A measure of uniformity

Rutger Kuyper

4 October 2015
Medvedev reducibility

**Definition.** Let $A, B \subseteq \omega^\omega$. Then we say that $A$ Medvedev reduces to $B$ ($A \leq_M B$) if there is a single Turing functional $\Phi$ such that $\Phi(B) \subseteq A$. 
Medvedev reducibility
Muchnik reducibility

**Definition.** Let $\mathcal{A}, \mathcal{B} \subseteq \omega^\omega$. Then we say that $\mathcal{A}$ *Muchnik reduces to* $\mathcal{B}$ ($\mathcal{A} \leq_w \mathcal{B}$) if and only if for every $g \in \mathcal{B}$ there exists $f \in \mathcal{A}$ with $f \leq_T g$. 
Muchnik reducibility
Where uniformity fails

**Theorem.** (Kučera) Let $n \in \omega$ and let $X$ be $n$-random. Then $X$ computes an $n$-DNC function.

**Proposition.** This does not hold uniformly.

**Proof.** Use the fact that the random reals are dense within $2^\omega$. 

**Theorem.** (Kautz) Every 2-random computes a function which is not computably dominated.

**Proposition.** This does not hold uniformly.

**Proof.** Majority vote.
Where uniformity fails

**Theorem.** (Kučera) Let $n \in \omega$ and let $X$ be $n$-random. Then $X$ computes an $n$-DNC function.

**Proposition.** This does not hold uniformly.

**Proof.** Use the fact that the random reals are dense within $2^\omega$.
Where uniformity fails

**Theorem.** (Kučera) Let $n \in \omega$ and let $X$ be $n$-random. Then $X$ computes an $n$-DNC function.

**Proposition.** This does not hold uniformly.

**Proof.** Use the fact that the random reals are dense within $2^\omega$. \qed
Where uniformity fails

**Theorem.** (Kučera) Let $n \in \omega$ and let $X$ be $n$-random. Then $X$ computes an $n$-DNC function.

**Proposition.** This does not hold uniformly.

**Proof.** Use the fact that the random reals are dense within $2^\omega$. 

**Theorem.** (Kautz) Every 2-random computes a function which is not computably dominated.
Where uniformity fails

**Theorem.** (Kučera) Let $n \in \omega$ and let $X$ be $n$-random. Then $X$ computes an $n$-DNC function.

**Proposition.** This does not hold uniformly.

*Proof.* Use the fact that the random reals are dense within $2^\omega$. □

**Theorem.** (Kautz) Every 2-random computes a function which is not computably dominated.

**Proposition.** This does not hold uniformly.
Where uniformity fails

**Theorem.** (Kučera) Let $n \in \omega$ and let $X$ be $n$-random. Then $X$ computes an $n$-DNC function.

**Proposition.** This does not hold uniformly.

*Proof.** Use the fact that the random reals are dense within $2^{\omega}$. □

**Theorem.** (Kautz) Every 2-random computes a function which is not computably dominated.

**Proposition.** This does not hold uniformly.

*Proof.** Majority vote. □
More failing uniformity

**Theorem.** (Jockusch) *We have that* $\text{DNC}_2 \leq_w \text{DNC}_3$, *but* $\text{DNC}_2 \not\leq_M \text{DNC}_3$. 
Theorem. (Jockusch) We have that $\text{DNC}_2 \leq_w \text{DNC}_3$, but $\text{DNC}_2 \not\leq_M \text{DNC}_3$.

Proof. A kind of majority vote.
Intermediate degree structures

**Definition.** Let $\mathcal{A}, \mathcal{B} \subseteq \omega^\omega$ and let $n \in \omega$. Then we say that $\mathcal{A}$ *n-uniformly reduces to* $\mathcal{B}$ (notation: $\mathcal{A} \leq_n \mathcal{B}$) if there exists a sequence $\mathcal{V}_0, \mathcal{V}_1, \ldots$ of uniformly $\Pi^0_n$ sets with $\mathcal{B} \subseteq \bigcup_{i \in \omega} \mathcal{V}_i$ and a uniformly computable sequence $e_0, e_1, \ldots$ such that for every $i \in \omega$ and every $f \in \mathcal{B} \cap \mathcal{V}_i$ we have $\Phi_{e_i}(f) \in \mathcal{A}$. 
Intermediate degree structures
Intermediate degree structures

Definition. Let $\mathcal{A}, \mathcal{B} \subseteq \omega^\omega$ and let $n \in \omega$. Then we say that $\mathcal{A}$ $n$-uniformly reduces to $\mathcal{B}$ (notation: $\mathcal{A} \leq_n \mathcal{B}$) if there exists a sequence $\mathcal{V}_0, \mathcal{V}_1, \ldots$ of uniformly $\Pi^0_n$ sets with $\mathcal{B} \subseteq \bigcup_{i \in \omega} \mathcal{V}_i$ and a uniformly computable sequence $e_0, e_1, \ldots$ such that for every $i \in \omega$ and every $f \in \mathcal{B} \cap \mathcal{V}_i$ we have $\Phi_{e_i}(f) \in \mathcal{A}$. Note that $\leq_n$ induces a degree structure $M_n$ in the usual way, the $n$-uniform degrees. For $n = 1$, this structure was also studied by Higuchi and Kihara, although in a different setting and with a different (but equivalent) definition.
Intermediate degree structures

**Definition.** Let $\mathcal{A}, \mathcal{B} \subseteq \omega^\omega$ and let $n \in \omega$. Then we say that $\mathcal{A}$ $n$-uniformly reduces to $\mathcal{B}$ (notation: $\mathcal{A} \leq_n \mathcal{B}$) if there exists a sequence $\mathcal{V}_0, \mathcal{V}_1, \ldots$ of uniformly $\Pi^0_n$ sets with $\mathcal{B} \subseteq \bigcup_{i \in \omega} \mathcal{V}_i$ and a uniformly computable sequence $e_0, e_1, \ldots$ such that for every $i \in \omega$ and every $f \in \mathcal{B} \cap \mathcal{V}_i$ we have $\Phi_{e_i}(f) \in \mathcal{A}$.

Note that $\leq_n$ induces a degree structure $\mathcal{M}_n$ in the usual way, the $n$(-uniform)-degrees.
Intermediate degree structures

**Definition.** Let \( \mathcal{A}, \mathcal{B} \subseteq \omega^\omega \) and let \( n \in \omega \). Then we say that \( \mathcal{A} \ n\)-uniformly reduces to \( \mathcal{B} \) (notation: \( \mathcal{A} \leq_n \mathcal{B} \)) if there exists a sequence \( \mathcal{V}_0, \mathcal{V}_1, \ldots \) of uniformly \( \Pi^0_n \) sets with \( \mathcal{B} \subseteq \bigcup_{i \in \omega} \mathcal{V}_i \) and a uniformly computable sequence \( e_0, e_1, \ldots \) such that for every \( i \in \omega \) and every \( f \in \mathcal{B} \cap \mathcal{V}_i \) we have \( \Phi_{e_i}(f) \in \mathcal{A} \).

Note that \( \leq_n \) induces a degree structure \( \mathcal{M}_n \) in the usual way, the \( n\)(-uniform)-degrees.

For \( n = 1 \), this structure was also studied by Higuchi and Kihara, although in a different setting and with a different (but equivalent) definition.
Some elementary results

**Proposition.** Medvedev reducibility and 0-reducibility coincide.

We will write $\leq_{\infty}$ for $\leq_w$. 
Some elementary results

**Proposition.** Medvedev reducibility and $0$-reducibility coincide.

We will write $\leq_{\infty}$ for $\leq_w$.

**Proposition.** For every $n \in \omega \cup \{\infty\}$, $\mathcal{M}_n$ is a distributive lattice. In fact, it is even a Brouwer algebra: there is an operation $\to_n$ such that

$$\mathcal{A} \oplus C \geq_n \mathcal{B} \iff C \geq_n \mathcal{A} \to_n \mathcal{B}.$$
Going back and forth

**Proposition.** Let $n, m \in \omega \cup \{\infty\}$ with $n \leq m$. Then the natural surjection from $\mathcal{M}_n$ onto $\mathcal{M}_m$ (induced by the identity map) preserves $\oplus$ and $\otimes$, but not necessarily $\to$.

**Theorem.** ($m = 0, n = \infty$: Sorbi; $m = 0, n = 1$: Higuchi and Kihara) Let $n, m \in \omega \cup \{\infty\}$ with $n \leq m$. Then there is an embedding of $\mathcal{M}_n$ into $\mathcal{M}_m$ preserving $\oplus$ and $\to$, but not necessarily $\otimes$.

\[
\mathcal{M}_0 \leftrightarrow \mathcal{M}_1 \leftrightarrow \mathcal{M}_2 \leftrightarrow \cdots \leftrightarrow \mathcal{M}_\infty
\]
Levels of uniformity

**Definition.** Let $\mathcal{A} \leq_w \mathcal{B}$. Then we say that the *uniformity of $\mathcal{A}$ to $\mathcal{B}$* is the least $n \in \omega \cup \{\infty\}$ such that $\mathcal{A} \leq_n \mathcal{B}$. 

---

**Proposition.** (Higuchi and Kihara) Let $\mathcal{A} \leq_w \mathcal{B}$ be such that $\mathcal{A}$ is $\Sigma^0_{n+1}$. Then the uniformity of $\mathcal{A}$ to $\mathcal{B}$ is at most $\max(n, 2)$. 

Levels of uniformity

**Definition.** Let $\mathcal{A} \leq_w \mathcal{B}$. Then we say that the *uniformity of $\mathcal{A}$ to $\mathcal{B}$* is the least $n \in \omega \cup \{\infty\}$ such that $\mathcal{A} \leq_n \mathcal{B}$.

**Proposition.** (Higuchi and Kihara) Let $\mathcal{A} \leq_w \mathcal{B}$ be such that $\mathcal{A}$ is $\Sigma^0_{n+1}$. Then the uniformity of $\mathcal{A}$ to $\mathcal{B}$ is at most $\max(n, 2)$. 
Levels of uniformity in randomness

**Theorem.** (Effective 0-1-law, Kučera) Let $n \in \omega$, let $\mathcal{V}$ be a $\Pi^0_n$-class of positive measure and let $X$ be $n$-random. Then there is a $k \in \omega$ with $X \upharpoonright [k, \infty) \in \mathcal{V}$.
Levels of uniformity in randomness

**Theorem.** (Effective 0-1-law, Kučera) Let $n \in \omega$, let $\mathcal{V}$ be a $\Pi^0_n$-class of positive measure and let $X$ be $n$-random. Then there is a $k \in \omega$ with $X \upharpoonright [k, \infty) \in \mathcal{V}$.

**Theorem.** Let $n \in \omega$, let $\mathcal{A}$ be a mass problem and let $n$–Random be the class of $n$-randoms. Assume there exists a $\Pi^0_n$-class $\mathcal{V}$ of positive measure such that $\mathcal{A} \leq_M \mathcal{V}$. Then $\mathcal{A} \leq_n n$-Random.
Levels of uniformity in randomness

**Theorem.** (Effective 0-1-law, Kučera) Let $n \in \omega$, let $\mathcal{V}$ be a $\Pi^0_n$-class of positive measure and let $X$ be $n$-random. Then there is a $k \in \omega$ with $X \upharpoonright [k, \infty) \in \mathcal{V}$.

**Theorem.** Let $n \in \omega$, let $\mathcal{A}$ be a mass problem and let $n$–Random be the class of $n$-randoms. Assume there exists a $\Pi^0_n$-class $\mathcal{V}$ of positive measure such that $\mathcal{A} \leq_M \mathcal{V}$. Then $\mathcal{A} \leq_n n$-Random.

**Theorem.** Let $n \in \omega \cup \{\infty\}$ with $n \geq 1$. Then $n$-DNC Muchnik-reduces to $n$-randomness, with uniformity $n$. 

Levels of uniformity in randomness

**Theorem.** (Effective 0-1-law, Kučera) Let $n \in \omega$, let $\mathcal{V}$ be a $\Pi^0_n$-class of positive measure and let $X$ be $n$-random. Then there is a $k \in \omega$ with $X \upharpoonright [k, \infty) \in \mathcal{V}$.

**Theorem.** Let $n \in \omega$, let $\mathcal{A}$ be a mass problem and let $n$–Random be the class of $n$-randoms. Assume there exists a $\Pi^0_n$-class $\mathcal{V}$ of positive measure such that $\mathcal{A} \leq^M \mathcal{V}$. Then $\mathcal{A} \leq_n n$-Random.

**Theorem.** Let $n \in \omega \cup \{\infty\}$ with $n \geq 1$. Then $n$-DNC Muchnik-reduces to $n$-randomness, with uniformity $n$.

**Corollary.** If $n \neq m$ then $n$-reducibility and $m$-reducibility differ.
Levels of uniformity in randomness

**Theorem.** (Effective 0-1-law, Kučera) Let $n \in \omega$, let $\mathcal{V}$ be a $\Pi^0_n$-class of positive measure and let $X$ be $n$-random. Then there is a $k \in \omega$ with $X \upharpoonright [k, \infty) \in \mathcal{V}$.

**Theorem.** Let $n \in \omega$, let $\mathcal{A}$ be a mass problem and let $n-$Random be the class of $n$-randoms. Assume there exists a $\Pi^0_n$-class $\mathcal{V}$ of positive measure such that $\mathcal{A} \leq_M \mathcal{V}$. Then $\mathcal{A} \leq_n n$-Random.

**Theorem.** Let $n \in \omega \cup \{\infty\}$ with $n \geq 1$. Then $n$-DNC Muchnik-reduces to $n$-randomness, with uniformity $n$.

**Corollary.** If $n \neq m$ then $n$-reducibility and $m$-reducibility differ.

**Theorem.** The uniformity of the non-computably-dominated functions to the 2-random sets is 2.
More levels of uniformity

**Theorem.** (Higuchi and Kihara)

\[ \operatorname{DNC}_2 \not\leq_1 \operatorname{DNC}_3. \]

**Corollary.** *The uniformity of* \( \operatorname{DNC}_2 \) *to* \( \operatorname{DNC}_3 \) *is* 2.
Comparing to layerwise computability

Fix a universal Martin-Löf test $U_0, U_1, \ldots$. Let us say that $A$ layerwise reduces to 1-randomness if there is a uniformly computable sequence $e_0, e_1, \ldots$ such that $\Phi_{e_i}(2^\omega \setminus U_i) \subseteq A$.

**Question.** Is this weaker than 1-uniform reducibility to $n$-Random?
Comparing to layerwise computability

Fix a universal Martin-Löf test $U_0, U_1, \ldots$. Let us say that $A$ layerwise reduces to 1-randomness if there is a uniformly computable sequence $e_0, e_1, \ldots$ such that $\Phi_{e_i}(2^\omega \setminus U_i) \subseteq A$.

**Question.** Is this weaker than 1-uniform reducibility to $n$-Random?

**Theorem.** Let $n \in \omega \cup \{\infty\}$ with $n \geq 1$. Then $n$-DNC Muchnik-reduces to $n$-randomness, with uniformity $n$. 
Comparing to layerwise computability

Fix a universal Martin-Löf test $U_0, U_1, \ldots$. Let us say that $A$ \textit{layerwise reduces to 1-randomness} if there is a uniformly computable sequence $e_0, e_1, \ldots$ such that $\Phi_{e_i}(2^\omega \setminus U_i) \subseteq A$.

**Question.** Is this weaker than 1-uniform reducibility to $n$-Random?

**Theorem.** Let $n \in \omega \cup \{\infty\}$ with $n \geq 1$. Then $n$-DNC$_{2m}$ Muchnik-reduces to $n$-randomness, with uniformity $n$. 
Comparing to layerwise computability

Fix a universal Martin-Löf test $U_0, U_1, \ldots$. Let us say that $A$ layerwise reduces to $1$-randomness if there is a uniformly computable sequence $e_0, e_1, \ldots$ such that $\Phi_{e_i}(2^\omega \setminus U_i) \subseteq A$.

**Question.** Is this weaker than $1$-uniform reducibility to $n$-Random?

**Theorem.** Let $n \in \omega \cup \{\infty\}$ with $n \geq 1$. Then $n$-DNC$_{2m}$ Muchnik-reduces to $n$-randomness, with uniformity $n$.

**Proposition.** We do not have that $n$-DNC$_{2m}$ reduces layerwise to $n$-randomness.
Elementary (in)equivalence

**Theorem.** Let $n, m \in \omega \cup \{\infty\}$ with $m < n$ and $\{n, m\} \neq \{0, 1\}$. Then $\mathcal{M}_n$ and $\mathcal{M}_m$ are not elementarily equivalent.
Theorem. Let $n, m \in \omega \cup \{\infty\}$ with $m < n$ and $\{n, m\} \neq \{0, 1\}$. Then $\mathcal{M}_n$ and $\mathcal{M}_m$ are not elementarily equivalent.

Proof. Easy case: $n = \infty$. Muchnik reducibility is definable in $\mathcal{M}_m$ (Dyment). Since $m$-reducibility and Muchnik reducibility do not coincide, form the sentence expressing this.
Elementary (in)equivalence

Hard case: \( n \in \omega \). We use the following two lemmas.

**Lemma.** If \( f, g \) are \( \Delta_0^n \), then \( C(\{f\}) \otimes C(\{g\}) \equiv_n C(\{f, g\}) \).

**Lemma.** Let \( X \oplus Y \) be \( \max(m, 1) \)-random. Then \( C(\{X\}) \otimes C(\{Y\}) \not\leq_m C(\{X, Y\}) \).

Furthermore:

- The Medvedev degrees of \( \{\{f\} | f \in \omega^\omega\} \) are isomorphic to the Turing degrees (Medvedev).
- They are definable in \( M(Dyment) \).
- \( C \) is definable in the Medvedev degrees (essentially Dyment).
- The \( \Delta_0^n \)-degrees are definable in the Turing degrees (Shore and Slaman).

Using this, express that “there are \( \Delta_0^n X \) and \( Y \) such that \( C(\{X\}) \otimes C(\{Y\}) \not\leq_m C(\{X, Y\}) \).”
Elementary (in)equivalence

Hard case: \( n \in \omega \). We use the following two lemmas.

**Lemma.** If \( f, g \) are \( \Delta^0_n \), then \( C(\{f\}) \otimes C(\{g\}) \equiv_n C(\{f, g\}) \).

**Lemma.** Let \( X \oplus Y \) be \( \max(m, 1) \)-random. Then \( C(\{X\}) \otimes C(\{Y\}) \not\leq_m C(\{X, Y\}) \).

Furthermore:

- The Medvedev degrees of \( \{\{f\} \mid f \in \omega^\omega\} \) are isomorphic to the Turing degrees (Medvedev).
Elementary (in)equivalence

Hard case: $n \in \omega$. We use the following two lemmas.

**Lemma.** If $f, g$ are $\Delta^0_n$, then $C(\{f\}) \otimes C(\{g\}) \equiv_n C(\{f, g\})$.

**Lemma.** Let $X \oplus Y$ be max$(m, 1)$-random. Then $C(\{X\}) \otimes C(\{Y\}) \nleq_m C(\{X, Y\})$.

Furthermore:
- The Medvedev degrees of $\{\{f\} \mid f \in \omega^\omega\}$ are isomorphic to the Turing degrees (Medvedev).
- They are definable in $\mathcal{M}$ (Dyment).
Elementary (in)equivalence

Hard case: \( n \in \omega \). We use the following two lemmas.

**Lemma.** If \( f, g \) are \( \Delta^0_n \), then \( C(\{f\}) \otimes C(\{g\}) \equiv_n C(\{f, g\}) \).

**Lemma.** Let \( X \oplus Y \) be \( \text{max}(m, 1) \)-random. Then
\[
C(\{X\}) \otimes C(\{Y\}) \not\leq_m C(\{X, Y\})
\]

Furthermore:
- The Medvedev degrees of \( \{\{f\} \mid f \in \omega^\omega\} \) are isomorphic to the Turing degrees (Medvedev).
- They are definable in \( \mathcal{M} \) (Dyment).
- \( C \) is definable in the Medvedev degrees (essentially Dyment).
Elementary (in)equivalence

Hard case: $n \in \omega$. We use the following two lemmas.

**Lemma.** If $f, g$ are $\Delta^0_n$, then $C(\{f\}) \otimes C(\{g\}) \equiv_n C(\{f, g\})$.

**Lemma.** Let $X \oplus Y$ be $\max(m, 1)$-random. Then $C(\{X\}) \otimes C(\{Y\}) \nleq_m C(\{X, Y\})$.

Furthermore:

- The Medvedev degrees of $\{\{f\} | f \in \omega^\omega\}$ are isomorphic to the Turing degrees (Medvedev).
- They are definable in $M$ (Dyment).
- $C$ is definable in the Medvedev degrees (essentially Dyment).
- The $\Delta^0_n$-degrees are definable in the Turing degrees (Shore and Slaman).

Using this, express that “there are $\Delta^0_n X$ and $Y$ such that $C(\{X\}) \otimes C(\{Y\}) \nleq C(\{X, Y\})$”.
Thank you

Thank you!