Uniform reduction and reverse mathematics
Preliminary report

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Motivation

**Goal:** Explore the relationship between uniform (also called Weihrauch) reducibility and results in reverse mathematics.

**Observation:** Some reducibility results and reverse mathematics results have proofs with significant common content.

For example, in [1], Gura, Hirst, and Mummert prove:

\[ \text{RCA}_0 \vdash \text{FC1} \iff \text{FC3} \text{ and FC1} \equiv_{sW} \text{FC3} \]

where FC1 says: every infinite graph in which every connected component is finite has a sequence of canonical indices of different components.

FC3 says: every infinite graph in which every connected component is finite has an infinite totally disconnect set.
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- **FC3** says: every infinite graph in which every connected component is finite has an infinite totally disconnect set.
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**Observation:** Some reducibility results and reverse mathematics results have proofs with significant common content.

We can reduce duplication in our arguments if we can prove single results that have both desired consequences as immediate corollaries.
Formalizing \( sW \) reduction

One characterization of \( sW \) reduction is to consider \textit{problems}:

The problem \( P \) is a sentence \( \forall X \exists Y \ p(X, Y) \), where \( p(X, Y) \) is a formula of second order arithmetic.

If \( p(X_P, Y_P) \), we say \( X_P \) is an instance of the problem \( P \) and \( Y_P \) is a solution of \( X_P \).

In this setting \( Q \leq_{sW} P \) means there are computable functionals \( \psi \) and \( \phi \) such that

\[
\begin{array}{c}
X_Q \\
\downarrow \\
Y_Q
\end{array} \xrightarrow{\psi} \begin{array}{c}
X_P \\
\downarrow \\
Y_P
\end{array} \xleftarrow{\phi}
\]
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In this setting \( Q \leq_{sW} P \) means there are computable functionals \( \psi \) and \( \phi \) of type \( 1 \rightarrow 1 \) such that

\[
\begin{array}{c}
\psi \\
X_Q \rightarrow X_P \\
\downarrow \\
Y_Q \\
\phi \leftarrow Y_P
\end{array}
\]
Kohlenbach’s axioms

Kohlenbach [3] presents axioms for reverse mathematics in higher types.

- $\text{RCA}_0^\omega$ consists of $\text{E-HA}_1^\omega$ plus law of the exclude middle plus QF-AC$^{1,0}$:

$$\forall X \exists y A(X, y) \rightarrow \exists Y \forall X A(X, Y(X))$$

for $A$ quantifier free.

- $\text{E-HA}_1^\omega$ is intuitionistic arithmetic in all finite types. (See §3.4 of Kohlenbach [4]).

- $\text{E-HA}_1^\omega$ includes combinators allowing $\lambda$-abstraction.
Formalizing sW reduction

Since the functionals defining sW reduction are of finite type, statements about their existence can be formulated in higher order reverse mathematics.

Since the type structure ECF is a model of RCA$^\omega_0$ that contains only computable functionals (see [3]), if RCA$^\omega_0 \vdash Q \leq_{sW} P$, then $Q \leq_{sW} P$.

By composition of functionals,

\[ \text{RCA}_0^\omega \vdash Q \leq_{sW} P \rightarrow (P \rightarrow Q \land \hat{P} \rightarrow \hat{Q}) \]

where $\hat{P}$ is the infinite parallelization of $P$.

By Proposition 3.1 of Kohlenbach [3]:

If $\text{RCA}_0^\omega \vdash \theta$ then $\text{RCA}_0 \vdash \theta$. 
A sample problem

**Goal:** Prove $\text{RCA}_0^\omega \vdash \widehat{\text{LPO}} \leq_{sW} \text{RAN}$.

$\widehat{\text{LPO}}$ is $\forall \langle p_n \rangle \exists g \ (g(i) = 1 \iff \exists t \ p_i(t) = 0)$

So $g$ selects those $i$ such that 0 is in the range of $p_i$. Infinite parallelization of the limited principle of omniscience.

$\text{RAN}$ is “Every injective function has a range.”

$\forall f \ \exists \chi_f \ \forall y \ (\chi_f(y) = 1 \iff \exists t \ f(t) = y)$
$\widehat{\text{LPO}} \leq_{sW} \text{RAN}: \text{Construction of } \phi \text{ in } \text{RCA}_0^\omega$

Given $\langle p_n \rangle$ for $\widehat{\text{LPO}}$, define an injection $f$ by $f((i, j)) = k$ if and only if the following formula (denoted $\theta(\langle p_n \rangle, (i, j), k)$) holds:

$$(k = 2i + 1 \land p_i(j) = 0 \land \forall t < j p_i(t) \neq 0) \lor (k = 2(i, j) \land (p_i(j) \neq 0 \lor \exists t < j p_i(t) = 0))$$

Note that $2i + 1 \in \text{RAN}(f)$ if and only if $\exists t p_i(t) = 0$, so

$$\chi_{\text{RAN}(f)}(2i + 1) = \begin{cases} 0 & \text{if } \forall t p_i(t) \neq 0 \\ 1 & \text{if } \exists t p_i(t) = 0 \end{cases}$$

which is the solution to the instance $\langle p_n \rangle$ of $\widehat{\text{LPO}}$.

Define $\phi$ by $\phi(\chi_{\text{RAN}(f)}) = \chi_{\text{RAN}(f)}(2i + 1)$. 
Working in RCA$_0^\omega$, we need to prove the existence of the functional $\psi$ mapping $\langle p_n \rangle$ to $f$ (as defined on the previous slide).

Our main tool is QF-AC$^{1,0}$: $\forall X \exists y \ A(X, y) \rightarrow \exists Y \forall X \ A(X, Y(X))$

$\theta(\langle p_n \rangle, (i, j), k)$ is $\Sigma^0_0$ and $\forall(\langle p_n \rangle, (i, j)) \exists k \ \theta(\langle p_n \rangle, (i, j), k)$, so QF-AC$^{1,0}$ proves the existence of a functional $F$ such that $\theta(\langle p_n \rangle, (i, j), F(\langle p_n \rangle, (i, j)))$.

$F$ maps $\langle p_n \rangle, (i, j)$ to $f((i, j))$.

Thus $f$ is $\lambda(i, j).F((\langle p_n \rangle, (i, j)))$

and $\psi = \lambda\langle p_n \rangle.[\lambda(i, j).F((\langle p_n \rangle, (i, j)))]$. 

LPO $\leqslant_{sW}$ RAN: Construction of $\psi$ in RCA$_0^\omega$
Summarizing the demonstration problem:

We showed $\text{RCA}_0^\omega \vdash \widehat{\text{LPO}} \leq_{sW} \text{RAN}$. Similar techniques can be used to prove $\text{RCA}_0^\omega \vdash \text{RAN} \leq_{sW} \widehat{\text{LPO}}$, so

$$\text{RCA}_0^\omega \vdash \widehat{\text{LPO}} \equiv_{sW} \text{RAN}.$$ 

Consequently,

Because every functional in the ECF model of $\text{RCA}_0^\omega$ is computable,

$$\widehat{\text{LPO}} \equiv_{sW} \text{RAN}$$

By Kohlenbach’s conservation result,

$$\text{RCA}_0 \vdash \widehat{\text{LPO}} \leftrightarrow \text{RAN} \quad \text{and} \quad \text{RCA}_0 \vdash \widehat{\text{LPO}} \leftrightarrow \widehat{\text{RAN}}$$

Because $\text{RCA}_0^\omega \vdash \widehat{P} \equiv_{sW} \widehat{P}$,

$$\text{RCA}_0 \vdash \text{ACA}_0 \leftrightarrow \widehat{\text{LPO}} \leftrightarrow \widehat{\text{LPO}} \leftrightarrow \text{RAN} \leftrightarrow \widehat{\text{RAN}}$$
Questions

- How unfaithful is this formalization of sW reduction? Find good examples where $P \leq_{sW} Q$ but $\text{RCA}_0^\omega \not\vdash P \leq_{sW} Q$. In particular, what about statements that are equivalent to the pigeonhole principle or to $\Sigma^0_2$ induction?

- If $\text{RCA}_0^\omega \not\vdash P \leq_{sW} Q$, then we can view the formalization of $P \leq_{sW} Q$ as a “functional existence axiom” which is not provable in $\text{RCA}_0^\omega$. What is the logical strength of these functional existence axioms? How are they related to the $\rightarrow$ operator on Weihrauch problems?

- What about other reducibilities?

Organizers: V. Brattka, A. Kawamura, A. Marcone, A. Pauly

Associated bibliography:  
http://cca-net.de/publications/weibib.php

Summaries of talks will eventually appear in *Dagstuhl Reports*
Some references


