Copy vs Diagonalize

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Let $\mathcal{K}$ be a class of structures on a relational language.

**The players:** $C$ and $D$.

They build sequences of finite structures alternatively:

<table>
<thead>
<tr>
<th>Player $D$</th>
<th>$D[0] \subseteq D[1] \subseteq \cdots$</th>
<th>let $D = \bigcup_{D[s]}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player $C$</td>
<td>$C[0] \subseteq C[1] \subseteq C[2] \cdots$</td>
<td>let $C = \bigcup_{C[s]}$</td>
</tr>
</tbody>
</table>

**note 1:** It’s allowed to repeat previous play (ex: $C[s+1] = C[s]$).

- If $C, D \in \mathcal{K}$ are isomorphic then $C$ wins.
- If $C, D \in \mathcal{K}$ are not isomorphic then $D$ wins.
- If $C \in \mathcal{K}$, but $D \not\in \mathcal{K}$, then $C$ wins.
- If $D \in \mathcal{K}$, but $C \not\in \mathcal{K}$, then $D$ wins.
- If $C \not\in \mathcal{K}$, $D \not\in \mathcal{K}$, then $D$ wins.

**note 2:** To play a finite structure legally,

a player has to eventually mark it with a move ‘□’.

**Def:** $\mathcal{K}$ is *copyable* if $C$ has a computable winning strategy.

$\mathcal{K}$ is *diagonalizable* if $D$ has a computable winning strategy.
Examples

Theorem ([Kach, M])

Linear orderings are diagonalizable.

The ideas in this proof are due to:
[Jockusch, Soare 91]: Not every low linear order has computable copy.
[R. Miller 01]: There is an $\mathcal{L}$ with $\text{Spec}(\mathcal{L}) \cap \Delta^0_2 = \Delta^0_2 \setminus \{0\}$.

Theorem

The class $\mathbb{K}$ of Boolean algebras with a predicate for atom, and with infinitely many atoms, is copyable.

The ideas in this proof are due to:
[Downey Jockusch 94]: Every low Boolean algebra has computable copy.

Theorem

The class $\mathbb{K}$ of Boolean algebras with predicates for atom, infinite, atomless, and with infinitely many atoms, is copyable.

The ideas in this proof are due to:
[Thurber 95]: Every low$_2$ Boolean algebra has low copy.
**Def:** \( K \) is *computably listable* if there exists a computable list of all computable structures in \( K \).

**Definition**

**K** is *listable* if there exists a Turing functional \( \Phi \), s.t., \( \forall X \in 2^{\omega} \), \( \Phi^X \) lists all the \( X \)-computable structures in \( K \).

**Theorem (M)**

*If \( K \) is copyable, it’s listable.*

The theorem doesn’t reverse but...
Now, player $C$ builds infinitely many structures $C^0$, $C^1$, $C^2$, ....

<table>
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<tr>
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<th>let $D = \bigcup_s D[s]$</th>
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<tbody>
<tr>
<td>$C^0[0]$</td>
<td>$C^0[1]$ $\subseteq$ $C^0[2]$ $\cdots$</td>
<td>let $C^0 = \bigcup_s C^0[s]$</td>
</tr>
<tr>
<td>$C^1[0]$</td>
<td>$C^1[1]$ $\cdots$</td>
<td>let $C^1 = \bigcup_s C^1[s]$</td>
</tr>
<tr>
<td>$C^2[0]$</td>
<td>$\cdots$</td>
<td>let $C^2 = \bigcup_s C^1[s]$</td>
</tr>
</tbody>
</table>

- If $D, C^0, C^1, \ldots \in K$, and for all $i$, $D \not\approx C^i$ then $D$ wins.
- If $D, C^0, C^1, \ldots \in K$, and for some $i$, $D \cong C^i$, then $C$ wins.
- If for some $i$, $C^i \not\in K$, then $D$ wins.
- If for all $i$, $C \in K$, but $D \not\in K$, then $C$ wins.

**Def:** $K$ is $\infty$-copyable if $C$ has a computable winning strategy. $K$ is $\infty$-diagonalizable if $D$ has a computable winning strategy.

**Theorem (M)**

$K$ is listable if and only if it's $\infty$-copyable.
The $0^{(k)}$-Game

Player $C$ now builds infinitely many structures $C^0, C^1, \ldots$, but needs to choose a single one, $C^j$, using $k$-jumps.

<table>
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<tr>
<th>Player $D$</th>
<th>$D[0]$ $\subseteq$ $D[1]$ $\subseteq$ $\cdots$</th>
<th>let $D = \bigcup_s D[s]$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$e$ $\subseteq$ $C^0[0]$ $\subseteq$ $C^1[0]$ $\subseteq$ $\cdots$</td>
<td>defining $f : \omega \to \omega$</td>
</tr>
<tr>
<td>Player $C$</td>
<td>$f(0)$ $\subseteq$ $f(1)$ $\subseteq$ $\cdots$</td>
<td>let $C^0 = \bigcup_s C^0[s]$</td>
</tr>
<tr>
<td></td>
<td>$C^0[1]$ $\subseteq$ $C^0[2]$ $\subseteq$ $\cdots$</td>
<td>let $C^1 = \bigcup_s C^1[s]$</td>
</tr>
<tr>
<td></td>
<td>$C^1[1]$ $\subseteq$ $\cdots$</td>
<td>let $C^2 = \bigcup_s C^2[s]$</td>
</tr>
<tr>
<td></td>
<td>$C^2[0]$ $\subseteq$ $\cdots$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\vdots$</td>
<td></td>
</tr>
</tbody>
</table>

- If $\{e\}^{f(k)}(0) \uparrow$, then $D$ wins, otherwise, let $j = \{e\}^{f(k)}(0)$.
- If $D, C^0, C^1, \ldots \in \mathbb{K}$, and $D \not\equiv C^j$, then $D$ wins.
- If $D, C^0, C^1, \ldots \in \mathbb{K}$, and $D \equiv C^j$, then $C$ wins.
- If for some $i$, $C^i \not\in \mathbb{K}$, then $D$ wins.
- If for all $i$, $C \in \mathbb{K}$, but $D \not\in \mathbb{K}$, then $C$ wins.

**Def:** $\mathbb{K}$ is $k$-**copyable** if $C$ has a computable winning strategy.

$\mathbb{K}$ is $k$-**diagonalizable** if $D$ has a computable winning strategy.
The 0\(^{(k)}\)-Game

**Obs:**

\[ \text{copyable} \implies 1\text{-copyable} \implies 2\text{-copyable} \implies \cdots \implies \infty\text{-copyable}. \]
\[ \text{diagonalizable} \iff 1\text{-diagonalizable} \iff 2\text{-diagonalizable} \iff \cdots \iff \infty\text{-diagonalizable}. \]

**Theorem**

**Boolean algebras with predicate for atom** are **1-diagonalizable and 2-copyable.**

This can be used to prove:

**[M]**: Not every low BA is 0\(^{(2)}\)-isomorphic to a computable one.

Recall:

**[Downey, Jockusch 94]**: Every low BA is 0\(^{(3)}\)-iso. to a computable one.

**Theorem**

**[M]**: Linear orderings are 4-copyable.

**[Kach, M]**: Linear orderings are 2-diagonalizable.

**Question** Are linear orderings 3-diagonalizable?
**Def:** The *jump of a structure* $\mathcal{A}$ is another structure $\mathcal{A}'$ built by adding relations $\mathcal{A}$, one for each $\Sigma_1$-formula.

**Example:**
- For a linear ordering, $\mathcal{L}' \equiv (\mathcal{L}, \text{succ}, 0')$.
- For a Boolean alg. $\mathcal{B}' \equiv (\mathcal{B}, \text{atom}, 0')$.
- For a Boolean alg. $(\mathcal{B}, \text{atom})' \equiv (\mathcal{B}, \text{atom}, \text{infinite}, \text{atomless}, 0')$.
- For a vector space $\mathcal{V}' \equiv (\mathcal{V}, \text{LinDep}, 0')$.

**Thm:** [Soskov][M 09] $\text{Spec}(\mathcal{A}') = \{ x' : x \in \text{Spec}(\mathcal{A}) \}$.

**Def:** For a class of structures $\mathcal{K}$, let $\mathcal{K}' = \{ \mathcal{A}' : \mathcal{A} \in \mathcal{K} \}$
The low property

**Definition**

We say that $\mathcal{A}$ has the *low property* if, $\forall X, Y \in 2^\omega$ with $X' \equiv_T Y'$, $\mathcal{A}$ has an $X$-computable copy $\iff$ $\mathcal{A}$ has a $Y$-computable copy. $\mathcal{K}$ has the *low property* if every $\mathcal{A}$ in $\mathcal{K}$ does.

**Thm:** $[M \ 09]$ $\mathcal{A}$ has the low property if and only if, $\forall X \in 2^\omega$, $\mathcal{A}'$ has an $X'$-computable copy $\iff$ $\mathcal{A}$ has a $X$ computable copy.

**Theorem (M)**

Assume that $\mathcal{K}$ is $\Pi_2^c$-axiomatizable, then if $\mathcal{K}$ has the low property, $\mathcal{K}'$ is listable.
Let $\mathcal{B}A$ be the class of Boolean algebras

**Example** [Downey, Jockusch 95][Thurber 95][Knight, Stob 00] $\mathcal{B}A$, $\mathcal{B}A'$, $\mathcal{B}A''$ and $\mathcal{B}A'''$ have the low property, and hence $\mathcal{B}A$ has the low$_4$ property.

**Question:** Does $\mathcal{B}A^{(n)}$ have the low property for all $n$?

**Theorem (Harris–M)**

*There is a low$_5$ $\mathcal{B}A$ not $0^{(7)}$-isomorphic to any computable one.*

**Ideas in the proof:**
Let $\mathcal{K}$ be the class of structures $(\mathcal{B}, \text{atom}, P)$ where $\mathcal{B} \in \mathcal{B}A$, and $P$ is a unary relation that defines a c.e. subset of the atoms.

- Then $\mathcal{K}$ is 2-diagonalizable.
- $\mathcal{K}$ embeds, in a sense, in $\mathcal{B}A^{(5)}$. 
We say that $\mathbb{K}$ has a **computable 1-back-and-forth structure** if there is effective listing $t_1, t_2, \ldots$ of all the $\Sigma_1$-types realized in $\mathbb{K}$, and the set $\{ \langle i, j \rangle : t_i \subseteq t_j \}$ is computable.

**Example:** The following class of structures have computable 1-back-and-forth structures:

- linear orderings,
- Boolean algebras,
- $\mathbb{Q}$-vector spaces,
- equivalence structures.
Theorem (M)

Let $\mathbb{K}$ be a $\Pi^c_2$-axiomatizable class of structures with a computable 1-back-and-forth structure.

The following are equivalent:

- $\mathbb{K}$ has the low property.
- $\mathbb{K}'$ is listable.
- $\mathbb{K}'$ is $\infty$-copyable.