Dominating and unbounded reals in Hechler extensions

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Dominating and unbounded reals

**Definition**

If $V$ is a model of set theory and $V[G]$ is a generic extension, a real $d \in V[G] \cap \omega^\omega$ is called *dominating* if for every $f \in V \cap \omega^\omega$ we have $f \leq^* d$.

We will also be interested in *unbounded* reals.

**Definition**

A real $x \in V[G] \cap \omega^\omega$ is called *unbounded* if for every $f \in V \cap \omega^\omega$ we have $x \not\leq^* f$. 

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Here $\leq^*$ is the preorder of *eventual domination*.
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Here $\leq_*$ is the preorder of eventual domination

$$f \leq_* g \iff (\forall \infty n) f(n) \leq g(n).$$
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Definition

A real $x \in V[G] \cap \omega^\omega$ is called unbounded if for every $f \in V \cap \omega^\omega$ we have $x \not\leq^* f$. 
The most basic method of adding a dominating real to the universe is *Hechler forcing* $\mathbb{D}$.
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The ordering is given by $\langle s', f' \rangle \leq \langle s, f \rangle$ if:

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The point of this definition is that $A$ is a dense set exactly when every nondecreasing $s \in \omega^{<\omega}$ gets a rank.

Using the rank analysis Baumgartner and Dordal proved:

Theorem (Baumgartner, Dordal, 1985)

Say $V \models \text{CH}$. Let $G$ be generic for the finite support iteration of $D_{nd}$. Then $V[G] \models s = \omega_1 \land b = 2^\omega$. In particular $s < b$ is consistent.
\( \mathbb{D}_{\text{nd}} \) admits a *rank analysis*. Let \( A \subseteq \omega^{<\omega} \). For each nondecreasing \( s \in \omega^{<\omega} \) we define \( \text{rk}_A(s) \in \text{ON} \cup \{\infty\} \) by recursion:

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In “Combinatorial properties of Hechler forcing” Brendle, Judah and Shelah used this same rank analysis to prove:

**Theorem (Brendle, Judah and Shelah, 1992)**

Forcing with $D_{nd}$ adds a MAD family of size $\omega_1$ and a Luzin set of size $2^{\omega}$. The existence of a Luzin set of size $2^{\omega}$ completely determines Cichoń’s diagram of cardinal characteristics; it sets the left half equal to $\omega_1$ and the right half equal to the continuum. They also introduced a rank analysis for $D$ and showed that their theorem holds for the usual Hechler extension. It was an open question whether $D$ and $D_{nd}$ are equivalent as forcing notions.
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Brendle and Löwe (in “Eventually different functions and inaccessible cardinals”) used a further variant of Hechler forcing. Conditions in $D_{\text{tree}}$ are trees $T \subseteq \omega^{<\omega}$ with a distinguished stem $s = \text{stem}(T)$ so that:

1. $(\forall t \in T) s \subseteq t$ or $t \subseteq s$.
2. $t \in T$ with $s \subseteq t$ implies that $(\forall \infty n) t \upharpoonright n \in T$.

The ordering is inclusion: $T' \leq T$ whenever $T' \subseteq T$. 

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Both the Baumgartner-Dordal and the Brendle-Judah-Shelah theorems go through for $\mathbb{D}_{\text{tree}}$; the proofs are the same, but easier.

Since $\mathbb{D}$, $\mathbb{D}_{\text{nd}}$, and $\mathbb{D}_{\text{tree}}$ all admit a rank analysis and all have the same effect on the common cardinal characteristics, it is natural to ask: how do these forcings relate to each other? Are they actually distinct as forcing notions?
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Since $D$, $D_{\text{nd}}$, and $D_{\text{tree}}$ all admit a rank analysis and all have the same effect on the common cardinal characteristics, it is natural to ask: how do these forcings relate to each other? Are they actually distinct as forcing notions?
Theorem (Neeman, P.)

\( D \) and \( D_{nd} \) are equivalent as forcing notions.
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\( \mathbb{D} \) and \( \mathbb{D}_{\text{nd}} \) are equivalent as forcing notions.

The strategy of the proof is to first show that \( \mathbb{D}_{\text{nd}} \ast \mathbb{C} \) and \( \mathbb{D} \) are equivalent.
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Proving this is complicated by the fact that each poset is a subforcing of the other: forcing with \( \mathcal{D} \) adds a \( \mathcal{D}_{\text{tree}} \)-generic real and vice versa.

Thus \( \mathcal{D} \) and \( \mathcal{D}_{\text{tree}} \) provide a counterexample to the natural Cantor-Bernstein theorem in the category of forcing notions.
To separate the two notions of forcing, we give a comparison of the relationship between dominating reals and the unbounded reals in the two extensions. We have the following two results:

**Theorem (P.)**

Let $G$ be $D$-generic over $V$. There is an unbounded real $x$ in $V \[ G \]$ so that $x \leq^* y$ for every dominating real $y \in V \[ G ]$.

**Theorem (P.)**

Let $G$ be $D$-tree-generic over $V$. Let $x$ be an unbounded real in $V \[ G \]$. Then there is a dominating real $y \in V \[ G \]$ so that $(\exists \infty n) y(n) < x(n)$. 

(That is, $x \not\leq^* y$).
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Let \( G \) be \( \mathbb{D} \)-generic over \( V \). There is an unbounded real \( x \) in \( V[G] \) so that \( x \leq^* y \) for every dominating real \( y \in V[G] \).

Theorem (P.)

Let \( G \) be \( \mathbb{D}_{\text{tree}} \)-generic over \( V \). Let \( x \) be an unbounded real in \( V[G] \). Then there is a dominating real \( y \in V[G] \) so that \((\exists^\infty n)y(n) < x(n)\).
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A conjecture of Brendle and Löwe

Brendle and Löwe proved a dichotomy theorem for the possible reals in the extension by $\mathbb{D}_{\text{tree}}$:

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Theorem (Brendle and Löwe, 2009)

Every real added by $\mathbb{D}_{\text{tree}}$ is either dominating or infinitely equal to some ground model real.
\end{quote}

Motivated by this, they made an analogous dichotomy-style conjecture on the possible subforcings of $\mathbb{D}_{\text{tree}}$:

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The only nontrivial subforcings of $\mathbb{D}_{\text{tree}}$ are Cohen forcing $\mathbb{C}$ and $\mathbb{D}_{\text{tree}}$ itself.
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We can see now that this conjecture is false. Forcing with $\mathbb{D}_{\text{tree}}$ adds a $\mathbb{D}_{\text{tree}}$-generic real, which is neither equivalent to $\mathbb{D}_{\text{tree}}$ nor to $\mathbb{C}$.

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Let $d$ be a $\mathbb{D}_{nd}$-generic real, and suppose $y \in V[d]$ is dominating. Then there are $z_0, z_1 \in V \cap \omega \uparrow \omega$ so that $z_0 \circ d \circ z_1 \leq^* y$.

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**Theorem (P.)**

Let $d$ be a $\mathbb{D}_{nd}$-generic real, and suppose $y \in V[d]$ is dominating. Then there are $z_0, z_1 \in V \cap \omega \rightarrow^\omega$ so that $z_0 \circ d \circ z_1 \leq^* y$. We can view this theorem as saying that $d$ generates all the dominating reals in $V[d]$. 
This result has strong consequences for the cofinal structure of $\mathcal{D}$, the collection of dominating reals in $V[d]$. 

**Corollary**

The structures $(V \cap \omega \omega, \leq \ast)$ and $(\mathcal{D}, \ast \geq)$ are cofinally isomorphic.

Using this fact, one can extend work of Laflamme ("Bounding and dominating numbers of families of functions on $\mathbb{N}$", 1993), and give new consistently achievable values of the following three cardinal characteristics for bounded $F \subseteq \omega \omega$:

1. $b(F) = \min \{|H| : H \subseteq F \text{ is unbounded in } F\}$
2. $d(F) = \min \{|H| : H \subseteq F \text{ is dominating in } F\}$
3. $b_{\downarrow}(F) = \min \{|H| : H \subseteq F \downarrow \text{ is unbounded in } (F \downarrow, \ast \geq)\}$

Here $F \downarrow \subseteq \omega \omega$ is the set of functions dominating $F$. (So if $F = V \cap \omega \omega$ then $F \downarrow = \mathcal{D}$.)
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http://www.math.ucla.edu/~justinpa/
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