Orbits of $D$-maximal sets in $E$

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Outline

1 Background

2 Automorphisms
   - Basic Work
   - Advanced Automorphism Methods

3 $\mathcal{D}$-maximal sets
Notation

- $\omega$ is the natural numbers. $\overline{X} = w - X$
- $p[X]$ denotes the image of $X$ under $p$.
- $W_e$ is the domain of the $e$-th Turing machine
- $A_s$ is the set of elements enumerated into $A$ by stage $s$.
- All sets are c.e. unless otherwise noted. $R_i$ is assumed to be computable
Lattice of C.E. Sets

Definition (Lattice of c.e. sets)

1. \( \langle \{ W_e, e \in \omega \} . \subseteq \rangle = \mathcal{E} \) are the c.e. sets under inclusion.
2. \( \mathcal{E}^* \) is \( \mathcal{E} \) modulo the ideal \( \mathcal{F} \) of finite sets.

Question (Motivating Questions)

1. What are the automorphisms of \( \mathcal{E} \)? \( \mathcal{E}^* \)?
2. What are the definable classes? Orbits?
Simple Automorphisms

- Permutations $p$ of $\omega$ induce maps $\Upsilon(A) = p[A]$ respecting $\subseteq$.
- Any permutation taking c.e. sets to c.e. sets is automatically an automorphism.
- Computable permutations (aka recursive isomorphisms) induce ($\omega$ many) automorphisms.

**Theorem**

_All creative sets belong to the same orbit._

**Proof.**

It is well known that the creative sets are recursively isomorphic.
How Many Automorphisms?

**Theorem (Lachlan)**

There are $2^\omega$ automorphisms of $\mathcal{E}^*$ (and $\mathcal{E}$).

**Idea**

- Build permutations as limit of computable permutations $p_f = \bigcup_{\sigma \in 2^{<\omega}} p_\sigma$ (Respects $\subseteq$).
- Ensure that $\Upsilon(W_e) = R \cup p_\sigma[W_{e_1}] \cup p_\sigma[W_{e_2}]$ where $W_e = R \cup W_{e_1} \cup W_{e_2}$. (Ensures images are c.e.).
- Build so if $\sigma \mid \tau$ then for some $A$, $p_\sigma(A) \not\equiv^* p_\tau(A)$.
Building Continuum Many Automorphisms

Idea

Build $R_0 \supset R_1 \supset \ldots$ with members of $R_e$ sharing the same $e$-state and leaving us free to define permutation on $R_e$ as we wish. But first we see we have two choices for this permutation in non-trivial cases.

Lemma

If $R \supset \infty R \cap W \supset \infty \emptyset$ then there are computable permutations $p_0, p_1$ of $R$ with $p_0[W \cap R] \neq^* p_1[W \cap R]$.

Proof.

Let $S \subset W \cap R$ infinite computable subset. Pick $p_0$ to be the identity and $p_1$ to exchange $S$ and $R - S$, i.e., $p_1$ moves infinitely many points outside of $W$ into $W$. 

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Glueing Permutations

Construction

Assume $R_n, p_\sigma$ are defined. ($R_0 = \omega, p_\lambda = \emptyset$)

1. If $W_n$ almost avoids or contains $R_n$ finitely modify $R_{n+1}, p_\sigma$ to eliminate the exceptions.

2. Otherwise let $R_{n+1} \subset W_n \cap R_n$. $W_n, R_n - R_{n+1}$ satisfy the lemma.
   
   - For each maximal $\sigma$ with $p_\sigma$ defined let $p_\sigma \langle j \rangle = p_j \cup p_\sigma, j = 0, 1$.
   
   - WLOG we insist $W_{2n}$ is always a split of $R_{2n}$ so this case occurs infinitely.
Summarizing Construction

- \( \bigcap R_n = \emptyset \) (infinitely often we lose the least element).
- \( p_f = \bigcup_{\sigma \subseteq f} p_\sigma \) is a permutation of \( \omega \)
- Images of c.e. sets are given by finitely many computable permutations on disjoint computable sets.
- \( R_{k+1} \) isn’t split by \( \{ W_i \mid i \leq k \} \) so we can redefine/extend permutation on \( R_{k+1} \).

Remark

Nifty but as Soare points out if \( p[A] = B \) built in this fashion then \( (p_1 \circ p_2 \circ \ldots \circ p_n)[A] = B \).
Permutations and Automorphisms

Question

Are all automorphisms of $E^*$ induced by a permutation?

Remark

Since permutations respect $\subseteq$, this would show every $\Upsilon^* \in \text{Aut } E^*$ is induced by some $\Upsilon \in \text{Aut } E$.

Theorem (?)

Every automorphism $\Upsilon(W_e) = W_{v(e)}$ is induced by a permutation $p \leq_T v(e) \oplus 0'$.
Proof Idea

Idea

After some point map $x$ to $y$ only if for all $i \leq n$

$x \in W_i \iff y \in W_{v(e)}$.

Definition

The $e$-state ($e$-hat-state) of $x$ is $\sigma(e, x)$ ($\hat{\sigma}(e, x)$) where:

$$
\sigma(e, x) = \{ i \leq e \mid x \in W_i \}
$$

$$
\hat{\sigma}(e, x) = \{ i \leq e \mid x \in W_{v(i)} \}
$$
Proof

- At stage $2n$ define $p(x)$ for least $x \notin \text{dom } p$.
- Let $e_{2n}$ be max s.t. $(\exists y)(y \notin \text{rng } p \land \sigma(e, x) = \hat{\sigma}(e, y))$.
- Let $p(x)$ be least such $y$.
- At odd stages define $p^{-1}(y)$ for least $y \notin \text{dom } p^{-1}$.
- $\liminf_{n \to \infty} e_n = \infty$ so $p(W_e) =^* W_{v(e)}$
  - $|W_i| < \omega$ then eventually $W_i \subseteq \text{dom } p, \text{rng } p$
Often we have $A, B$ and want to build $\Upsilon$ with $\Upsilon(A) = B$.

Difficult to directly build permutation with $p[A] = B$ while sending c.e. set to c.e. sets.

Easier to work in $\mathcal{E}^*$ and effectively construct $W_v(e)$.

Problem is respecting $\subseteq^*$.

- Must ensure that $W_v(e)$ has same lattice of c.e. subsets/supersets as $W_e$.
- Have to build $W_v(e)$ without knowing if $|W_e \cap A| = \infty, W_e \supseteq A, W_e \subseteq A$ or $W_e \supseteq \overline{A}$ at any stage.
- To ensure $\Upsilon(W_e)$ is c.e. we need a somewhat effective grip on $\Upsilon$
The Extension Theorem and $\Delta^0_3$ Automorphisms

Definition

$L(A)$ are the c.e. sets containing $A$ and $E(A)$ are the c.e. sets contained in $A$ (under inclusion).

- Want to build automorphism $\Upsilon$ with $\Upsilon(A) = B$.
- The Extension Theorem (Soare) and Modified Extension Theorem (Cholak) break up construction.
  - Build (sufficiently effective) automorphism of $L^*(A)$ with $L^*(B)$.
  - Ensure (roughly) that (mod finite) elements fall into $A$ and $B$ in same $e$-state, $e$-hat-state.
- The $\Delta^0_3$ automorphism method uses a complicated $\Pi^0_2$ tree construction to build $\Delta^0_3$ automorphisms.
Some Results

- (Martin) h.h.s. sets and complete sets aren’t invariant.
- (Soare) The maximal sets form an orbit.
- (Downey, Stob) The hemi-maximal sets form an orbit.
- (Cholak, Downey, and Herrmann) The Hermann sets form an orbit.
- (Soare) Every (non-computable) c.e. set is automorphic to a high set.
- Hodgepodge of results about orbits of other classes of sets.
Completeness

Question

Is every $W_e$ automorphic to a Turing complete r.e. set?

Theorem (Harrington-Soare)

There is an $\mathcal{E}$ definable property $Q(A)$ satisfied only by incomplete sets.

Theorem (Cholak-Lange-Gerdes)

There are disjoint properties $Q_n(A), n \geq 2$ satisfied only by incomplete sets.
**D-maximal sets**

**Definition (Sets disjoint from A)**

\[ \mathcal{D}(A) = \{ B : \exists W (B \subseteq^* A \cup W \text{ and } W \cap A =^* \emptyset) \} \]

Let \( \mathcal{E}_{\mathcal{D}(A)} \) be \( \mathcal{E} \) modulo \( \mathcal{D}(A) \), i.e., \( B = C \mod \mathcal{D}(A) \) if

\[ (\exists D_1, D_2 \text{ s.t. } D_1 \cap A =^* D_2 \cap A =^* \emptyset)[B \cup A \cup D_1 =^* C \cup A \cup D_2] \]

**Definition**

1. A is *hh-simple* iff \( \mathcal{L}^*(A) = \{ B \mid B \supseteq^* A \} \) is a \( (\Sigma^0_3) \) Boolean algebra.

2. A is *\( \mathcal{D} \)-hhsimple* iff \( \mathcal{E}_{\mathcal{D}(A)} \) is a \( (\Sigma^0_3) \) Boolean algebra.

3. A is *\( \mathcal{D} \)-maximal* iff \( \mathcal{E}_{\mathcal{D}(A)} \) is the trivial Boolean algebra iff

\[ (\forall B)(\exists D \text{ s.t. } D \cap A =^* \emptyset)[B \subset^* A \cup D \text{ or } B \cup A \cup D =^* \omega]. \]
**Definition**

A is $\mathcal{D}$-maximal if

$$(\forall B)(\exists D \text{ s.t. } D \cap A \neq \emptyset)[B \subset^* A \cup D \text{ or } B \cup A \cup D =^* \omega].$$

**Example**

Maximal and hemi-maximal sets are $\mathcal{D}$-maximal.

A set that is maximal on a computable set is $\mathcal{D}$-maximal.

**Question**

- What are the orbits of $\mathcal{D}$-maximal sets?
- Do they form finitely many orbits?
Many ways to get $\mathcal{D}$-maximal sets already covered.

(Proper) splits of maximal sets are in a single orbit.

Maximal subsets of computable sets in a single orbit.

(Cholak-Harrington) $\mathcal{D}$-maximal sets in $A$-special lists fall in a single orbit.

Full consideration leaves only the case of $\mathcal{D}$-maximal sets contained in atomless $r$-maximal sets as potential source for infinitely many orbits.
Background
Automorphisms
\( \mathcal{D} \)-maximal sets

Infinitely many orbits for \( \mathcal{D} \)-maximal sets

- Borrow technique from Nies and Cholak for showing atomless \( r \)-maximal sets aren’t automorphic.
- Technique reveals structure of \( \mathcal{L}^*(A) \) by giving it tree structure.
- In particular \( \mathcal{L}^*(A) \) has structure \( T \) if there is a homomorphism \( \phi \) from \( \langle \mathcal{L}^*(A) - \omega, \subset_\infty \rangle \) to \( \langle T, \subset \rangle \) s.t.

\[
|W_e \cap \phi^{-1}(\sigma) - \phi^{-1}(\sigma^-)| = \infty \implies \hspace{1cm} |W_e \cap \phi^{-1}(\sigma^-) - \phi^{-1}(\sigma^{--})| = \infty
\]

- Define infinite sequence of trees \( T^n \) which guarantee that incompatible structure of supersets.
Non-automorphic $\mathcal{D}$-maximal sets

- Build $A$ $\mathcal{D}$-maximal subset of $r$-maximal set $A_r$ with structure $T^n$.
- Build $B$ $\mathcal{D}$-maximal subset of $r$-maximal set $B_r$ with structure $T^{n+1}$.
- If $\gamma(A) = B$ then there is superset of $B_r$ with structure given by subtree of $T^n$.
- Incompatibility result ensures this is impossible.
- Gives us infinite number of orbits for $\mathcal{D}$-maximal sets.