Department of Mathematics, University of Wisconsin-Madison
Math 114
Worksheet Sections 3.1, 3.3, and 3.5

1. For \( f(x) = -5x + 4 \)
   (a) Determine the slope and the y-intercept.

   **Solution:** \( f(x) = -5x + 4 \) is of the form \( y = mx + b \), so slope is \( m = -5 \) and y-intercept is the \( y \)-value when \( x = 0 \): \( y = -5 \cdot 0 + 4 = 4 \).

(b) Use the slope and the y-intercept to graph the function.

![Graph of the function \( f(x) = -5x + 4 \)]

(c) Determine the average rate of the function.

   **Solution:** The average rate of any function \( f \) from \( x \) to \( x + h \) is \( \frac{f(x+h) - f(x)}{h} \). For our \( f \), we get \( \frac{(-5(x+h) + 4) - (-5x + 4)}{h} = \frac{-5x - 5h + 4 + 5x - 4}{h} = \frac{-5h}{h} = -5 \).
   Note: for a linear function, the average rate is the slope.

(d) Determine if the function is increasing, decreasing, or constant.

   **Solution:** Since the average rate of change/slope is always \(-5\), a negative number, the function is decreasing.
2. Suppose $f(x) = 3x + 5$ and $g(x) = -2x + 15$

(a) Solve $f(x) = 0$.

**Solution:**

$f(x) = 0 = 3x + 5 \implies x = -\frac{5}{3}$

(b) Solve $f(x) > 0$.

**Solution:**

$3x + 5 > 0 \implies x > -\frac{5}{3}$

(c) Solve $f(x) > g(x)$.

**Solution:**

$3x + 5 > -2x + 15 \implies 5x > 10 \implies x > 2$

(d) Solve $f(x) = g(x)$.

**Solution:**

$3x + 5 = -2x + 15 \implies x = 2$

(e) Graph $y = f(x)$ and $y = g(x)$ and label the point that represents the solution to the equation $f(x) = g(x)$.

![Graph of functions](image)

3. For the following functions, (a) graph each quadratic function by determining whether its graph opens up of down and by finding its vertex, axis of symmetry, y-intercept, and x-intercepts, if any. (b) Determine the domain and the range of the function. (c) Determine where the function is increasing and where it is decreasing. (d) Determine whether the quadratic function has a maximum value or a minimum value, and then find that value.
(a) \( h(x) = x^2 - 2x - 3 \)

\[ \text{Solution:} \]

(b) \( f(x) = -3x^2 + 3x - 2 \)

\[ \text{Solution:} \]

4. Solve each inequality.

(a) \( x^2 + 3x - 10 > 0 \)

\[ \text{Solution:} \] Factoring we get \((x - 2)(x + 5) > 0\). The sign of the expression can only change at zeroes at \( x = 2 \) and \( x = -5 \) (and discontinuities, but we don’t have such). There are three possible ways to proceed. Testing values around the zeroes we get \((-6 - 2)(-6 + 5) = 8\) for \( x = -6 \), \((0 - 2)(0 + 5) = -10\) for \( x = 0 \), and \((3 - 2)(3 + 5) = 8\) for \( x = 3 \). Hence the solution set of positive values of the expression is \((-\infty, -5) \cup (2, \infty)\).

Alternatively, we reason:

- If \( x < -5 \), then \( x + 5 < 0 \) and \( x - 2 < 0 \), so \((x - 2)(x + 5)\) is the product of negative numbers so positive.
- If \(-5 < x < 2\), then \( x + 5 > 0 \) and \( x - 2 < 0 \), so \((x - 2)(x + 5)\) is the product of a positive and a negative, so negative.
- If \( 2 < x \), then \( x + 5 > 0 \) and \( x - 2 > 0 \), so \((x - 2)(x + 5)\) is the product of a positive and a positive, so positive.

Again, we would get that the solutions (where the quadratic is positive) are \((-\infty, -5) \cup (2, \infty)\).

As another alternative, we could notice that the coefficient of \( x^2 \) in the quadratic is positive, so the graph of the quadratic expression opens upward, hence the function has to be larger than 0 away from the zeroes and smaller than 0 between the zeroes. Again, we get \((-\infty, -5) \cup (2, \infty)\).

(b) \( 4x^2 + 9 < 6x \)

\[ \text{Solution:} \] Rearranging we get \(4x^2 - 6x + 9 < 0\). Factoring using the quadratic formula we get zeroes at \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{6 \pm \sqrt{6^2 - 4 \cdot 4 \cdot 9}}{2 \cdot 4} = \frac{6 \pm \sqrt{36 - 4 \cdot 36}}{2 \cdot 4} = \frac{6 \pm \sqrt{36(1 - 3)}}{2 \cdot 4} = \frac{6 \pm 6\sqrt{-3}}{2 \cdot 4} \). These are complex numbers so we have no real zero. There are two ways to proceed.

The sign of an expression changes only at zeroes or points of discontinuity. Since we have neither, the sign of the expression is constant, so we test at 0: \( 4 \cdot 0^2 - 6 \cdot 0 + 9 = 9 \). It follows that the expression is always positive, so the inequality has no solution.

Alternatively, notice that the coefficient of \( x^2 \) is positive, so the graph of the quadratic expression opens upwards. Hence, if it has no real \( x \)-intercepts, the graph must be completely above the \( x \)-axis and there are no solution to the inequality.
5. A rectangle has one vertex on the line $y = 10 - x$, $x > 0$, another at the origin, one on the positive x-axis, and one on the positive y-axis. Find the largest area $A$ that can be enclosed by the rectangle.

**Solution:** The vertices of the rectangle are $(0,0)$, $(x,0)$, $(0,y)$, and $(x,y)$ where $y = 10 - x$ and $x > 0$ and $y > 0$. The rectangle thus has sides of length $x$ and $y$, and its area is $A = xy$.

Using the fact that $y = 10 - x$, we can substitute to get $A = x(10 - x) = 10x - x^2$, as long as $x > 0$ and $10 - x > 0$, i.e. as long as $0 < x < 10$.

Because the coefficient of $x^2$ is negative, the graph of the area $A = 10x - x^2$ is a parabola opening downwards, with $x$-intercepts $x = 0$ and $x = 10$, axis of symmetry $x = \frac{-10}{2 \cdot (-1)} = 5$ and vertex $(5, 10 \cdot 5 - 5^2) = (5, 25)$. Since the parabola opens downwards, the $y$-coordinate 25 of the vertex is the largest value obtained by the quadratic. Since the largest value occurs at $x = 5$ satisfying our constraint that $0 < x < 10$, this 25 is also the largest value of the area when $0 < x < 10$.

6. For $f(x) = -x^2 + 1$ and $g(x) = 2x + 1$

(a) Graph $f(x)$ and $g(x)$ in the same cartesian plane.

![Graph of f(x) and g(x)](image)

(b) Solve $f(x) = g(x)$

**Solution:**

\[-x^2 + 1 = 2x + 1\]
\[0 = x^2 + 2x\]
\[0 = x(x + 2)\]

$x = 0$ or $x = -2$
(c) Solve \( f(x) \leq 0 \)

**Solution:** We need to find the x-intercepts (zeroes).

\[
-x^2 + 1 = 0 \\
x^2 = 1
\]

\( x = 1 \) or \( x = -1 \). The parabola opens down so \( g(x) \leq 0 \) when \( x \in (-\infty, -1) \cup (1, \infty) \).

(d) Solve \( f(x) > g(x) \)

**Solution:** According to the graphs, \( f(x) > g(x) \) between the two intersection points so when \(-2 < x < 0\).

We could also solve it algebraically:

\[
-x^2 + 1 > 2x + 1 \\
0 > x^2 + 2x \\
x^2 + 2x < 0 \\
x(x + 2) < 0
\]

The zeroes of the function \( y = x(x + 2) \) are \( x = -2 \) or \( x = 0 \) and because the graph of the function \( y = x(x + 2) \) is a parabola that opens up, the function is negative between the zeroes so when \(-2 < x < 0\).

7. For \( f(x) = x^2 - 2x + 1 \) and \( g(x) = -x^2 + 1 \)

(a) Graph \( f(x) \) and \( g(x) \) in the same cartesian plane.

**Solution:**

(b) Solve \( f(x) = g(x) \)
(c) Solve $f(x) > g(x)$

**Solution:** According to the graphs, $f(x) > g(x)$ "outside" the two intersection points so when $x \in (-\infty, 0) \cup (1, \infty)$.

We could also solve it algebraically:

\[
\begin{align*}
x^2 - 2x + 1 &> -x^2 + 1 \\
2x^2 - 2x &> 0 \\
2x(x - 1) &> 0
\end{align*}
\]

The zeroes of the function $y = 2x(x - 1)$ are $x = 0$ or $x = 1$ and because the graph of the function $y = 2x(x - 1)$ is a parabola that opens up, the function is positive when $x \in (-\infty, 0) \cup (1, \infty)$. 

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\[
\begin{align*}
x^2 - 2x + 1 &= -x^2 + 1 \\
2x^2 - 2x &= 0 \\
2x(x - 1) &= 0
\end{align*}
\]

$x = 0$ or $x = 1$