Lebesgue density and cupping with K-trivial sets

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There are several notions of “effective randomness”. They are usually defined by isolating a countable collection of nice measure zero sets \( \{C_0, C_1, \ldots \} \).

Then:

**Definition**

\( X \in 2^\omega \) is random if \( X \notin \bigcup_n C_n \).

The most important example was given by Martin-Löf in 1966. We give a definition due to Solovay:

**Definition**

A Solovay test is a computable sequence \( \{\sigma_n\}_{n \in \omega} \) of elements of \( 2^{<\omega} \) (finite binary strings) such that \( \sum_n 2^{-|\sigma_n|} < \infty \). The test covers \( X \in 2^\omega \) if \( X \) has infinitely many prefixes in \( \{\sigma_n\}_{n \in \omega} \). \( X \in 2^\omega \) is Martin-Löf random if no Solovay test covers it.
Martin-Löf randomness

Why is Martin-Löf randomness a good notion?

1. It has nice properties
   - Satisfies all reasonable statistical tests of randomness
   - Plays well with computability-theoretic notions

2. It has several natural characterizations

Let $K$ denote *prefix-free (Kolmogorov) complexity*. Intuitively, $K(\sigma)$ is the length of the shortest (binary, self-delimiting) description of $\sigma$.

Theorem (Schnorr)

$X$ is Martin-Löf random iff $K(X \mid n) \geq n - O(1)$.

In other words, a sequence is Martin-Löf random iff its initial segments are *incompressible*.

Martin-Löf random sequences can also be characterized as *unpredictable*; it is hard to win money betting on the bits of a Martin-Löf random.
Other randomness notions

2-randomness
↓
weak 2-randomness
↓
difference randomness
↓
Martin-Löf randomness (1-randomness)
↓
Computable randomness
↓
Schnorr randomness
↓
Kurtz randomness (weak 1-randomness)

Randomness Zoo (Antoine Taveneaux)
A template for randomness and analysis

Many results in analysis and related fields look like this:

**Classical Theorem**
Given a mimsy borogove $M$, almost every $x$ is frabjous for $M$.

There are only countably many effective borogoves, so

**Corollary**
Almost every $x$ is frabjous for *every* effective mimsy borogove.

Thus a sufficiently strong randomness notion will guarantee being frabjous for every effective mimsy borogove.

**Question**
How much randomness is necessary?

Ideally, we get a characterization of a natural randomness notion:

**Ideal Effectivization of the Classical Theorem**
$x$ is *Alice* random iff $x$ is frabjous for every effective mimsy borogove.
Examples will clarify:

**Classical Theorem**

Every function $f: [0, 1] \rightarrow \mathbb{R}$ of bounded variation is differentiable at almost every $x \in [0, 1]$.

**Ideal Effectivization (Demuth 1975)**

A real $x \in [0, 1]$ is Martin-Löf random iff every computable $f: [0, 1] \rightarrow \mathbb{R}$ of bounded variation is differentiable at $x$.

**Classical Theorem (a special case of the previous example)**

Every monotonic function $f: [0, 1] \rightarrow \mathbb{R}$ is differentiable at almost every $x \in [0, 1]$.

**Ideal Effectivization (Brattka, M., Nies)**

A real $x \in [0, 1]$ is computably random iff every monotonic computable $f: [0, 1] \rightarrow \mathbb{R}$ is differentiable at $x$. 
Randomness and analysis (more examples)

An effectivization of a form of the Lebesgue differentiation theorem (also related to the previous examples):

**Theorem (Rute; Pathak, Rojas and Simpson)**

A real \( x \in [0, 1] \) is Schnorr random iff the integral of an \( L_1 \)-computable \( f: [0, 1] \to \mathbb{R} \) must be differentiable at \( x \).

An effectivization of (a form of) Birkhoff’s Ergodic Theorem:

**Theorem (Franklin, Greenberg, M., Ng; Bienvenu, Day, Hoyrup, Mezhirov, Shen)**

Let \( M \) be a computable probability space, and let \( T: M \to M \) be a computable ergodic map. Then a point \( x \in M \) is Martin-Löf random iff for every \( \Pi^0_1 \) class \( P \subseteq M \),

\[
\lim_{n \to \infty} \frac{\# \{ i < n : T^i(x) \in P \} }{n} = \mu(P).
\]

There are a handful of other examples.
We would like to do the same kind of analysis for (a form of) the Lebesgue Density Theorem.

**Definition**

Let $C \in 2^\omega$ be measurable. The *lower density of* $X \in C$ is

$$
\rho(X \mid C) = \lim \inf_n \frac{\mu([X \mid n] \cap C)}{2^{-n}}.
$$

Here, $\mu$ is the standard Lebesgue measure on Cantor space and $[\sigma] = \{Z \in 2^\omega \mid \sigma \prec Z\}$, so $\mu([X \mid n]) = 2^{-n}$.

**Lebesgue Density Theorem**

If $C \in 2^\omega$ is measurable, then $\rho(X \mid C) = 1$ for almost every $X \in C$.

We want to understand the density points of $\Pi^0_1$ classes.
Lebesgue density

We want to understand the density points of $\Pi_1^0$ classes.

**Question**

For which $X$ is it the case that $\rho(X \mid C) = 1$ for every $\Pi_1^0$ class $C$ containing $X$.

**Note.** Every 1-generic has this property. So this is not going to characterize a natural randomness class.

**Theorem (Bienvenu, Hölzl, M., Nies)**

Assume that $X$ is Martin-Löf random. Then $X \geq_T \emptyset'$ iff there is a $\Pi_1^0$ class $C$ containing $X$ such that $\rho(X \mid C) = 0$.

**Notes:**

- We have not been able to extend this to $\rho(X \mid C) < 1$.
- If $\mu(C)$ is computable, then by the effectivization of the Lebesgue differentiation theorem, every Schnorr random in $C$ is a density point of $C$. 
Difference randomness

Theorem (Bienvenu, Hölzl, M., Nies)

Assume that $X$ is Martin-Löf random. Then $X \geq_T \emptyset'$ iff there is a $\Pi^0_1$ class $C$ containing $X$ such that $\rho(X \mid C) = 0$.

The contrapositive lets us characterize the Martin-Löf randoms that do not compute $\emptyset'$ (which will be very useful!). It is not the first such characterization.

Definition (Franklin and Ng)

A (Solovay-rian) difference test is a $\Pi^0_1$ class $C$ and a computable sequence $\{\sigma_n\}_{n \in \omega}$ of elements of $2^{<\omega}$ such that $\sum_n \mu([\sigma_n] \cap C) < \infty$. The test covers $X \in C$ if $X$ has infinitely many prefixes in $\{\sigma_n\}_{n \in \omega}$. $X \in 2^\omega$ is difference random if no difference test covers it.

Essentially, a difference test is just a Solovay test (or usually, a Martin-Löf test) inside a $\Pi^0_1$ class.
difference randomness

**Theorem (Franklin and Ng)**

\[ X \text{ is difference random iff } X \text{ is Martin-Löf random and } X \not \geq_{T} \emptyset'. \]

It can be shown:

**Lemma**

Let \( C \) be a \( \Pi^0_1 \) class and \( X \in C \) Martin-Löf random. TFAE:

1. \( \rho(X \mid C) = 0. \)
2. There is a computable sequence \( \{\sigma_n\}_{n \in \omega} \) such that \( C \) and \( \{\sigma_n\}_{n \in \omega} \) form a difference test.

From which our result follows immediately:

**Theorem (Bienvenu, Hölzl, M., Nies)**

Assume that \( X \) is Martin-Löf random. Then \( X \geq_{T} \emptyset' \) iff there is a \( \Pi^0_1 \) class \( C \) containing \( X \) such that \( \rho(X \mid C) = 0. \)
The previous result has an application to K-triviality.

**Theorem (variously Nies, Hirschfeldt, Stephan, …)**

The following are equivalent for \( A \in 2^\omega \):

1. \( K(A \upharpoonright n) \leq K(n) + O(1) \) (\( A \) is K-trivial).
2. Every Martin-Löf random \( X \) is Martin-Löf random relative to \( A \) (\( A \) is low for random).
3. There is an \( X \geq_T A \) that is Martin-Löf random relative to \( A \).

\[ \vdots \]

17. For every \( A \)-c.e. set \( F \subseteq 2^{<\omega} \) such that \( \sum_{\sigma \in F} 2^{-|\sigma|} < \infty \), there is a c.e. set \( G \supseteq F \) such that \( \sum_{\sigma \in G} 2^{-|\sigma|} < \infty \).

**Other Facts**

- [Solovay 1975] There is a non-computable K-trivial set.
- [Chaitin] Every K-trivial is \( \leq_T \emptyset' \).
- [Nies, Hirschfeldt] Every K-trivial is low (\( A' \leq_T \emptyset' \)).
(Weakly) ML-cupping

Definition (Kučera 2004)

$A \in 2^\omega$ is weakly ML-cuppable if there is a Martin-Löf random sequence $X \not\leq_T \emptyset'$ such that $A \oplus X \geq_T \emptyset'$. If one can choose $X <_T \emptyset'$, then $A$ is ML-cuppable.

Question (Kučera)

Can the K-trivial sets be characterized as either

1. not weakly ML-cuppable, or
2. $\leq_T \emptyset'$ and not ML-cuppable?

Compare this to:

Theorem (Posner and Robinson)

For every $A >_T \emptyset$ there is a 1-generic $X$ such that $A \oplus X \geq_T \emptyset'$. If $A \leq_T \emptyset'$, then also $X \leq_T \emptyset'$.
Question (Kučera 2004)

Can the $K$-trivial sets be characterized as either
1. not weakly ML-cuppable, or
2. $\leq_T \emptyset'$ and not ML-cuppable?

Answer (Day and M.)

Yes, both.

Partial results

- If $A \leq_T \emptyset'$ and not $K$-trivial, it is weakly ML-cuppable (by $\Omega^A$).
- If $A$ is low and not $K$-trivial, then it is ML-cuppable (by $\Omega^A$). (Also any $A$ that can be shown to compute a low non-$K$-trivial.)
- [Nies] There is a non-computable $K$-trivial c.e. set that is not weakly ML-cuppable.
Answering Kučera’s question

**Theorem (Day and M.)**

If $A$ is not $K$-trivial, then it is weakly ML-cuppable (i.e., there is a Martin-Löf random sequence $X \not \leq_T \emptyset'$ such that $A \oplus X \geq_T \emptyset'$). If $A \prec_T \emptyset'$ is not $K$-trivial, then it is ML-cuppable (i.e., we can take $X \leq_T \emptyset'$ too).

These are proved by straightforward constructions.

**Idea.** Given $A$, we (force with positive measure $\Pi^0_1$ classes to) construct a Martin-Löf random $X$ that is not Martin-Löf random relative to $A$. We code the settling-time function for $\emptyset'$ into $A \oplus X$ by alternately making $X$ look $A$-random for long stretches and then dropping $K^A(X|n)$ for some $n$.

It is the other direction I want to focus on.

**Theorem (Day and M.)**

If $A$ is $K$-trivial, then it is not weakly ML-cuppable.

This involves the work on Lebesgue density and $\Pi^0_1$ classes.
Answering Kučera’s question

**Theorem (Day and M.)**

If \( A \) is K-trivial, then it is not weakly ML-cuppable.

**Proof.**

Let \( A \) be K-trivial, \( X \) Martin-Löf random, and \( A \oplus X \succeq_T \emptyset' \). We will show that \( X \succeq_T \emptyset' \).

Because \( A \) is K-trivial it is low (\( \emptyset' \succeq_T A' \)), hence \( A \oplus X \succeq_T A' \). It is also low for random, so \( X \) is Martin-Löf random relative to \( A \). Therefore, by the Bienvenu et al. result relativized to \( A \), there is a \( \Pi^0_1[A] \) class \( C \) containing \( X \) such that \( \rho(X | C) = 0 \).

Let \( F \subseteq 2^{<\omega} \) be an \( A \)-c.e. set such that

\[
2^\omega \setminus C = [F] = \bigcup_{\sigma \in F} [\sigma].
\]

We may assume that \( F \) is prefix-free, hence

\[
\sum_{\sigma \in F} 2^{-|\sigma|} \leq 1 < \infty.
\]
Theorem (Day and M.)

If $A$ is $K$-trivial, then it is not weakly ML-cuppable.

Proof continued.

... 

By characterization \(^{17}\) of $K$-triviality, there is a c.e. set $G \supseteq F$ such that

$$\sum_{\sigma \in G} 2^{-|\sigma|} < \infty.$$ 

This $G$ is a \textit{Solovay test}. Because $X$ is Martin-Löf random, there are only finitely many $\sigma \in G$ such that $\sigma \prec X$. No such $\sigma$ is in $F$, so without loss of generality, we may assume that no such $\sigma$ is in $G$.

Consider the $\Pi^0_1$ class $D = 2^\omega \setminus [G]$. Note that $X \in D$. Also, $D \subseteq C$, so $\rho(X \mid D) = 0$. Therefore, by the Bienvenu et al. result, $X \geq_T \emptyset'$. 

In other words, $X$ does not witness the weak ML-cuppability of $A$. \hfill $\square$
Kučera’s question answered

Theorem (various)
The following are equivalent for \( A \in 2^\omega \):

1. \( K(A \upharpoonright n) \leq K(n) + O(1) \) (\( A \) is \( K \)-trivial).

\[ \vdots \]

18. \( A \) is not weakly ML-cuppable.

19. \( A \leq_T \emptyset' \) and \( A \) is not ML-cuppable.

These are the first characterizations of \( K \)-triviality in term of their interactions in the Turing degrees with the degrees of ML-randoms.

By improving the cupping direction, we can even remove any mention of \( \emptyset' \).

20. There is a \( D >_T \emptyset \) such that if \( X \) is Martin-Löf random and \( A \oplus X \geq_T D \), then \( X \geq_T D \). (also with Adam Day)
Suppose that $C$ is a $\Pi^0_1$ class and $X \in C$.

We know that if $X$ is difference random, then $\rho(X \mid C) > 0$. But we wanted to characterize the $X$ such that $\rho(X \mid C) = 1$.

**Definition**

Call $X \in 2^\omega$ a *non-density point* if there is a $\Pi^0_1$ class $C$ such that $X \in C$ and $\rho(X \mid C) < 1$.

**Lemma (Bienvenu, Hölzl, M., Nies)**

Assume that $X$ is a Martin-Löf random non-density point. Then $X$ computes a function $f$ (witnessing its non-density) such that for every $A$ either:

- $f$ dominates every $A$-computable function, or
- $X$ is not Martin-Löf random relative to $A$. 

Taking $A = \emptyset$, this shows that a Martin-Löf random non-density point computes a function that dominates every computable function. In other words:

**Theorem (Bienvenu, Hölzl, M., Nies)**

If $X$ is a Martin-Löf random non-density point, then $X$ is high ($X' \geq_T \emptyset''$).

In fact, $X$ is Martin-Löf random relative to almost every $A$, so $f$ must dominate every $A$-computable function for almost every $A$.

**Theorem (Bienvenu, Hölzl, M., Nies)**

If $X$ is a Martin-Löf random non-density point, then $X$ is (uniformly) almost everywhere dominating.

So for Martin-Löf random sequences:

not a.e.d $\implies$ density point for $\Pi^0_1$ classes $\implies$ not $\geq_T \emptyset'$. 

If $A$ is a computably enumerable set, then $A$ computes a function $g$ (its settling-time function) such that every function dominating $g$ computes $A$. Therefore:

**Lemma**

If $X$ is a Martin-Löf random non-density point and $A$ is c.e., then either $X \geq_T A$ or $X$ is not Martin-Löf random relative to $A$.

So if $A$ is $K$-trivial (hence low for random) and c.e., then $X$ must compute $A$! But every $K$-trivial is bounded by a c.e. $K$-trivial (Nies), so:

**Theorem (Greenberg, Nies, Turetsky??)**

If $X$ is a Martin-Löf random non-density point, then $X$ computes *every* $K$-trivial.

This is related to another open question about the $K$-trivial sets.
Question (Stephan 2004)

If $A$ is $K$-trivial, must there be a Martin-Löf random $X \geq_T A$ such that $X \not\leq_T \emptyset'$?

Together with the following result, this would give a new characterization of the c.e. $K$-trivial sets:

Theorem (Hirschfeldt, Nies, Stephan)

If $A$ is c.e., $X$ is Martin-Löf random, $X \geq_T A$ but $X \not\leq_T \emptyset'$, then $A$ is $K$-trivial.

But now we see that this is connected to Lebesgue density:

Fact

If there a Martin-Löf random non-density point $X \not\leq_T \emptyset'$, then the question has a positive answer: every $K$-trivial is below a Martin-Löf random that does not compute $\emptyset'$ (because they are all below $X$!).
Thank You!