On the number of infinite sequences with trivial initial segment complexity

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The map $G$ takes $c$ to the number of $K$-trivial streams with constant $c$.

What is the arithmetical complexity of $G$?

... or equivalently

*How hard is to compute $G$?*
A stream is **random** if it has high initial segment complexity.  

*To describe the first $n$ bits of the sequence you need to use $n$ bits (modulo a constant)*

On the other end of the spectrum:

A stream is **trivial** if the complexity of its first $n$ bits is as low as the complexity of $0^n$. 
Chaitin asked if there are non-computable streams whose initial segment complexity is as low as a computable stream.

Solovay gave a positive answer.

The world of $K$-trivial streams

Computable from the halting problem i.e. $\Delta^0_2$ (Chaitin 70s)

Incomplete, and in fact low (Downey/Hirschfeldt/Nies/Stephan)

Downward closed under $\leq_T$ (Hirschfeldt/Nies 2005)

Form an ideal in the Turing degrees.
Provide a ‘natural’ solution to Post’s problem.

\[ A = \{ n \mid \exists e, s \left( W_{e,s} \cap A_s = \emptyset \land n > 2e \land n \in W_{e,s} \land \sum_{n < j < s} 2^{-K_s(j)} < 2^{-e} \right) \} \]

Post’s simple set

**Scott sets:** Turing incomparability using the \( K \)-trivial degrees.

(Kučera and Slaman)
Cumulative hierarchy of $K$-trivial streams

A stream $X$ is $K$-trivial if $K(X \upharpoonright n) \leq K(n) + c$ for all $n$, some $c$.

$K$-trivial streams are stratified in a hierarchy of length $\omega$

\ldots whose $c$-level contains the $K$-trivial streams with constant $c$. 
The map $G$ takes $c$ to the number of $K$-trivial strings with constant $c$.

What is the arithmetical complexity of $G$?

... or equivalently

How hard is to compute $G$?
Basic facts about $G$, by DMNY

- Computable from $0^{(3)} \ldots$ i.e. $\Delta^0_4$

- Not computable i.e. not $\Delta^0_1$

- Not computable from the halting problem, i.e. not $\Delta^0_2$

*Is it computable from $0^{(2)}$ i.e. is it $\Delta^0_3$?*
The classes of $K_c$-trivial streams

- They are uniformly $\Pi^0_1$ in the halting set.
- The set of infinite paths through a $0'$-computable tree.
- The width of these trees is computably bounded since

$$|\{\sigma \in 2^n \mid K(\sigma) \leq K(|\sigma|) + c\}| < 2^c$$

\ldots by the coding theorem
The number of infinite paths through a tree \( T \) with bounded width can be computed from \( T'' \).

This is optimal!

If a family of trees is computable from a low\(_2\) oracle \( A \) then the number of paths is computable from \( 0^{(2)} \).

Oracle \( A \) is low\(_2\) if \( A'' \) is computable from \( 0^{(2)} \); \( \Sigma^0_2(A) \subseteq \Delta^0_3 \).
Theorem (B. and Tom Sterkenburg)

Given a $\Delta^0_2$ tree $T$ which only has $K_c$-trivial paths we can compute the index of another $\Sigma^0_1$ tree which is $K$-trivial and has the same infinite paths as the original tree.

The new trees have trivial initial segment complexity.

Fact: $0^{(2)}$ can compute a low$_2$ index of a $K_c$-trivial stream given $c$ and the $\Delta^0_2$ index of the stream.
Computation of $G(c)$ from $0^{(2)}$

- Get the index of the original $\Delta^0_2$ tree representing the class $K_c$-trivial.

- Compute the index of the $K$-trivial tree representing this class.

- Use $0^{(2)}$ to compute a low$_2$-ness index of the new tree.

- Use $0^{(2)}$ again to compute the number of infinite paths through this tree.

- This is $G(c)$
A related class: low for $K$ streams

If a computer is given access to a powerful oracle, it will achieve better compression for many strings.

$X$ is called low for $K$ if $K^X = K$.

...... if as far as prefix-free complexity is concerned, it is not better than a computable oracle.

This class was defined by Muchnik in 1999, who also exhibited non-computable elements in it.
Low for \( K \) streams are stratified in a cumulative hierarchy of finite classes.

Hirschfeldt and Nies showed that they coincide with the \( K \)-trivial streams.

Our methodology applies to this class, showing that

\[ \ldots \text{the corresponding function giving the cardinality of the hierarchy classes is } \Delta^0_3. \]
A consequence of the main result is that \( 0'' \) can obtain the indices of the \( K_c \)-trivial strings.

This can be used to show that a number of \( K \)-related objects have lower complexity.

For example, gap functions for \( K \)-triviality.
These are non-decreasing unbounded functions $f$ such that

$$\forall n \ [K(X \upharpoonright n) \leq K(n) + f(n) + c] \Rightarrow X \text{ is } K\text{-trivial.}$$

- Constructed by Csima and Montalbán in 2006
- Used to obtain minimal pairs in the degrees of randomness
- Complexity: $\Delta^0_4$
- Downey raised the question about their complexity
Complexity of gap functions

Theorem (Barmpalias/Baartse and Bienvenu/Merkle/Nies)

If \( f \) is \( \Delta^0_2 \) unbounded and non-decreasing then there are uncountably many streams \( X \) such that

\[
K(X \upharpoonright n) \leq K(n) + f(n) \quad \text{for all } n.
\]

Theorem (Barmpalias and Martijn Baartse)

There is a \( \Delta^0_3 \) gap function for \( K \)-triviality.
References

Barmpalias/Sterkenburg
On the number of infinite sequences with trivial initial segment complexity

Barmpalias/Baartse
On the gap between trivial and nontrivial initial segment prefix-free complexity
Submitted.

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