Computability strength of the field of real numbers

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Definition (Schweber)

Let $\mathcal{A}$ and $\mathcal{B}$ be structures, potentially uncountable. Then $\mathcal{A} \leq^*_w \mathcal{B}$ if, after a forcing collapse that causes $\mathcal{A}$ and $\mathcal{B}$ to both become countable, every copy of $\mathcal{B}$ computes a copy of $\mathcal{A}$.

Under reasonable hypotheses, this reducibility does not depend on the forcing used.

This agrees with $\leq_w$ on countable structures.

In practice, very little set theory is involved: most proofs can be written by just imagining that $\mathcal{A}$ and $\mathcal{B}$ were countable, and seeing what would happen.
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In practice, very little set theory is involved: most proofs can be written by just imagining that $\mathcal{A}$ and $\mathcal{B}$ were countable, and seeing what would happen.
Our Structures

- $\mathcal{R}_{\text{exp}} = (\mathbb{R}, +, \cdot, <, \text{exp})$
- $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$
- $\mathcal{R}_+ = (\mathbb{R}, +, <)$
- $\mathcal{R}_Q = (\mathbb{R}, \text{constants } \langle q \rangle (q \in \mathbb{Q}), <)$
- $\mathcal{B} = (\omega^\omega, \text{predicates } \langle f(n) = m \rangle) \equiv^*_{w} (\mathbb{R}, \text{binary expansion})$
- $\mathcal{W} = (P(\omega), \text{predicates } \langle n \in \rangle)$

Observation (Knight, Montalban, Schweber; IKS)

$\mathcal{W} \leq^*_{w} \mathcal{B} \leq^*_{w} \mathcal{R}_Q \leq^*_{w} \mathcal{R}_+ \leq^*_{w} \mathcal{R} \leq^*_{w} \mathcal{R}_{\text{exp}}$
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\[ R_{\text{exp}} = (\mathbb{R}, +, \cdot, <, \exp) \]
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The Relations

Theorem (Igusa, Knight, Schwebber)
\[ \mathcal{W} \prec_w B \equiv_w \mathcal{R}_Q \equiv_w \mathcal{R}_+ \equiv_w \mathcal{R} \equiv_w \mathcal{R}_{\exp} \]

Theorem (Downey, Greenberg, Miller)
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\[ \mathcal{W} <^*_w \mathcal{B} \equiv^*_w \mathcal{R}_+ \equiv^*_w \mathcal{R} \]
Every countable Scott set $S$ has a real closed field realizing exactly the types in $S$ that is equicomputable with $S$. (Macintyre and Marker)

If $S$ is $\mathcal{W}$, then this real closed field, $\tilde{\mathcal{R}}$, is a recursively saturated extension of $\mathcal{R}$, so $\mathcal{R}$ is the residue field of $\tilde{\mathcal{R}}$

**Theorem (Igusa, Knight)**

Let $K$ be a countable recursively saturated real closed field with residue field $k$. Then $k \not\leq_w K$. 
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Effective structure theory!

- A property is computable \( \Sigma_2 \) definable in a structure if and only if it is \( \Sigma_0^2 \) in every copy of the structure. (Ash, Knight, Manasse, Slaman)

- If \( k \leq_w K \), then \( FT(K) \), the set of finite elements of \( K \) in transcendental Dedekind cuts, is computable \( \Sigma_2 \) definable in \( K \). (A direct 0′-style construction)

- If \( K \) is recursively saturated, then \( FT(K) \) is not computable \( \Sigma_2 \) definable in \( K \). (Next Slide)
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- If $K$ is recursively saturated, then $FT(K)$ is not computable $\Sigma_2$ definable in $K$. (Next Slide)
Proving that $FT(K)$ is not computable $\Sigma_2$ definable in $K$:

- If $FT(K)$ were definable via a $\exists\forall -$ formula, then each $\forall -$ part would have to omit all algebraic Dedekind cuts.

- **Recursive saturation** + compactness of first order logic means that it must omit a neighborhood of that cut, and that it must do so at a finite stage.

- We then dive into that neighborhood, but not that cut, dodging both the $\forall -$ formula and the algebraic cut.

- We then repeat this infinitely many times to produce a transcendental element that does not satisfy any of the $\forall -$ formulas.

- **Recursive saturation** guarantees that this element exists in $K$. 
Theorem (DGM)

If \( I \) is a countable Scott ideal, then to list all the functions in \( I \), you must be able to compute a function dominating all of them, but you can list all the sets in \( I \) without doing so.

(A forcing proof.)

Theorem (DGM)

If \( I \) is a countable Scott ideal, then from a list of all the functions in \( I \), you can compute the field of reals whose Turing degrees are in \( I \).

(Uses quantifier elimination and decidability of \( Th(RCF) \).)

(Note, in both of these, \( I \) is the set of all Turing degrees in \( \mathcal{W} \) or equivalently \( B \).)
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Properties from $\mathcal{R}_Q$:

- Ability to code $Th(\mathbb{R}, +, \cdot, \exp)$ using a parameter.
- Ability to code infinite paths through a tree in a $\Pi^0_1$ manner.
- Ability to code open rational boxes.
- Ability to list all the reals, and compute which boxes they are in.

Properties from $\mathcal{R}_{\exp}$:

- o-minimal.
- Any copy of $\mathcal{R}_Q$ has a unique expansion to $\mathcal{L}(\mathbb{R}_{\exp})$.

Theorem (Igusa, Knight, Schwebber)

$\mathcal{R}_Q \equiv^*_w \mathcal{R}_{\exp}$

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**Theorem (Igusa, Knight, Schwebber)**

$\mathcal{R}_Q \equiv^*_w \mathcal{R}_{exp}$
Choose a parameter coding $Th(\mathbb{R}, +, \cdot, \exp)$.

Using this parameter, we find an algebraicity basis for $\mathcal{R}_{\exp}$ in $\mathcal{R}_{\mathbb{Q}}$ in a $\Delta^0_2$ way.

(Uses the fact that a tuple is algebraically independent if and only if every formula that is true about it is true on a rational box around it.)

Using Dedekind approximations to a basis, together with a parameter for the theory, can build a copy of $\mathcal{R}_{\exp}$.

Add finite injury to the construction because we have a $\Delta^0_2$ approximation to the basis.
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The function \( \exp \) can be replaced by any other \( f \) such that \( \mathcal{R}_f \) is o-minimal.

Using restrictions to compact intervals, we can get any analytic \( f \). (van den Dries, Gabrielov)

Classically, \( f : \mathbb{R} \to \mathbb{R} \) is continuous if and only if \( f \) is computable from a parameter.

In our context, this only shows that \( \mathcal{R}_Q \equiv^*_w \mathcal{R}_{Q,f} \).

Question:

Is \( \mathcal{R} \equiv^*_w \mathcal{R}_{Q,f} \) for an arbitrary continuous \( f \)?
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