Four Related Questions
How common are minimal degrees?

Question 1
What is the (classical) Hausdorff dimension of the set of minimal Turing degrees?

Notes
No 1-random has minimal degree, so the measure of the minimal degrees is zero.

Even better, the degrees that compute a minimal degree have measure zero (Paris).

In particular, no 2-random computes a minimal degree (Barmpalias, Day and Lewis improving on work of Kurtz).

The packing dimensions of the set of minimal Turing degrees is 1 (Downey, Greenberg).
How common are minimal degrees?

Question 1
What is the (classical) Hausdorff dimension of the set of minimal Turing degrees?

Notes
- No 1-random has minimal degree, so the measure of the minimal degrees is zero.
How common are minimal degrees?

Question 1

What is the (classical) Hausdorff dimension of the set of minimal Turing degrees?

Notes

- No 1-random has minimal degree, so the measure of the minimal degrees is zero.
- Even better, the degrees that compute a minimal degree have measure zero (Paris).
How common are minimal degrees?

Question 1
What is the (classical) Hausdorff dimension of the set of minimal Turing degrees?

Notes
- No 1-random has minimal degree, so the measure of the minimal degrees is zero.
- Even better, the degrees that compute a minimal degree have measure zero (Paris).
- In particular, no 2-random computes a minimal degree (Barmpalias, Day and Lewis improving on work of Kurtz).
How common are minimal degrees?

Question 1
What is the (classical) Hausdorff dimension of the set of minimal Turing degrees?

Notes
- No 1-random has minimal degree, so the measure of the minimal degrees is zero.
- Even better, the degrees that compute a minimal degree have measure zero (Paris).
- In particular, no 2-random computes a minimal degree (Barmpalias, Day and Lewis improving on work of Kurtz).
- The packing dimensions of the set of minimal Turing degrees is 1 (Downey, Greenberg).
How common are minimal degrees?

Question 1

What is the (classical) Hausdorff dimension of the set of minimal Turing degrees?
How common are minimal degrees?

**Question 1**

What is the (classical) Hausdorff dimension of the set of minimal Turing degrees?

How might we answer this?
Question 1

What is the (classical) Hausdorff dimension of the set of minimal Turing degrees?

How might we answer this?

If for every oracle $X$, there is a real of minimal degree and effective Hausdorff dimension $1$ relative to $X$, then $\dim_H(\text{Minimal}) = 1$. 

Proposition (Greenberg and M.)

There is a computable order function $h : \omega \to \omega \setminus \{0, 1\}$ such that every $h$-bounded DNC function computes a real of effective Hausdorff dimension $1$. 

How common are minimal degrees?

Question 1

What is the (classical) Hausdorff dimension of the set of minimal Turing degrees?

How might we answer this?

If for every oracle $X$, there is a real of minimal degree and effective Hausdorff dimension 1 \textit{relative to} $X$, then $\dim_H(\text{Minimal}) = 1$.

Proposition (Greenberg and M.)

There is a computable order function $h: \omega \rightarrow \omega \setminus \{0, 1\}$ such that every $h$-bounded DNC function computes a real of effective Hausdorff dimension 1.
Can minimal DNC functions grow slowly?

Proposition (Greenberg and M.)
There is a computable order function $h: \omega \to \omega \setminus \{0, 1\}$ such that every $h$-bounded DNC function computes a real of effective Hausdorff dimension 1.
Can minimal DNC functions grow slowly?

Proposition (Greenberg and M.)

There is a computable order function \( h: \omega \to \omega \setminus \{0, 1\} \) such that every \( h \)-bounded DNC function computes a real of effective Hausdorff dimension 1.

There is a DNC function of minimal degree (Kumabe, Lewis).
Proposition (Greenberg and M.)

There is a computable order function $h : \omega \to \omega \setminus \{0, 1\}$ such that every $h$-bounded DNC function computes a real of effective Hausdorff dimension 1.

There is a DNC function of minimal degree (Kumabe, Lewis). Can such a function grow slowly?

Question 2

Is there an $h$-bounded DNC function of minimal degree?
Can minimal DNC functions grow slowly?

Proposition (Greenberg and M.)

There is a computable order function $h: \omega \to \omega \setminus \{0, 1\}$ such that every $h$-bounded DNC function computes a real of effective Hausdorff dimension 1.

There is a DNC function of minimal degree (Kumabe, Lewis). Can such a function grow slowly?

Question 2

Is there an $h$-bounded DNC function of minimal degree?

We would actually need this in a partially relativized form:

Question $2^X$

For an oracle $X$, is there an $h$-bounded function that is DNC relative to $X$ and has minimal degree?
Can minimal DNC functions grow slowly?

**Proposition (Greenberg and M.)**
There is a computable order function $h: \omega \to \omega \setminus \{0, 1\}$ such that every $h$-bounded DNC function computes a real of effective Hausdorff dimension 1.

There is a DNC function of minimal degree (Kumabe, Lewis). Can such a function grow slowly?

**Question 2**
Is there an $h$-bounded DNC function of minimal degree?

We would actually need this in a partially relativized form:

**Question $2^X$**
For an oracle $X$, is there an $h$-bounded function that is DNC relative to $X$ and has minimal degree?

Question $2^X$ implies that $\dim_H(\text{Minimal}) = 1$. 
Can minimal DNC functions grow slowly?

There are connections between what can be computed from a slow growing DNC function and what can be computed *uniformly* from a bounded DNC function:
Can minimal DNC functions grow slowly?

There are connections between what can be computed from a slow growing DNC function and what can be computed \textit{uniformly} from a bounded DNC function:

**Facts (Greenberg and M.)**

- There is a uniform way to compute a real of Hausdorff dimension 1 from a DNC$_k$ function.
- There is a computable order function $h$ such that every $h$-bounded DNC function computes a real of Hausdorff dimension 1.
Can minimal DNC functions grow slowly?

There are connections between what can be computed from a slow growing DNC function and what can be computed \textit{uniformly} from a bounded DNC function:

**Facts (Greenberg and M.)**

- There is a uniform way to compute a real of Hausdorff dimension 1 from a DNC\(_k\) function.
- There is a computable order function \(h\) such that every \(h\)-bounded DNC function computes a real of Hausdorff dimension 1.

Also:

- (Downey, Greenberg, Jockusch, Milans) There is no uniform way to compute a Kurtz random from a DNC\(_3\) function.
- (Greenberg, M.; Khan, M.) For any computable order function \(h\), there is an \(h\)-bounded DNC that computes no Kurtz random.
Can we uniformly witness the non-minimality of DNC\(_3\) functions?

There are connections between what can be computed from a slow growing DNC function and what can be computed uniformly from a bounded DNC function:
Can we uniformly witness the non-minimality of DNC<sub>3</sub> functions?

There are connections between what can be computed from a slow growing DNC function and what can be computed uniformly from a bounded DNC function:

So this:

**Question 2**

Is there an h-bounded DNC function of minimal degree?
Can we uniformly witness the non-minimality of DNC$_3$ functions?

There are connections between what can be computed from a slow growing DNC function and what can be computed uniformly from a bounded DNC function:

So this:

**Question 2**

Is there an $h$-bounded DNC function of minimal degree?

…is related to the uniform question for bounded DNC functions:
Can we uniformly witness the non-minimality of $\text{DNC}_3$ functions?

There are connections between what can be computed from a slow growing DNC function and what can be computed uniformly from a bounded DNC function:

So this:

Question 2
Is there an $h$-bounded DNC function of minimal degree?

... is related to the uniform question for bounded DNC functions:

Question 3.$k$
Fix $k \geq 3$. Is there a functional $\Gamma$ such that $\emptyset <_T \Gamma^f <_T f$ for every $\text{DNC}_k$ function $f : \omega \to k$?
Can we uniformly witness the non-minimality of $\text{DNC}_3$ functions?

There are connections between what can be computed from a slow growing DNC function and what can be computed uniformly from a bounded DNC function:

So this:

**Question 2**

Is there an $h$-bounded DNC function of minimal degree?

…is related to the uniform question for bounded DNC functions:

**Question 3.k**

Fix $k \geq 3$. Is there a functional $\Gamma$ such that $\emptyset <_T \Gamma^f <_T f$ for every $\text{DNC}_k$ function $f: \omega \rightarrow k$?

It is not hard to see that $\text{DNC}_k$ functions are non-minimal, but no uniform proof is known.
Are continuous functions either injective on a big set or constant on a big(ish) set?

We might want to modify Kumabe, Lewis to answer Questions 2.

\[ f : \omega \rightarrow 2^\omega \text{ is continuous}, \text{ is } f \text{ either } 1\text{-injective on a } 2\text{-bushy tree, or } 2\text{-constant on an eventually } 2\text{-bushy tree.} \]

A tree \( T \) is 2-bushy if every \( \sigma \in T \) has at least two immediate extensions. \( T \) is eventually 2-bushy if this holds for sufficiently long strings \( \sigma \). 

17 is an arbitrary number (greater than 3).
Are continuous functions either injective on a big set or constant on a big(ish) set?

We might want to modify Kumabe, Lewis to answer Questions 2.

For this, we would need to prove an appropriate (delayed) splitting lemma.
Are continuous functions either injective on a big set or constant on a big(ish) set?

We might want to modify Kumabe, Lewis to answer Questions 2.

For this, we would need to prove an appropriate (delayed) splitting lemma. In purely combinatorial form:

**Question 4**

If $f : 17^\omega \to 2^\omega$ is continuous, is $f$ either

1. injective on a 2-bushy tree, or
2. constant on an eventually 2-bushy tree.
Are continuous functions either injective on a big set or constant on a big(ish) set?

We might want to modify Kumabe, Lewis to answer Questions 2.

For this, we would need to prove an appropriate (delayed) splitting lemma. In purely combinatorial form:

**Question 4**

If \( f : 17^\omega \to 2^\omega \) is continuous, is \( f \) either

1. injective on a 2-bushy tree, or
2. constant on an eventually 2-bushy tree.

- A tree \( T \) is **2-bushy** if every \( \sigma \in T \) has at least two immediate extensions.
Are continuous functions either injective on a big set or constant on a big(ish) set?

We might want to modify Kumabe, Lewis to answer Questions 2.

For this, we would need to prove an appropriate (delayed) splitting lemma. In purely combinatorial form:

**Question 4**

If $f : 17^{\omega} \rightarrow 2^{\omega}$ is continuous, is $f$ either

1. injective on a 2-bushy tree, or
2. constant on an eventually 2-bushy tree.

- A tree $T$ is **2-bushy** if every $\sigma \in T$ has at least two immediate extensions.
- $T$ is **eventually 2-bushy** if this holds for sufficiently long strings $\sigma$. 
Are continuous functions either injective on a big set or constant on a big(ish) set?

We might want to modify Kumabe, Lewis to answer Questions 2.

For this, we would need to prove an appropriate (delayed) splitting lemma. In purely combinatorial form:

**Question 4**

If \( f : 17^\omega \to 2^\omega \) is continuous, is \( f \) either

1. injective on a 2-bushy tree, or
2. constant on an eventually 2-bushy tree.

- A tree \( T \) is **2-bushy** if every \( \sigma \in T \) has at least two immediate extensions.
- \( T \) is **eventually 2-bushy** if this holds for sufficiently long strings \( \sigma \).
- 17 is an arbitrary number (greater than 3).
Are continuous functions either injective on a big set or constant on a big(ish) set?

Question 4
If $f : 17^\omega \rightarrow 2^\omega$ is continuous, is $f$ either
1. injective on a 2-bushy tree, or
2. constant on an eventually 2-bushy tree.
Are continuous functions either injective on a big set or constant on a big(ish) set?

**Question 4**

If \( f : 17^\omega \to 2^\omega \) is continuous, is \( f \) either

1. injective on a 2-bushy tree, or
2. constant on an eventually 2-bushy tree.

It should be noted that:

**Kumar, private communication**

There is a continuous \( f : [0, 1] \to \mathbb{R} \) such that

1. \( f \) is non-injective on every positive measure set, and
2. \( f \) is non-constant on every positive measure set.