$\mathsf{RT}^{1}_k$, $\mathsf{SRT}^{2}_\ell$ and $\leq_{sc}$ reducibility

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Set up for reverse mathematics

Second order arithmetic: number and set variables, $+, \cdot, \leq, \in, 0, 1$.

$Z_2$: $\text{PA}^-$ + set induction + comprehension scheme.

Models: $\mathcal{M} = (M, S_M, +_M, \cdots)$ with $S_M \subseteq \mathcal{P}(M)$.

Project: Prove implications (or equivalences) between theorems (or subsystems) over a weak base theory.

$\text{RCA}_0$: $\text{PA}^- + \Sigma^0_1$ induction + $\Delta^0_1$ comprehension scheme.

If $M = \omega$, $\mathcal{M}$ is an $\omega$-model and we identify it with $S \subseteq \mathcal{P}(\omega)$.

An $\omega$ model satisfies $\text{PA}^-$ and full induction.

$(\omega, S) \models \text{RCA}_0 \iff S$ closed under $\oplus$ and $\leq_T$.

$\text{REC}$ is the minimal $\omega$-model.
Ramsey’s theorem for singletons and induction

$$\text{RT}_k^1 : \forall f : \mathbb{N} \to k \exists X (X \text{ infinite } \land f \upharpoonright X \text{ constant})$$

For all $k \in \omega$, REC $\models \text{RT}_k^1$ and RCA$_0 \vdash \text{RT}_k^1$.

$$\text{RT}_{\infty}^1 : \forall k (\text{RT}_k^1)$$

Again, REC $\models \text{RT}_{\infty}^1$.

Theorem (Hirst)

Over RCA$_0$, $\text{RT}_{\infty}^1$ is equivalent to $B\Sigma^0_2$.

For any $\omega$-model $S$ of RCA$_0$, $S \models \text{RT}_{\infty}^1$.

But RCA$_0 \nvdash \text{RT}_{\infty}^1$.
COH

COH: For every sequence of sets \( \langle R_i \mid i \in \mathbb{N} \rangle \), there is an infinite \( C \) s.t.

\[
\forall i \left( C \subseteq^* R_i \lor C \subseteq^* \overline{R}_i \right)
\]

You can think of COH as an infinite collection of RT\(_2^1\) instances

\[
f_i(x) = \begin{cases} 
0 & \text{if } x \in \overline{R}_i \\
1 & \text{if } x \in R_i 
\end{cases}
\]

to solve simultaneously but each allowing finitely many errors. Or as an infinite collection of RT\(_4^1\) instances

\[
f_i(x) = \begin{cases} 
0 & \text{if } x \in \overline{R}_{2i} \cap \overline{R}_{2i+1} \\
1 & \text{if } x \in \overline{R}_{2i} \cap R_{2i+1} \\
2 & \text{if } x \in R_{2i} \cap \overline{R}_{2i+1} \\
3 & \text{if } x \in R_{2i} \cap R_{2i+1}
\end{cases}
\]
Let \( f : [\mathbb{N}]^2 \to 2 \). \( H \) is homogeneous for \( f \) if \( f \restriction [H]^2 \) is constant.

\[
\text{RT}_2^2 : \quad \forall f : [\mathbb{N}]^2 \to 2 \exists H (H \text{ is infinite and homogenous})
\]

\((\text{RT}_2^2 \text{ can be proved using } \omega + 1 \text{ successive applications of } \text{RT}_1^2.)\)

We say \( f \) is stable if \( \lim_y f(x, y) \) exists for every \( x \).

\[
\text{SRT}_2^2 : \quad \forall \text{ stable } f : [\mathbb{N}]^2 \to 2 \exists H (H \text{ is infinite and homogenous})
\]

We say \( H \) is limit-homogeneous for a stable \( f \) if there is a color \( i \) such that \( \lim_y f(x, y) = i \) for every \( x \in H \).

\[
\text{D}_2^2 : \quad \forall \text{ stable } f : [\mathbb{N}]^2 \to 2 \exists H (H \text{ is infinite and limit homogenous})
\]
Connecting $D_2^2$, $RT_2^2$, $SRT_2^2$ and COH

$D_2^2$ is equivalent to $SRT_2^2$ over $\text{RCA}_0 + B\Sigma_2^0$.

(Cholak, Jockusch and Slaman) $RT_2^2$ is equivalent to $SRT_2^2 + \text{COH}$ over $\text{RCA}_0$. COH does not imply $RT_2^2$ over $\text{RCA}_0$ (for induction reasons).

(Hirschfeldt and Shore) There is an $\omega$-model of $\text{RCA}_0 + \text{COH}$ which does not satisfy $RT_2^2$.

(Chong, Slaman and Yang) There is a nonstandard model of $\text{RCA}_0 + SRT_2^2$ which is not a model of $RT_2^2$.

Question: Does $SRT_2^2$ imply $RT_2^2$ on $\omega$-models of $\text{RCA}_0$?
ω-reducibility

Let $P$ be a true $\Pi^1_2$ sentence of the form

$$\forall X \ (\varphi(X) \rightarrow \exists \hat{X} \ \psi(X, \hat{X}))$$

where $\varphi$ and $\psi$ are arithmetic statements. We refer to an $X$ such that $\varphi(X)$ as a $P$-instance and we refer to the corresponding witnesses $\hat{X}$ such that $\psi(X, \hat{X})$ as solutions to $X$.

Given two principles of this form $P$ and $Q$, rather than asking if $\text{RCA}_0 \vdash Q \rightarrow P$, we ask how they compare on $\omega$ models.

$$P \leq_\omega Q \iff \forall \text{ Turing ideal } S \ (S \models Q \rightarrow S \models P)$$

$$\iff \text{ every } \omega\text{-model of } \text{RCA}_0 + Q \text{ is an } \omega\text{-model of } \text{RCA}_0 + P$$

Question: Does $\text{RT}^2_2 \leq_\omega \text{SRT}^2_2$? Does $\text{COH} \leq_\omega \text{SRT}^2_2$?
Stronger reducibility: \( P \leq_c Q \)

\( P \leq_c Q \iff \) for every \( P \)-instance \( X \), there is a \( Q \)-instance \( Y \leq_T X \) such that for every solution \( \hat{Y} \) of \( Y \), there is a solution \( \hat{X} \) of \( X \) with \( \hat{X} \leq_T X \oplus \hat{Y} \).

\[
\begin{array}{ccc}
X & \xrightarrow{\Phi^X} & Y \\
\downarrow & & \downarrow \\
\hat{X} & \leftarrow & \hat{Y}
\end{array}
\]

\( \Delta^{X \oplus \hat{Y}} \)
Example 1

$\text{SRT}_2^2 \leq_c \text{D}_2^2$

Given an $\text{SRT}_2^2$ instance $f : [\omega]^2 \rightarrow 2$, we view $f$ as a $\text{D}_2^2$ instance. Let $H$ be an infinite limit-homogeneous set for $f$ with color $i$. We thin $H$ to a homogeneous set:

- Set $h_0 =$ the least element of $H$.
- Let $h_{n+1} =$ the least element $x \in H$ such that $x > h_n$ and $f(h_m, x) = i$ for all $m \leq n$.
- The set $\{h_0, h_1, \ldots\}$ is homogeneous for $f$.

This procedure uses both the $\text{D}_2^2$ solution $H$ and the $\text{SRT}_2^2$ instance $f$ to compute the $\text{SRT}_2^2$ solution to $f$. 
Stronger reducibility: \( P \leq_{sc} Q \)

\[ P \leq_{sc} Q \iff \text{for every } P\text{-instance } X, \text{ there is a } Q\text{-instance } Y \leq_T X \text{ such that for every solution } \hat{Y} \text{ of } Y, \text{ there is a solution } \hat{X} \text{ of } X \text{ with } \hat{X} \leq_T \hat{Y}. \]

\[
\begin{array}{c}
X \xrightarrow{\Phi^X} Y \\
\downarrow \\
\hat{X} \xleftarrow{\Delta^Y} \hat{Y}
\end{array}
\]
Example 2

$RT^1_{<\infty} \leq_{sc} RT^2_2$

Given $f : \omega \to k$. Define $f$-computable coloring $g : [\omega]^2 \to 2$

$$g(x, y) = \begin{cases} 
0 & \text{if } f(x) = f(y) \\
1 & \text{if } f(x) \neq f(y)
\end{cases}$$

Let $H$ be an infinite homogeneous set for $H$. We must have $g \upharpoonright [H]^2 = 0$, so $H$ is homogeneous for $f$ as well.

Once we have $RT^2_2$ solution $H$, we do not need to use $RT^1_{<\infty}$ instance $f$ to help compute $RT^1_{<\infty}$ solution.

$RT^1_k$, SRT$^2_\ell$ and $\leq_{sc}$ reducibility
Example 3

\[ \text{RT}_k^1 \leq_{sc} D_k^2 \leq_{sc} \text{SRT}_k^2 \]

Given \( f : \omega \to k \). Define \( f \)-computable coloring \( g : [\omega]^2 \to k \) by \( g(x, y) = f(x) \). The coloring \( g \) is stable and any limit-homogeneous set for \( g \) is monochromatic for \( f \).

Theorem (Dzhafarov)

- \( \text{SRT}_2^2 \not\leq_{sc} D_2^2 \) and in fact \( \text{SRT}_2^2 \not\leq_{sc} D_{<\infty}^2 \)
- \( \text{COH} \not\leq_{sc} \text{SRT}_2^2 \)
A list of questions from Hirschfeldt, Jockusch and Dzhafarov:

1. Motivating Question: Does $RT^2_2 \leq \omega SRT^2_2$? Does $COH \leq \omega SRT^2_2$?
2. Does $COH \leq_c SRT^2_2$?
3. Does $RT^1_3 \leq_{sc} SRT^2_2$?
4. Does $RT^1_k \leq_{sc} SRT^2_{\ell}$ when $k < \ell$?
5. Does $COH \leq_{sc} SRT^2_{\ell}$ for $\ell > 2$? Does $COH \leq_{sc} SRT^2_{<\infty}$?

Theorem (Dzhafarov, Patey, Solomon and Westrick)

- If $k > \ell$, then $RT^1_k \not\leq_{sc} SRT^2_{\ell}$.
- $COH \not\leq_{sc} SRT^2_{<\infty}$ (almost certainly).
Thank you!