T^k f(Y^k(t),v)| \Gamma_1(dt \times dv) \right] = 0.
\]

We are now ready to prove the main results of this section.

**Theorem 5.7.** Suppose \( \nu(x)^{\epsilon} a(x) \nu(x) > 0 \) for all \( x \in \partial D \). Let \( \mathcal{T} \) be the operator defined by (1.11), with domain \( C^2(\bar{D}) \), and let \( \gamma \) be the vector field on \( \partial D \) defined by (1.15). Then, if \( \{X_0\} \) is relatively compact, any limit point, as \( \epsilon \to 0 \), of \( \{Y_0, X_0, \lambda_1^\epsilon\} \) is a solution of the patchwork martingale problem for \( (L, D, (\gamma, \nabla), \partial D) \).

Suppose \( a(x) \) is strictly positive definite for every \( x \in \bar{D} \), and \( \gamma \) is Lipschitz continuous and satisfies
\[
\inf_{x \in \partial D} \langle \gamma(x), \nu(x) \rangle > 0.
\]
Then, for any initial condition \( X_0, X_0 \in \bar{D} \) almost surely, there is a unique solution, \( (Y, \lambda_0, \lambda_1) \) to the patchwork martingale problem for \( (\mathcal{T}, D, (\gamma, \nabla), \partial D) \) and \( \lambda_0 \) is strictly increasing and divergence to infinity, almost surely. If \( X_0^\epsilon \Rightarrow x_0 \), then \( Y^\epsilon \Rightarrow x_0 \).

**Proof.** The proof is similar to the proof of Theorem 4.6 combined with the proofs of Lemmas 2.4 and 4.5. As in the proof of Theorem 2.3, for \( f \in C^2_b(\mathbb{R}^d) \), let
\[
f^\epsilon(x,v) = f(x) + \epsilon \partial f(x) h(x,v).
\]
Then, with \( L \) defined by (2.13),
\[
f(Y^\epsilon(t)) + \epsilon \partial f(Y^\epsilon(t)) h(Y^\epsilon(t),U^\epsilon(t)) - \int_{[0,t] \times \mathbb{V}} Lf(Y^\epsilon(s,v)) \Gamma_0^\epsilon(ds \times dv)
- \int_{[0,t] \times \mathbb{V}} \partial f(Y^\epsilon(t)) Bh(Y^\epsilon(s,v)) \Gamma_1^\epsilon(ds \times dv)
\]
is a martingale. Let \( (Y, \lambda_0, \lambda_1) \) be a limit point of \( \{Y^\epsilon, X_0^\epsilon, \lambda_1^\epsilon\} \). Then \( (Y, \Gamma_0, \Gamma_1, \lambda_0, \lambda_1) \), where \( \Gamma_0 \) and \( \Gamma_1 \) are given by (4.3) and (4.4), is a limit point of \( \{Y, \Gamma_0^\epsilon, \Gamma_1^\epsilon, X_0^\epsilon, \lambda_1^\epsilon\} \) and
\[
\begin{align*}
&f(Y(t)) - \int_{[0,t] \times \mathbb{V}} Lf(Y(s,v)) \Gamma_0(ds \times dv) \\
&\quad - \int_{[0,t] \times \mathbb{V}} \nabla f(Y(t)) Bh(Y(s,v)) \Gamma_1(ds \times dv)
\end{align*}
\]
is a martingale. By Lemma 5.2 the second summand in (5.31) can be written as
\[ \int_0^t \hat{L} f(Y(s)) d\lambda_0(s), \]
where the operator \( \hat{L} \) is given by (1.11). As far as the third summand is concerned, first of all note that, by (5.5), the left hand side of (5.25) must be a martingale. Since the right hand side of (5.25) is continuous and of bounded variation, it follows (e.g. by Corollary II.6.1, Protter (1990)) that it must be zero, i.e. (5.2).

Therefore, in this section we state all the lemmas and the theorem that are needed to prove Theorem 1.4, but we write down only the parts of the proofs that are somewhat different from the corresponding ones of Section 5. We use the notation and all the assumptions of Sections 3, 4 and 5.

The rest of the proof is the same as for Theorem 4.6. \( \square \)

**Proof of Theorem 1.4.** Theorem 1.4 follows from Theorem 5.7 by exactly the same argument as Theorem 1.3 follows from Theorem 4.6. \( \square \)

**6. General boundary conditions.** Finally, we now turn to general boundary conditions of the form (1.2). We assume \((A_C), (A_U)\) and \((A_{LU})\). Our goal is to prove the diffusion approximation of Theorem 1.4. The proof of Theorem 1.4 in the general case follows closely the proof in the case of cool boundary reflection. Therefore, in this section we state all the lemmas and the theorem that are needed to prove Theorem 1.4, but we write down only the parts of the proofs that are somewhat different from the corresponding ones of Section 5. We use the notation and all the assumptions of Sections 3, 4 and 5.

We are interested in the behavior, as \( \epsilon \to 0 \), of a solution \( X^\epsilon \) of the constrained martingale problem for \((L', O \cup \partial_+ O, B^\epsilon, \partial_- O)\) (in particular a solution of (3.6)), with initial condition \((X_0^\epsilon, V_0^\epsilon)\) supported in \(O \cup \partial_+ O\). As in Sections 4 and 5, we first prove convergence of \( Y^\epsilon \), where \((Y^\epsilon, U^\epsilon, \lambda_0^\epsilon, \lambda_1^\epsilon)\) is a solution of the patchwork martingale problem for \((L', O \cup \partial_+ O, B^\epsilon, \partial_- O)\). \((Y^\epsilon, U^\epsilon, \lambda_0^\epsilon, \lambda_1^\epsilon)\) can be approximated by the pure
jump Markov processes \((Y^{\epsilon,n}, U^{\epsilon,n})\) and by their occupation times \((\lambda^{\epsilon,n}_0, \lambda^{\epsilon,n}_1)\) (Lemma 3.8). The generator of \((Y^{\epsilon,n}, U^{\epsilon,n})\) is given by (3.32)-(3.37). In particular the boundary operator \(B^{\epsilon,n}\) is defined as
\[
B^{\epsilon,n} f(x, v) \equiv \epsilon^{-1} \mu^{\epsilon,n}_b(p(x, v)) \left[ K^{\epsilon,n}_b f(p(x, v), v) - f(x, v) \right],
\]
where \(p(x, v)\) is the projection from \(\hat{O}\) on \(\partial^- O\) defined in (3.32), and \(\mu^{\epsilon,n}_b\) and \(K^{\epsilon,n}_b\) are given by
\[
\mu^{\epsilon,n}_b(x, v) \equiv \alpha(x, v) \left[ p(x, v) \vee \frac{1}{\epsilon n} \right] + 1 - \alpha(x, v),
\]
and
\[
K^{\epsilon,n}_b f(x, v) \equiv \mathbb{E} \left[ f(x, \Pi_V(v + I_{[0,1]}(x,v))(Y) | p(x, v) \vee \frac{1}{\epsilon n} \right] \left[ \zeta^{\epsilon,n}_\alpha(x,v) + I_{(\alpha(x,v),1]}(Y) \left[ \delta^{\epsilon,n}_x + \frac{1}{\epsilon n} \nu(x) \right] \right],
\]
Here \(\Pi_V\) is the normal projection on the ball \(V\), \(\zeta^{\epsilon,n}_\alpha\) and \(\delta^{\epsilon,n}_\alpha\) are random variables with laws \(\eta^{\epsilon}_b(x,v,\cdot)\) and \(\eta^{\epsilon}_b(x,v,\cdot)\) respectively, and \(Y\) is a \([0,1]\) uniform random variable, independent of \((\zeta^{\epsilon,n}_\alpha, \delta^{\epsilon,n}_\alpha)\).

The proofs of the following lemmas are exactly the same as for Lemmas 5.1 and 5.2.

**Lemma 6.1.** If \(\{X^\epsilon_0\}\) is relatively compact as \(\epsilon \to 0\), then \(\{Y^\epsilon\}\) is relatively compact in \(C_{\overline{\Omega}}[0,\infty)\) as \(\epsilon \to 0\).

**Lemma 6.2.** The families \(\{\Gamma^\epsilon_0\}\) and \(\{\Gamma^\epsilon_1\}\) are relatively compact in \(L(V)\) as \(\epsilon \to 0\). For every limit point \((Y, \Gamma_0, \Gamma_1, \lambda_0, \lambda_1)\), as \(\epsilon \to 0\), of \(\{(Y^\epsilon, \Gamma^\epsilon_0, \Gamma^\epsilon_1, \lambda^\epsilon_0, \lambda^\epsilon_1)\}\), \(\Gamma_0\) has the form
\[
\Gamma_0([0,t] \times A) = \int_0^t \pi(Y(s), A) d\lambda_0(s),
\]
where \(\pi\) is defined in (A1).

Lemma 5.3 is a statement about the unconstrained process, so that it still holds. As far as Lemma 5.4 is concerned, it is sufficient to modify its statement as follows. Recall that \(\psi_\delta\) and \(q(\delta)\) are defined by (5.12) and (5.13) respectively, i.e., for \(\delta > 0\),
\[
\psi_\delta(x, v) \equiv I_{\hat{O}}(x, v) I_{(\delta, \infty)}(\rho(p(x, v), v)), \quad \text{for} \ (x, v) \in \hat{O},
\]
where \(\hat{O}\) is defined in (3.32)-(3.33),
\[
q(\delta) > 0 \ \text{nondecreasing}, \ \lim_{\delta \to 0} q(\delta) = 0,
\]
and that (3.31) holds, i.e.
\[
\mu^{\epsilon,n}_b(x, v) \leq \frac{\|\rho\| \vee 1}{\rho(x, v) \vee \frac{1}{\epsilon n}}, \quad (x, v) \in \partial^- O.
\]

**Lemma 6.3.** Let \(\delta_0\) satisfy \(\delta_0 < 1\), \(q(\delta_0) < 1\) and let \(\psi_{\delta_0}\) be defined as above. Let \(0 < \delta < 1\). Then for \(\epsilon \leq \epsilon_0 = \epsilon_0(\delta)\) and \(\epsilon n \geq \epsilon_0 = \epsilon_0(\delta)\), for all \((x, v) \in \hat{O},\)
\[
(1 - \delta) \frac{1}{\rho(p(x, v), v) \vee \delta} \leq \frac{1}{c_2(1 - q(\delta_0))} \left( \epsilon B^{\epsilon,n} T^{\epsilon,n} \psi_{\delta_0}(x, v) + \frac{\|\rho\| \vee 1}{\delta_0} \right).
\]

**Proof.** The proof of Lemma 5.4 still works, taking into account (6.4).

**Lemma 6.4.**
Let \(\psi_\delta\) be the function defined by (6.3). Then
\[
\lim_{\delta \to 0} \limsup_{\epsilon n \to \infty} \mathbb{E} \left[ \int_{(t, v), t \leq t_0, \rho(p(Y^{\epsilon,n}(t), v), v) \leq \delta} \epsilon B^{\epsilon,n} T^{\epsilon,n} \psi_\delta(Y^{\epsilon,n}(t), v) \left| \Gamma_1^{\epsilon,n}(dt \times dv) \right. \right] = 0.
\]
Proof. The proof is the same as for Lemma 5.5, taking into account (6.4) and (6.5).

**Lemma 6.5.** Let \((Y, \Gamma_1)\) be a limit point, as \(\epsilon \to 0\), of \(\{(Y^\epsilon, \Gamma_1^\epsilon)\}\). Then
\[
\Gamma_1\{s, v) : (Y(s), v) \in \partial O, \rho(Y(s), v) = 0\} = 0,
\]
almost surely, and for each \(t_0 > 0\),
\[
E \left[ \int_{[0, t_0] \times V} \frac{1}{\alpha(Y(s), v)\rho(Y(s), v) + 1 - \alpha(Y(s), v)} \Gamma_1(ds \times dv) \right] < \infty.
\]
Let \(\{\epsilon_k, n_k\}\), \(\epsilon_k \to 0\), \(\epsilon_k n_k \to \infty\) be such that \(\{Y^{\epsilon_k,n_k}, \Gamma_1^{\epsilon_k,n_k}\}\) converges to \((Y, \Gamma_1)\) (recall that \((Y, \Gamma_1)\) is also a limit point of \(\{(Y^{\epsilon}, \Gamma_1^{\epsilon})\}\) as \(\epsilon \to 0\), \(n \to \infty\). Then for each function \(f \in C_b(R^d \times R^d)\) such that \(f\) vanishes on \(\{(x, v) \in \hat{O}, \rho(p(x), v) < \delta_f\}\) for some \(\delta_f > 0\),
\[
\lim_{k \to \infty} \int_{[0, t] \times V} \epsilon_k B^{\epsilon_k,n_k} T^{\epsilon_k,n_k} f(Y^{\epsilon_k,n_k}(s), v) \Gamma_1^{\epsilon_k,n_k}(ds \times dv)
= \int_{[0, t] \times V} \alpha(Y(s), v)\rho(Y(s), v) + 1 - \alpha(Y(s), v) \Gamma_1(ds \times dv)
\]
weakly as a stochastic process.

**Proof.** Notice that, for all \((x, v) \in \partial O\),
\[
\rho(x, v) \leq 1 \implies \alpha(x, v)\rho(x, v) + 1 - \alpha(x, v) \geq \rho(x, v),
\]
\[
\rho(x, v) > 1 \implies \alpha(x, v)\rho(x, v) + 1 - \alpha(x, v) \geq 1 = \frac{\rho(x, v)}{\|\rho\| \vee 1},
\]
(6.6)
\[
\alpha(x, v)\rho(x, v) + 1 - \alpha(x, v) \geq \frac{\rho(x, v)}{\|\rho\| \vee 1}
\]
Taking into account (6.6) and (6.5), the proof is the same as for Lemma 5.6.

**Theorem 6.6.** Suppose \(\nu(x)^T a(x)\nu(x) > 0\) for all \(x \in \partial D\). Let \(L\) be the operator defined by (1.11), with domain \(C^2(\overline{D})\), and let \(\gamma\) be the vector field on \(\partial D\) defined by (1.15). Then, if \(\{X_0^\epsilon\}\) is relatively compact, any limit point, as \(\epsilon \to 0\), of \(\{(Y^\epsilon, \lambda_0^\epsilon, \lambda_1^\epsilon)\}\) is a solution of the patchwork martingale problem for \((L, D, \langle \gamma, \nabla \rangle, \partial D)\).

Suppose \(a(x)\) is strictly positive definite for every \(x \in \overline{D}\), and \(\gamma\) is Lipschitz continuous and satisfies
\[
\inf_{x \in \partial D} \langle \gamma(x), \nu(x) \rangle > 0.
\]
Then, for any initial condition \(X_0^\epsilon, \lambda_0^\epsilon \in \overline{D}\) almost surely, there is a unique solution, \((Y, \lambda_0, \lambda_1)\) to the patchwork martingale problem for \((\overline{L}, D, \langle \gamma, \nabla \rangle, \partial D)\) and \(\lambda_0\) is strictly increasing and diverging to infinity, almost surely. If \(X_0^\epsilon \Rightarrow X_0\) as \(\epsilon \to 0\), then \(Y^\epsilon \Rightarrow Y\).

**Proof.** The proof is the same as for Theorem 5.7, taking into account Lemmas 6.1, 6.2 and 6.5 and the fact that, letting \(\gamma_1\) be the measure valued stochastic process such that
\[
\Gamma_1([0, t] \times A) = \int_0^t \gamma_1(s, A)d\lambda_1(s), \ A \in \mathcal{B}(R^d),
\]
it holds
\[
\int_V \frac{1}{\alpha(Y(s), v)\rho(Y(s), v) + 1 - \alpha(Y(s), v)} \gamma_1(s, dv) \geq \frac{1}{\|\rho\| \vee 1}, \ d\lambda_1 - a.e., \ a.s.
\]

**Proof of Theorem 1.4** In the general case, Theorem 1.4 still follows from Theorem 6.6 by exactly the same argument as Theorem 1.3 follows from Theorem 4.6.
REFERENCES


