The complexity within well-partial-orderings

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Madison, March 2012
1. Background on WQOs

2. WQOs in Proof Theory
   - Kruskal’s theorem and the graph-minor theorem
   - Linear orderings and Fraïssé’s Conjecture

3. WPOs in Computability Theory
Definition: A well-quasi-ordering (WQO), is quasi-ordering which has no infinite descending sequences and no infinite antichains.
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Definition: A well-partial-ordering (WPO), is a WQO which is a partial ordering.
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Well-partial-orders

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- $\mathcal{P}$ is well-founded and has no infinite antichains;

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- every subset of $\mathcal{P}$ has a finite set of minimal elements;
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Closure properties of WPOs

The sum and disjoint sum of two WPOs are WPO.
The product of two WPOs is WPO.
Finite strings over a WPO are a WPO (Higman, 1952).
Finite trees with labels from a WPO are a WPO (Kruskal, 1960).
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($\leq_L$ is a linearization of $(P, \leq_P)$ if it's linear and $x \leq_P y \Rightarrow x \leq_L y$. )

Definition: The length of $P = (P, \leq_P)$ is $\omega(P) = \sup\{\text{ordType}(W, \leq_L) : \text{where } \leq_L \text{ is a linearization of } P\}$.

Note: $P$ is a WPO $\iff$ $\text{B}ad(P)$ is well-founded.

Theorem: [De Jongh, Parikh 77] $\omega(P) + 1 = \text{rk}(\text{B}ad(P))$. 

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\[ o(\mathcal{P}) = \sup \{ \text{ordType}(W, \leq_L) : \text{where } \leq_L \text{ is a linearization of } \mathcal{P} \} \]
**Length**

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**Def:** \( \text{Bad}(P) = \{ \langle x_0, \ldots, x_{n-1} \rangle \in P^\omega : \forall i < j \ (x_i \not\leq_P x_j) \} \),

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**Note:** \(\mathcal{P}\) is a WPO \(\iff\) \(\text{Bad}(\mathcal{P})\) is well-founded.

**Theorem:** [De Jongh, Parikh 77] \(o(\mathcal{P}) + 1 = \text{rk}(\text{Bad}(\mathcal{P}))\).
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Kruskal’s theorem

**Theorem:** [Kruskal 60] Let $T$ be the set of finite trees ordered by $T \preceq S$ if there is an embedding $: T \rightarrow S$ preserving $<$ and $g.l.b.$ Then $T$ is a WQO.
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**Theorem:** [Friedman] The length of $\mathcal{T}$ is $\geq \Gamma_0$, the Feferman–Schütte ordinal.

Corollary: [Friedman] $(\text{RCA}_0)$ Kruskal’s theorem $\Rightarrow \Gamma_0$ well-ordered. Therefore, $\text{ATR}_0 \not\vdash$ Kruskal’s theorem.
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**Corollary:** [Friedman] (RCA$_0$) Kruskal’s theorem $\Rightarrow \Gamma_0$ well-ordered. Therefore,

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The “big five” subsystems of 2nd-order arithmetic

**Axiom systems:**

\( \text{RCA}_0: \)

\( \text{WKL}_0: \)

\( \text{ACA}_0: \)

\( \text{ATR}_0: \)

\( \Pi^1_1-\text{CA}_0: \)
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**RCA$_0$:** Recursive Comprehension + $\Sigma^0_1$-induction + Semiring ax.

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\( \Leftrightarrow \) “for every set \( X \), \( X' \) exists”.

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\( \Pi^1_1\)-\( \text{CA}_0 \): \( \Pi^1_1 \)-Comprehension + \( \text{ACA}_0 \).

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The exact reversals

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**Thm:** [Rathjen–Weiermann 93] The length of $\mathcal{T}$ is $\theta \Omega^\omega$, the Ackerman ordinal. The following are equivalent over RCA$_0$

- Kruskal’s theorem.
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**Thm:** [M.–Weiermann 2006] The following are equivalent over $\text{RCA}_0$

- $\text{ATR}_0$
- For every $\mathcal{P}$, *if $\mathcal{P}$ is a WQO, then so is $\mathcal{T}(\mathcal{P})$*,
  where $\mathcal{T}(\mathcal{P})$ is the set of finite trees with labels in $\mathcal{P}$, ordered by
  $T \preceq S$ if $\exists f : T \to S$ which preserves $<$ and increasing on labels.
Theorem: [Robertson–Seymour]

Let $\mathcal{G}$ be the set of finite graphs ordered by the minor relation. Then $\mathcal{G}$ is a WQO.
The minor-graph theorem

**Theorem:** [Robertson–Seymour] Let $G$ be the set of finite graphs ordered by the minor relation. Then $G$ is a WQO.

**Theorem:** [Friedman–Robertson–Seymour] The length of $G$ is $\geq \phi_0(\epsilon_{\Omega_\omega+1})$.

$(\text{where } \phi_0(\epsilon_{\Omega_\omega+1}), \text{the Takeuti-Feferman-Buchholz ordinal, is the the proof-theoretic ordinal of } \Pi_{1}^1-\text{CA}_0).$
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(Where \( \phi_0(\epsilon_{\Omega^\omega+1}) \), the Takeuti-Feferman-Buchholz ordinal, is the proof-theoretic ordinal of \( \Pi_1^1-CA_0 \).

\( \Pi_1^1-CA_0 \) – is the system that allows \( \Pi_1^1 \)-comprehension.)
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**Corollary:** [Friedman, Robertson, Seymour] (RCA$_0$) The minor-graph theorem $\Rightarrow \phi_0(\epsilon_{\Omega+1})$ well-ordered. Therefore,

$$\Pi_1^1$$-CA$_0$ $\not\vdash$ minor-graph theorem.
**Theorem** [Fraïssé’s Conjecture ’48; Laver ’71]

**FRA:** The countable linear orderings are WQO under embeddability.
Fraïssé’s Conjecture

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FRA: The countable linear orderings are WQO under embeddablity.

**Theorem** [Shore ’93]

FRA implies $\text{ATR}_0$ over $\text{RCA}_0$.

**Conjecture:** [Clote ’90][Simpson ’99][Marcone]

FRA is equivalent to $\text{ATR}_0$ over $\text{RCA}_0$. 

\[
\begin{align*}
\Pi^1_2-\text{CA}_0 & \downarrow \quad \Pi^1_1-\text{CA}_0 \quad \text{FRA} \\
\text{ATR}_0 & \downarrow \quad \text{ACA}_0 \quad \text{FRA} \\
\text{WKL}_0 & \downarrow \quad \text{RCA}_0
\end{align*}
\]
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**$\Pi^1_1$-CA$_0$:** $\Pi^1_1$-Comprehension + ACA$_0$.

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Fraïssé’s conjecture.

**Claim**

$\text{RCA}_0 + \text{FRA}$ is the least system where it is possible to develop a reasonable theory of embeddability of linear orderings.
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\[ \text{a reasonable theory of embeddability of linear orderings.} \]

**Theorem ([M. 05])**

The following are equivalent over \( \text{RCA}_0 \)

- \( \text{FRA} \);
- Every scattered lin. ord. is a finite sum of indecomposables;
- Every indecomposable lin. ord. is either an \( \omega \)-sum or an \( \omega^* \)-sum of indecomposable l.o. of smaller rank;
- Jullien’s characterization of extendible linear orderings
A Partition theorem

**Theorem:** [Folklore] If we color $\mathbb{Q}$ with finitely many colors, there exists an embedding $\mathbb{Q} \to \mathbb{Q}$ whose image has only one color.
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**Theorem (\( \ast \)):** [Laver ’72]

For every countable \( \mathcal{L} \), there exists \( n_{\mathcal{L}} \in \mathbb{N} \), such that:

If \( \mathcal{L} \) is colored with finitely many colors,

there is an embedding \( \mathcal{L} \to \mathcal{L} \) whose image has at most \( n_{\mathcal{L}} \) colors.

Theorem (\( \ast \)): [M. 2005]
FRA is implied by Theorem (\( \ast \)) over RCA\(_0\).

Theorem (\( \ast \)): [Kach–Marcone–M.–Weiermann 2011]
FRA is equivalent to Theorem (\( \ast \)) over RCA\(_0\).
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*FRA is implied by Theorem (*)& over RCA$_0$.\)
A Partition theorem

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**Theorem ([M. 2005])**

*FRA is implied by Theorem (*) over RCA$_0$.*

**Theorem ([Kach–Marcone–M.–Weiermann 2011])**

*FRA is equivalent to Theorem (*) over RCA$_0$.*
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1. [Laver 71] For countable $\alpha$, $L_\alpha$ is countable.

Question: Given $\alpha$, what is the length of $L_\alpha$? Given $\alpha$, what is the rank of $L_\alpha$ as a well-founded poset?
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1. [Laver 71] For countable $\alpha$, $\mathbb{L}_\alpha$ is countable.
2. [M. 05] For computable $\alpha$, $(\mathbb{L}_\alpha, \preccurlyeq)$ is computably presentable.
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3. (This was used to prove that every hyparithmetic linear ordering is bi-embeddable with a computable one in [M. 05])
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3. (This was used to prove that every hypearithmetic linear ordering is bi-embeddable with a computable one in [M. 05])
4. FRA is equivalent to “$\forall$ ordinal $\alpha < \omega_1$ (\mathbb{L}_\alpha$ is WQO).”
Def: Let $\mathbb{L}_\alpha$ be the set of linear orderings of Hausdorff rank $< \alpha$, quotiented by the bi-embeddability relation, and ordered by the embeddability relation.

1. [Laver 71] For countable $\alpha$, $\mathbb{L}_\alpha$ is countable.
2. [M. 05] For computable $\alpha$, $(\mathbb{L}_\alpha, \preccurlyeq)$ is computably presentable.
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Question: Given $\alpha$, what is the length of $\mathbb{L}_\alpha$?
Given $\alpha$, what is the rank of $\mathbb{L}_\alpha$ as a well-founded poset?
Finite Hausdorff rank

Theorem ([Marcone, M 08])

*The length of** $\mathbb{L}_\omega$ *is* $\varepsilon_\omega$, where

$\varepsilon_\alpha$ is the $(\alpha + 1)^{\text{st}}$ fixed point for the function $\beta \mapsto \omega^\beta$.

Note: $\varepsilon_\omega$ is the proof-theoretic ordinal of ACA$^+$, where ACA$^+$ is the system RCA$^0 + \forall X (X(\omega) \text{ exists})$.

(So $\varepsilon_\omega$ is the least ordinal that ACA$^+$ can’t prove is well-ordered.)

Theorem ([Marcone, M 08])

That $\mathbb{L}_\omega$ is a WQO, follows from ACA$^+$ + “$\varepsilon_\omega$ is well-ordered”, but not from ACA$^+$.
The length of $L_\omega$ is $\varepsilon\varepsilon\varepsilon...$, where $\varepsilon\varepsilon...$ is the first fixed point of the function $\alpha \mapsto \varepsilon_\alpha$, where $\varepsilon_\alpha$ is the $(\alpha + 1)$st fixed point for the function $\beta \mapsto \omega^\beta$. 

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1 Background on WQOs

2 WQOs in Proof Theory
   - Kruskal’s theorem and the graph-minor theorem
   - Linear orderings and Fraïssé’s Conjecture

3 WPOs in Computability Theory
complexity of maximal order types

Recall: \( o(\mathcal{P}) = \sup \{ \text{ordType}(\mathcal{P}, \leq_L) : \text{where } \leq_L \text{ is a linearization of } \mathcal{P} \} \).
complexity of maximal order types

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Theorem: [De Jongh, Parikh 77]
Every WPO $\mathcal{P}$ has a linearization of order type $o(\mathcal{P})$. 
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**Theorem:** [De Jongh, Parikh 77]
Every WPO \( \mathcal{P} \) has a linearization of order type \( o(\mathcal{P}) \).

We call such a linearization, a *maximal linearization* of \( \mathcal{P} \).
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**Question** [Schmidt 1979]:
Is the length of a computable WPO computable?
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We mentioned that $o(\mathcal{P}) + 1 = \text{rk}(\text{Bad}(\mathcal{P}))$, where

$$\text{Bad}(\mathcal{P}) = \{\langle x_0, ..., x_{n-1} \rangle \in W^{<\omega} : \forall i < j \ (x_i \nleq_P x_j)\},$$

Since $\text{Bad}(\mathcal{P})$ is computable and well-founded, it has rank $< \omega_1^{CK}$. So, $o(\mathcal{P})$ is a computable ordinal.
Q: Is the length of a computable WPO, computable?

We mentioned that $o(P) + 1 = \text{rk}(\mathcal{B}ad(P))$, where

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Q:
Does every computable WPO have a computable maximal linearization?
A computable maximal linearization

Theorem ([M 2007])

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There is computable procedure that given $\mathcal{P}$ produces a linearization $\mathcal{L}$ such that for some $\delta$

$$\omega^\delta \leq \mathcal{L} \leq o(\mathcal{P}) < \omega^{\delta+1}.$$
A computable maximal linearization

**Theorem ([M 2007])**

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\omega^\delta \leq L \leq o(P) < \omega^{\delta+1}.
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**Theorem ([M 2007])**

Let \( a \) be a Turing degree. TFAE:

1. \( a \) uniformly computes maximal linearizations of computable WPOs.
2. \( a \) uniformly computes \( 0^{(\beta)} \) for every \( \beta < \omega_1^{CK} \).
The height of a WPO

We denote by \( \text{Ch}(\mathcal{P}) \) the collection of all chains of \( \mathcal{P} \).

Theorem: [Wolk 1967] If \( \mathcal{P} \) is a WPO, there exists \( C \in \text{Ch}(\mathcal{P}) \) with order type \( \text{ht}(\mathcal{P}) \).

Such a chain is called a maximal chain of \( \mathcal{P} \).
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**Definition**

If $\mathcal{P}$ is well founded, its *height* is

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Q: How difficult is it to compute maximal chains?

Antonio Montalbán (U. of Chicago)

Well-Partial-Orderings

Madison, March 2012
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Every computable WPO $\mathcal{P}$ has a hyperarithmetic maximal chain.

(Recall: $X \subseteq \omega$ is hyperarithmetic iff it’s $\Delta^1_1$.)
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**Theorem ([Marcone–M.–Shore 2012])**

Let $\alpha < \omega_1^{CK}$.

*There exists a computable WPO $\mathcal{P}$ such that $0^{(\alpha)}$ does not compute any maximal chain of $\mathcal{P}$.***
Maximal chains are not easy to compute,

**Theorem** ([Marcone-M.-Shore 2012])

Let $G \in 2^{\omega}$ be hyperarithmetically generic. Then $G$ can compute a maximal chain in every computable WPO.

**Pf:**

• The key observation is that all downward closed subsets of $P$ are computable.

• Suppose that $P$ has cofinality $\omega_\alpha+1$.

• Then, build an operator $\Phi_{P, G_\alpha}$ that returns a sequence of computable sub-partial orderings $P_0 \leq P_1 \leq ...$, such that, if $G$ is generic, then infinitely many of the $P_i$ will have cofinality $\omega_\alpha$.

• Then use effective transfinite recursion.
Maximal chains are not easy to compute,
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