Maximal chains of computable well partial orders

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1. **Well-partial-orders**
   - Maximal linear extensions of wpos
   - Maximal chains of wpos

2. **Computing maximal chains**
   - Computing strongly maximal chains is hard
   - Computing maximal chains is not easy
   - A way of computing maximal chains

3. **Comparison with reverse mathematics**
Well-partial-orders

A partial order $\mathcal{P} = (P, \leq_P)$ is a well partial order (wpo) if for every $f : \mathbb{N} \to P$ there exists $i < j$ such that $f(i) \leq_P f(j)$.

There are many equivalent characterizations of wpos:

- $\mathcal{P}$ is well-founded and has no infinite antichains;
- every sequence in $P$ has a weakly increasing subsequence;
- every nonempty subset of $P$ has a finite set of minimal elements;
- all linear extensions of $\mathcal{P}$ are well-orders.

The reverse mathematics and computability theory of these equivalences has been studied in (Cholak-M-Solomon 2004).

All equivalences are provable in $\text{WKL}_0 + \text{CAC}$. 
Some examples of wpos

- Finite partial orders
- Well-orders
- Finite strings over a finite alphabet (Higman, 1952)
- Finite trees (Kruskal, 1960)
- Transfinite sequences with finite labels (Nash-Williams, 1965)
- Countable linear orders (Laver 1971, proving Fraïssé’s conjecture)
- Finite graphs (Robertson and Seymour, 2004)

The ordering is some kind of embeddability
Closure properties of wpos

- The sum and disjoint sum of two wpos are wpo
- The product of two wpos is wpo
- Finite strings over a wpo are a wpo (Higman, 1952)
- Finite trees with labels from a wpo are a wpo (Kruskal, 1960)
- Transfinite sequences with labels from a wpo which use only finitely many labels are a wpo (Nash-Williams, 1965)
The maximal order type of a wpo

\( \mathcal{P} \) is a wpo \( \iff \) all linear extensions of \( \mathcal{P} \) are well-orders

We denote by \( \text{Lin}(\mathcal{P}) \) the collection of all linear extensions of \( \mathcal{P} \).

**Definition**

If \( \mathcal{P} \) is a wpo, its maximal order type is

\[
\alpha(\mathcal{P}) = \sup \{ \alpha \mid \exists \mathcal{L} \in \text{Lin}(\mathcal{P}) \alpha = \text{ot}(\mathcal{L}) \}.
\]

**Theorem (de Jongh – Parikh, 1977)**

*The sup in the definition of \( \alpha(\mathcal{P}) \) is actually a max,* i.e. there exists \( \mathcal{L} \in \text{Lin}(\mathcal{P}) \) with order type \( \alpha(\mathcal{P}) \).

*In other words, every \( \mathcal{I} \in \text{Lin}(\mathcal{P}) \) embeds into \( \mathcal{L} \). \( \mathcal{L} \) is called a maximal linear extension of \( \mathcal{P} \).*
Computing maximal linear extensions

Theorem (Montalbán, 2007)

Every computable wpo has a computable maximal linear extension. However there is no hyperarithmetic function mapping the index of a computable wpo to the index of one of its maximal linear extensions.
The height of a well founded partial order

\( \mathcal{P} \) is a wpo \( \iff \mathcal{P} \) is well founded and all its chains are well-orders.

We denote by \( \text{Ch}(\mathcal{P}) \) the collection of all chains of \( \mathcal{P} \).

**Definition**

If \( \mathcal{P} \) is well founded, its **height** is

\[
\text{ht}(\mathcal{P}) = \sup\{ \alpha \mid \exists C \in \text{Ch}(\mathcal{P}) \alpha = \text{ot}(\mathcal{L}) \}.
\]

We can also define the height of \( x \in \mathcal{P} \):

\[
\text{ht}_\mathcal{P}(x) = \sup\{ \text{ht}_\mathcal{P}(y) + 1 \mid y <_\mathcal{P} x \}
\]

so that \( \text{ht}(\mathcal{P}) = \sup\{ \text{ht}_\mathcal{P}(x) + 1 \mid x \in \mathcal{P} \} \).
Wolk’s Theorem

**Theorem (Wolk 1967)**

*If \( \mathcal{P} \) is a wpo, the \( \sup \) in the definition of \( \text{ht}(\mathcal{P}) \) is actually a max, i.e. there exists \( C \in \text{Ch}(\mathcal{P}) \) with order type \( \text{ht}(\mathcal{P}) \). Such a chain is called a maximal chain of \( \mathcal{P} \).*

*Actually \( C \) can be chosen so that for every \( \alpha < \text{ht}(\mathcal{P}) \) there exists \( x \in C \) such that \( \text{ht}_\mathcal{P}(x) = \alpha \). Such a chain is called a strongly maximal chain of \( \mathcal{P} \).*
Two questions

In analogy with the Montalbán’s result we ask:

**Question**

If $\mathcal{P}$ is a computable wpo, how complicated must maximal and strongly maximal chains of $\mathcal{P}$ be?

It follows from previous work that a computable wpo always has a hyperarithmetic strongly maximal chain.

**Question**

How complicated must any function taking the computable wpo $\mathcal{P}$ to a maximal chain be?
Computing maximal chains

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Theorem

Let $\alpha < \omega^\text{CK}_1$.

There exists a computable wpo $\mathcal{P}$ such that every strongly maximal chain of $\mathcal{P}$ computes $0^{(\alpha)}$. 
Computing maximal chains is hard

The main tool

Theorem (Ash-Knight 1990)

Let $\alpha < \omega_1^{CK}$ and $A$ is a $\Pi^0_{2\alpha+1}$ set. There exists a uniformly computable sequence of linear orders $\mathcal{L}_n^A$ such that

$$\text{ot}(\mathcal{L}_n^A) = \begin{cases} \omega^\alpha & \text{if } n \in A; \\ \omega^{\alpha+1} & \text{if } n \not\in A. \end{cases}$$

This sequence of linear orderings can be computed uniformly in indices for $\alpha$ as a computable ordinal and $A$ as a $\Pi^0_{2\alpha+1}$ set.
Every strongly maximal chain of $\mathcal{P}$ computes $0^{(\alpha)}$: the global view
Every strongly maximal chain of $\mathcal{P}$ computes $0^{(\alpha)}$: zooming

When $n \in 0^{(\alpha)}$,
\[\text{ot}(\mathcal{L}_n^{0^{(\alpha)}}) = \omega^\alpha\] and
\[\text{ot}(\mathcal{L}_n^{0^{(\alpha)}}) = \omega^{\alpha+1};\]
when $n \notin 0^{(\alpha)}$,\n\[\text{ot}(\mathcal{L}_n^{0^{(\alpha)}}) = \omega^{\alpha+1}\]
and \[\text{ot}(\mathcal{L}_n^{0^{(\alpha)}}) = \omega^{\alpha}\]

The unique strongly maximal chain $C$ of $\mathcal{P}$ always picks the $\omega^{\alpha+1}$ side

$\text{ht}(\mathcal{P})(a_n) = \omega^{\alpha+1} \cdot n$ and $\text{ht}(\mathcal{P}) = \omega^{\alpha+2}$

$n \in 0^{(\alpha)}$ iff $c_n \in C$
Computing maximal chains is not easy

**Theorem**

Let $\alpha < \omega_1^{CK}$. There exists a computable wpo $\mathcal{P}$ such that $0^{(\alpha)}$ does not compute any maximal chain of $\mathcal{P}$.

We are not claiming that the maximal chains of $\mathcal{P}$ compute $0^{(\alpha)}$. 

$0^{(\alpha)}$ does not compute any maximal chain of $\mathcal{P}$: the global view
Computing maximal chains is not easy

$0^{(\alpha)}$ does not compute any maximal chain of $P$:
zooming

\[ n \in A_i \implies \text{ot}(L_{n}^{A_i}) = \omega^\alpha \]
\[ n \notin A_i \implies \text{ot}(L_{n}^{A_i}) = \omega^{\alpha+1} \]

where $n \in A_i$ iff
\[ \exists e < n \Phi_e^{0(\alpha)}(n) = i \]

\[ \{| i \mid n \in A_i \} \leq n \text{ and } \text{ot}(L_{n}^{A_i}) = \omega^{\alpha+1} \text{ for at least one } i \leq n \]

\[ \text{ht}_{P}(a_n) = \omega^{\alpha+1} \cdot n \text{ and } \text{ht}(P) = \omega^{\alpha+2} \]
$0^{(\alpha)}$ does not compute any maximal chain of $\mathcal{P}$:

concluding

Let $\mathcal{C}$ be a maximal chain. Define $\psi \leq_T \mathcal{C}$ by

$$\psi(n) = \begin{cases} i & \text{if } \exists x \in \mathcal{C} \ b_i^j \leq_P x < P \ a_{n+1}; \\ \uparrow & \text{otherwise.} \end{cases}$$

Infinitely often $\psi$ picks an $\omega^{\alpha+1}$ chain.

Fix $e$. There exists $n > e$ such that $n \notin A_{\psi(n)}$. Thus $\Phi^{0(\alpha)}_e(n) \neq \psi(n)$ and thus $\psi \neq \Phi^{0(\alpha)}_e$.

Therefore $\psi \not\leq_T 0^{(\alpha)}$ and $\mathcal{C} \not\leq_T 0^{(\alpha)}$. 
Generic sets for Cohen forcing

Definition

For $\alpha < \omega_1^{CK}$, a set $G$ is $\alpha$-generic if the conditions which are initial segments of $G$ suffice to decide all $\Sigma_\alpha$-questions. $G$ is hyperarithmetically generic if it is $\alpha$-generic for every $\alpha < \omega_1^{CK}$.

- Almost every set, in the sense of category, is hyperarithmetically generic
- A hyperarithmetically generic is not hyperarithmetic
- A hyperarithmetically generic does not compute any noncomputable hyperarithmetic set
Almost every set computes maximal chains

**Theorem**

Let \( G \) be hyperarithmetically generic.

*For every computable wpo \( P \), there exists a maximal chain \( C \) in \( P \) such that \( C \leq_T G \).*

If \( \text{ht}(P) < \omega^{\alpha+1} \), then \( 2 \cdot \alpha \)-genericity of \( G \) suffices.

- Almost every set, in the sense of category, computes maximal chains.
- Every computable wpo has a maximal chain that does not compute any noncomputable hyperarithmetic set, i.e. maximal chains cannot code any \( 0^{(\alpha)} \).
Nonuniformity

Our proof of the previous result has several nonuniform steps.

If $\mathcal{L}_0$ and $\mathcal{L}_1$ are computable well-orders consider $\mathcal{L}_0 \oplus \mathcal{L}_1$, which is a computable wpo.

A maximal chain of $\mathcal{L}_0 \oplus \mathcal{L}_1$ is included in some $\mathcal{L}_i$, and the $i$ is uniformly computable from the maximal chain and the wpo. Then $\mathcal{L}_{1-i}$ embeds in $\mathcal{L}_i$ and $\mathcal{L}_i$ is the longer chain.

By Ash-Knight this can uniformly code any hyperarithmetic set.

Theorem

There is no hyperarithmetic procedure which calculates a maximal chain of every computable wpo.

Suppose $f$ is such that, for every index $e$ for a computable wpo $\mathcal{P}$, $n \mapsto f(e, n)$ is a maximal chain of $\mathcal{P}$.

Then $f$ computes every hyperarithmetic set.
Comparison with reverse mathematics

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Some equivalences with $\text{ATR}_0$

**Theorem**

Over $\text{RCA}_0$, the following are equivalent to $\text{ATR}_0$:

1. the maximal linear extension theorem for wpos [M-Shore 2011];
2. the maximal chain theorem for wpos [M-Shore 2011];
3. the strongly maximal chain theorem for wpos [M-Shore 2011];

These are all statements of the form $\forall X (\varphi(X) \implies \exists Y \psi(X,Y))$. 
Different complexity

For statements of the form $\forall X (\varphi(X) \implies \exists Y \psi(X,Y))$ we ask

*if $X$ is computable, how complicated must $Y$ be?*

1. A computable wpos has a computable maximal linear extension
2. A computable wpos has a hyp maximal chain, but maximal chains can be incomparable with all noncomputable hyp sets
3. A computable wpos has a hyp strongly maximal chain, and strongly maximal chains can be of arbitrarily high complexity in the hyp hierarchy
4. There exists a computable bipartite graph such that any pair matching/cover satisfying König's duality computes every hyp set and hence is not hyp

These are four different levels of computational complexity for theorems all axiomatically equivalent to $\text{ATR}_0$.

The phenomena in 2 seems to be new.