The Stable Ramsey’s Theorem for Pairs

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Ramsey’s Theorem

Definition

For $X \subseteq \mathbb{N}$, let $[X]^n$ denote the size $n$ subsets of $X$. For $n, m > 0$ and $F : [\mathbb{N}]^n \to \{0, \ldots, m - 1\}$, $H \subseteq \mathbb{N}$ is homogeneous for $F$ iff $F$ is constant on $[H]^n$.

Theorem (Ramsey, 1930)

For all $n, m > 0$ and all $F : [\mathbb{N}]^n \to \{0, \ldots, m - 1\}$, there is an infinite set $H$ such that $H$ is homogeneous for $F$.

If we fix $n$ and $m$, then we represent that instance of Ramsey’s Theorem by $RT_n^m$.

Question

What are the first and second order consequences of $RT_n^m$?

Subsystems of Second Order Arithmetic: $\text{RCA}_0$

Definition

A model $\mathcal{M}$ of second-order arithmetic consists of a structure $\mathcal{N}$ for first-order arithmetic, called the numbers of $\mathcal{M}$, and a collection of subsets of $\mathcal{N}$, called the reals of $\mathcal{M}$.

Definition

$\text{RCA}_0$ is the second-order theory formalizing the following.

- $P^-$, the axioms for the nonnegative part of a discretely ordered ring.
- $I\Sigma_1$, for $\varphi$ a $\Sigma^0_1$ predicate, if $0$ is a solution to $\varphi$ and the solutions to $\varphi$ are closed under successor, then $\varphi$ holds of all numbers.
- The reals are closed under join and relative $\Delta^0_1$-definability.

In an $\omega$-model $\mathcal{M}$, $\mathcal{M} = \mathbb{N}$ and the reals of $\mathcal{M}$ form an ideal in the Turing degrees.

Recursion Theoretic Content of Ramsey’s Theorem

Theorem (Jockusch, 1972)

- There is a recursive partition of $F$ of pairs such that there is no $F$-homogeneous set which is recursive in $0'$.
- $\text{RCA}_0 \vdash RT_2^3$.

Theorem (Seetapun, 1995)

There is an ideal $J$ in the Turing degrees as follows.

- $0' \notin J$.
- For every $F : [\mathbb{N}]^2 \to 2$ in $J$, there is an infinite $F$-homogeneous $H$ in $J$.
- $RT_2^3 \not\vdash \text{ACA}_0$. 

(\rt_2^3 \not\vdash \text{ACA}_0)
**RT\(^2\), second order consequences**

**Definition**
- An infinite set \( X \) is **cohesive** for a family \( R_0, R_1, \ldots \) of sets iff for each \( i \), one of \( X \cap R_i \) or \( X \cap R_i^c \) is finite. \( \text{COH} \) is the principle stating that every family of sets has a cohesive set.
- A partition \( F : [\mathbb{N}]^2 \to \mathbb{N} \) is **stable** iff for all \( x, y \), \( \lim_{y \to \infty} F(x, y) \) exists. \( \text{SRT}^2_2 \) is the principle \( \text{RT}^2_2 \) restricted to stable partitions.

**Theorem (Cholak, Jockusch, and Slaman, 2001)**
\[
\text{RCA}_0 \vdash [\text{RT}^2_2 \iff (\text{SRT}^2_2 \& \text{COH})]
\]

**Stable Partitions**

**Question**
Does \( \text{SRT}^2_2 \) imply \( \text{RT}^2_2 \)?

**Observations:***
- If \( F \) is a stable partition of pairs, then each \( x \) has an **eventual color** given by \( \lim_{y \to \infty} F(x, y) \), which can be computed from \( F' \).
- If \( H \) is infinite and monochromatic with respect to eventual color, then \( H \) can compute an \( F \)-homogeneous set. Consequently, \( F' \) can compute one.
- \( \text{SRT}^2_2 \) is equivalent to “For every \( \Delta^0_2 \) property \( A \), there is an infinite set \( H \) contained in or disjoint from \( A \).”

**Proposal and Rejection**

**Proposal:**
- Show that for every \( \Delta^0_2 \) subset \( A \) of \( \mathbb{N} \), there is a low infinite set \( H \) contained in or disjoint from \( A \).
- Iterate this fact to build an ideal \( J \) in the Turing degrees consisting of only low sets such that for every \( \Delta^0_2 \) subset \( A \) of \( \mathbb{N} \), there is an infinite \( H \in J \) contained in or disjoint from \( A \).
- Conclude that \( (\mathbb{N}, J) \) is a model of \( \text{SRT}^2_2 \) which is not a model of \( \text{RT}^2_2 \).

**Rejection:**

**Theorem (Downey, Hirschfeldt, Lempp, and Solomon, 2001)**
There is a \( \Delta^0_2 \) set with no infinite low subset in either it or its complement.

**RT\(^2\), first order consequences**

The following two theorems bracket the first order theory of \( \text{RT}^2_2 \).

**Theorem (Hirst, 1987)**
\[
\text{RCA}_0 \vdash (\text{SRT}^2_2 \implies B\Sigma_2), \text{ where } B\Sigma_2 \text{ is the formalization of the assertion “If a } \Sigma_2 \text{ property holds of every element of a finite set then there is a uniform bound on the witnesses.”}
\]

**Theorem (Cholak, Jockusch, and Slaman, 2001)**
\( \text{RT}^2_2 \) is conservative over \( \text{RCA}_0 + I\Sigma_2 \) for \( \Pi^1_1 \)-sentences.

**Question**
Does either of \( \text{SRT}^2_2 \) or \( \text{RT}^2_2 \) imply \( I\Sigma_2 \)?
**SRT\(_2^2\) in a Customized Model**

**Theorem (Chong, Slaman, and Yang)**

There is a model \(\mathcal{M}\) of \(\text{RCA}_0\) with the following properties.

- \(\mathcal{M} \models SRT_2^2\)
- \(\mathcal{M} \models \neg I\Sigma_2\)
- Every real in \(\mathcal{M}\) is low in \(\mathcal{M}\).

**Corollary**

\(SRT_2^2\) proves neither \(I\Sigma_2\) nor \(RT_2^2\) over \(\text{RCA}_0\).

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**The Natural Numbers of \(\mathcal{M}\)**

We construct a model \(\mathcal{N}\) of \(\text{P}^\alpha + \Sigma^0_2\) within which there is a function \(g\) that is recursive in \(0'\) behaving as follows.

- \(\mathcal{N}\) is the union of a sequence of \(\Sigma_1\)-elementary end-extensions of models of \(\text{PA}\).
  
  \[
  \mathcal{N}_1 \prec_{\Sigma_1} \mathcal{N}_2 \prec_{\Sigma_1} \mathcal{N}_3 \prec_{\Sigma_1} \cdots \prec_{\Sigma_1} \mathcal{N}
  \]

- For each \(i \in \mathbb{N}\), \(g(i) \in \mathcal{N}_i \setminus \mathcal{N}_{i-1}\), hence \(\mathcal{N} \nvdash I\Sigma_2\).

- For each set \(Y \subseteq \mathbb{N}\), if \(Y\) is definable in \(\mathcal{N}\) then \(Y\) has an \(\mathcal{N}\)-finite end-extension in \(\mathcal{N}\).

- \(\mathcal{N}\) is countable.

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**Low Subsets of \(\Delta_2^0\) Sets in \(\mathcal{N}\)**

Suppose that \(A\) is \(\Delta_2^0\) in \(\mathcal{N}\). Build \(H_0 \subseteq A\) and \(H_1 \subseteq \overline{A}\).

- For \(a \in \mathcal{N}\), adapt Seetapun’s argument so as to decide \(H'_0 \upharpoonright a\) or of \(H'_1 \upharpoonright a\).

- Use reflection to the models \(\mathcal{N}_i\) of \(\text{PA}\) to show that the activity of a single step is bounded in \(\mathcal{N}\).

- Construct \(H_0\) and \(H_1\) by an \(\omega\)-length recursion. Each step has two parts:
  - an \(\mathcal{N}\)-finite extension, uniformly computed from \(0'\)
  - a global constraint, non-uniformly computed from \(0'\), depending on a \(\Sigma_2\)-boolean condition

- Conclude that the construction of \(H_0\) and \(H_1\) is \(\Delta_2^0\) in \(\mathcal{N}\), using the \(\mathcal{N}\)-finite parameter extending the sequence of \(\Sigma_2\)-boolean conditions that appear in the construction.

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**Extending to a Model of \(SRT_2^2\)**

We obtain an \(H\) as desired, but it does not preserve the \(\Sigma_1\)-reflection properties of \(\mathcal{N}\) that were used to construct it. So, we cannot simply iterate the argument.

We use a full-approximation construction to obtain a collection of subsets \(\mathcal{J}\) of \(\mathcal{N}\) with the desired properties:

- \((\mathcal{N}, \mathcal{J}) \models \text{RCA}_0\)

- For any \(X \in \mathcal{J}\), \(X\) is low in \(\mathcal{N}\).

- For every \(A\) which is \(\Delta_2^0\) in \(\mathcal{N}\), there is an infinite element of \(\mathcal{J}\) which is contained in it or in its complement.
Questions

Question

- Does “$\text{RCA}_0 + \text{SRT}_2^2 \not\vdash \text{RT}_2^2$” settle the issue of whether there is an effective proof of Ramsey’s Theorem for Pairs given that Ramsey’s Theorem for Pairs is true for stable partitions?

- For $\omega$-models, does $\text{SRT}_2^2$ imply $\text{RT}_2^2$? Perhaps $\text{I}\Sigma_2$ is sufficient to deduce the implication.

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