The extent of the failure of Ramsey’s theorem at successor cardinals

2012 North American Annual Meeting of the ASL
University of Wisconsin Madison
31-March-2012

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Ramsey’s theorem

The arrow notation
Let $\lambda \rightarrow (\lambda)^2_\kappa$ denote the assertion:
For every function $f : [\lambda]^2 \rightarrow \kappa$, there exists a subset $H \subseteq \lambda$ s.t.:

- $|H| = \lambda$;
- $f \upharpoonright [H]^2$ is constant.
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Theorem (Ramsey, 1929)
$\omega \rightarrow (\omega)^2_2$ holds.
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Theorem (Ramsey, 1929)
$\omega \to [\omega]^2_2$ holds.
Ramsey’s theorem (Cont.)

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\[ \omega \rightarrow [\omega]_2^2. \]

Ramsey’s theorem is very pleasing. Unfortunately, it does not generalize to higher cardinals.

Theorem (Sierpiński, 1933)

\[ \omega_1 \not\rightarrow [\omega_1]_2^2. \]

Sierpiński’s theorem is pleasing on its own! It tells us that \([\omega_1]_2^2\) admits a rather wild 2-valued coloring.
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Generalizing Sierpiński

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So, there exists a coloring \( f : [\omega_1]^2 \rightarrow 2 \) such that \([X]^2\) attains all colors for every uncountable \( X \subseteq \omega_1 \). This raises the question of whether an analogous statement concerning a coloring with more than two colors is valid.
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**Theorem (Erdős-Hajnal-Rado, 1965)**

*CH entails* \( \omega_1 \not\rightarrow [\omega_1]^{\omega_1}_2. \)
Generalizing Sierpiński

Theorem (Erdös-Hajnal-Rado, 1965)

$CH$ entails $\omega_1 \not\rightarrow [\omega_1]^2_{\omega_1}$.

**Question**: May the cardinal arithmetic hypothesis be eliminated?
Generalizing Sierpiński in ZFC

Theorem (Sierpiński, 1933)

\[ \omega_1 \nless \left[ \omega_1 \right]_2. \]
Generalizing Sierpiński in ZFC

Theorem (Sierpiński, 1933)
\[ \omega_1 \not\rightarrow [\omega_1]^2. \]

Theorem (Blass, 1972)
\[ \omega_1 \not\rightarrow [\omega_1]^3. \]
Generalizing Sierpiński in ZFC

Theorem (Sierpiński, 1933)
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$\omega_1 \nrightarrow [\omega_1]^2_3$.

Theorem (Galvin-Shelah, 1973)
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Theorem (Sierpiński, 1933)
\[ \omega_1 \not\rightarrow [\omega_1]^2_{\omega}. \]

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\textbf{Theorem (Todorčević, 1987)}
\[ \omega_1 \not\rightarrow [\omega_1]^\omega_1. \]
The rectangular square-bracket relation

Negative square-bracket relation

$\lambda \nrightarrow [\lambda]^2_{\kappa}$ asserts the existence of a function $f : [\lambda]^2 \rightarrow \kappa$ such that for every subset $X \subseteq \lambda$: if $|X| = \lambda$, then $f \upharpoonright [X]^2$ is onto $\kappa$. 
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**Negative square-bracket relation**

$\lambda \not
\rightarrow [\lambda]^2_\kappa$ asserts the existence of a function $f : [\lambda]^2 \rightarrow \kappa$ such that for every subset $X \subseteq \lambda$: if $|X| = \lambda$, then $f \upharpoonright [X]^2$ is onto $\kappa$.

**Negative rectangular square-bracket relation**

$\lambda \not
\rightarrow [\lambda; \lambda]^2_\kappa$ asserts the existence of a function $f : [\lambda]^2 \rightarrow \kappa$ s.t. for every subsets $X, Y$: if $|X| = |Y| = \lambda$, then $f \upharpoonright (X \odot Y)$ is onto $\kappa$. 
Theorem (Erdös-Hajnal-Rado, 1965)

$CH$ entails $\omega_1 \not\rightarrow [\omega_1]^2_{\omega_1}$.

Theorem (Todorčević, 1987)

$\omega_1 \not\rightarrow [\omega_1]^2_{\omega_1}$ holds in $ZFC$. 
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Theorem (Erdős-Hajnal-Rado, 1965)

\( CH \) entails \( \omega_1 \not\rightarrow [\omega_1; \omega_1]_{\omega_1}^2 \).

Theorem (Todorčević, 1987)

\( \omega_1 \not\rightarrow [\omega_1]_{\omega_1}^2 \) holds in ZFC.

Theorem (Moore, 2006)

\( \omega_1 \not\rightarrow [\omega_1; \omega_1]_{\omega_1}^2 \) holds in ZFC.
Negative square-bracket for higher cardinals

WRITE THE
LARGEST† NUMBER
YOU CAN:
Theorem (Erdös-Hajnal-Rado, 1965)

\[ 2^\lambda = \lambda^+ \ \text{entails} \ \lambda^+ \nrightarrow [\lambda^+; \lambda^+]_{\lambda^+}^2. \]
The rectangular square-bracket relation for higher cardinals

Theorem (Erdős-Hajnal-Rado, 1965)

$2^\lambda = \lambda^+ \text{ entails } \lambda^+ \not\rightarrow [\lambda^+; \lambda^+]_{\lambda^+}^2.$

Theorem (Todorčević, 1987)

$\lambda^+ \not\rightarrow [\lambda^+]_{\lambda^+}^2 \text{ holds for every infinite regular } \lambda.$
The rectangular square-bracket relation for higher cardinals

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\[ \lambda^+ \not\rightarrow [\lambda^+]^2_{\lambda^+} \text{ holds for every infinite regular } \lambda. \]

Open Problems

1. Does \( \lambda^+ \not\rightarrow [\lambda^+]^2_{\lambda^+} \) hold for every singular cardinal \( \lambda \)?
2. Does \( \lambda^+ \not\rightarrow [\lambda^+]^2_{\lambda^+} \) entail \( \lambda^+ \not\rightarrow [\lambda^+; \lambda^+]^2_{\lambda^+} \)?
A Solution to Problem #2
Main result: comparing squares with rectangles

Theorem

The following are equivalent for all cardinals $\lambda, \kappa$:

- $\lambda^+ \notightarrow [\lambda^+]^2_\kappa$
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Main result: comparing squares with rectangles

Theorem

The following are equivalent for all cardinals $\lambda, \kappa$:

1. $\lambda^+ \not\rightarrow [\lambda^+]^2_\kappa$
2. $\lambda^+ \not\rightarrow [\lambda^+; \lambda^+]^2_\kappa$

The above is a corollary of the following ZFC theorem.

Main technical result

Every infinite cardinal $\lambda$ admits a function $rts : [\lambda^+]^2 \rightarrow [\lambda^+]^2$ s.t.: for every cofinal subsets $X, Y$ of $\lambda^+$, there exists a cofinal subset $Z \subseteq \lambda^+$ such that $rts[X \odot Y] \supseteq Z \odot Z$. 
Shelah’s study of strong colorings

\[ \Pr_0(\lambda, \kappa, \theta) \]
\[ \Pr_1(\lambda, \kappa, \theta) \]
\[ \Pr_1^-(\lambda, \kappa, \theta) \]
\[ \Pr_2(\lambda, \kappa, \theta) \]
\[ \Pr_3^L(\lambda, \kappa, \theta) \]
\[ \Pr_3^S(\lambda, \kappa, \theta) \]
\[ \lambda \not\rightarrow [\lambda]_\kappa^2 \]
\[ \lambda \not\rightarrow \exists \lambda \text{-L space} \]
\[ \lambda \not\rightarrow \exists \lambda \text{-S space} \]
\[ \lambda \text{-c.c. not productive} \]
Comparing classic concepts with modern one

Our main technical result was the missing link to the following.

**Corollary (Eisworth+Shelah+R.)**

*TFAE for every uncountable cardinal* \( \lambda \):

- \( \lambda^+ \not\rightarrow [\lambda^+]_\lambda^+ \)
- \( \Pr_0(\lambda^+, \lambda^+, \omega) \)

**Definition (Shelah)**

\( \Pr_0(\lambda^+, \lambda^+, \omega) \) asserts the existence of a function \( f : [\lambda^+]^2 \rightarrow \lambda^+ \) satisfying the following.

For every \( n < \omega \), every \( g : n \times n \rightarrow \lambda^+ \), and every collection \( A \subseteq [\lambda^+]^n \) of mutually disjoint sets, of size \( \lambda^+ \), there exists some \( x, y \in A \) with \( \max(x) < \min(y) \) such that

\[
f(x(i), y(j)) = g(i, j) \quad \text{for all } i, j < n.
\]
Surprise, Surprise!!

\[ \lambda = \text{successor of uncountable} \]
\[ \kappa = \lambda \text{ i.e. maximal num. of colors} \]
\[ \theta = \omega \]

\[ \lambda \text{-c.c. not productive} \]

\[ \lambda \not
\subseteq [\lambda]^2_{\kappa} \]
Ingredients of the proof

Case 1. Successors of singulars
The proof of the existence of a function \( rts : [\lambda^+]^2 \rightarrow [\lambda^+]^2 \) for a singular \( \lambda \) is the heart of the matter.

- Shelah's club guessing theorems, and Eisworth's theorem on the existence of off-center club guessing matrices for singular cardinals of countable cofinality;
- A generalization of Todorćevićc method of walks on ordinals, where each ordinal \( \alpha \) admits a sequence of clubs, \( \langle C_i^\alpha \mid i < \text{cf}(\lambda) \rangle \), rather than a single one;
- Oscillation theory of \( \text{pcf} \) scales, plus coding, from which one can get essentially-generic guidelines on which clubs to visit throughout the generalized walks, and moreover, which ordinals to pick from these walks.
Successor of singulars — in ZFC

The proof of the existence of a function $rts : [\lambda^+]^2 \to [\lambda^+]^2$ for a singular $\lambda$ is the heart of the matter. The definition of the function and the verification of its properties involves the following:

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Ingredients of the proof

Case 2. Successors of regulars
Successors of regulars — in ZFC

Let $\lambda$ denote a regular cardinal. Then:

1. (Todorčević, 1987) $\lambda^+ \nrightarrow [\lambda^+]^2_{\lambda^+}$ [Partitioning pairs of countable ordinals]

Corollary (Shelah+Moore) $\lambda^+ \nrightarrow [\lambda^+]^2_{\lambda^+}$ holds for every regular cardinal $\lambda$.

Remark The proofs of 3, 4, 5 are entirely different, and it was unknown whether a uniform proof of 3 + 4 + 5 exists.
Let $\lambda$ denote a regular cardinal. Then:

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Successors of regulars — in ZFC

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5. (Moore, 2006) $\lambda^+ \not\rightarrow [\lambda^+; \lambda^+]^2_{\lambda^+}$, if $\lambda = \aleph_0$ [A solution to the L space problem]
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**Corollary (Shelah+Moore)**

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1. Moore’s proof involves the definition of a function $o : [\omega_1]^2 \rightarrow \omega$ that witnesses $\omega_1 \not\rightarrow [\omega_1; \omega_1]^2_{\omega_1}$. (Then, a stretching argument yields $\omega_1 \not\rightarrow [\omega_1; \omega_1]^2_{\omega_1}$.)
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2. We found a generalization of Moore’s definition that yields a function

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3. We then compose the generalized $o$ with the classic function $Tr : [\lambda^+]^2 \rightarrow <\omega \lambda^+$, and argue that this witnesses $\lambda^+ \not\rightarrow [\lambda^+; \lambda^+]_{\lambda^+}^2$. 
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2. We found a generalization of Moore’s definition that yields a function \( o : [\lambda^+]^2 \rightarrow \omega \) witnessing \( \lambda^+ \not\rightarrow [\lambda^+; \lambda^+]^2_\omega \) for every regular \( \lambda \);

3. We then compose the generalized \( o \) with the classic function \( \text{Tr} : [\lambda^+]^2 \rightarrow <\omega \lambda^+ \), and argue that this witnesses \( \lambda^+ \not\rightarrow [\lambda^+; \lambda^+]^2_{\lambda^+} \).

4. While \( \lambda^+ \not\rightarrow [\lambda^+; \lambda^+]^2_\omega \) has been established previously using other functions, the generalized \( o \) is the first function that is known to have this successful composition property.
Thank you!

The slides of this talk may be found at the following address:
http://assafrinot.com/talks/asl2012
More on successor of singulars — in ZFC

Theorem (Shelah, 1990’s)

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Theorem (Shelah, 1990’s)

If \( \lambda \) is a singular cardinal of uncountable cofinality, then \( E^{\lambda^+}_{\text{cf}(\lambda)} \) carries a club-guessing sequence of a very strong form.

Theorem (Eisworth, 2010)

If \( \lambda \) is a singular cardinal of countable cofinality, then \( E^{\lambda^+}_{\omega_1} \) carries a club-guessing matrix of a very strong form.
More on successor of singulars — in ZFC

Theorem (Shelah, 1990’s)

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If \( \lambda \) is a singular cardinal of countable cofinality, then \( E_{\omega_1}^{\lambda^+} \) carries a club-guessing matrix of a very strong form.

Still Open

Whether \( \lambda^+ \nrightarrow [\lambda^+]^2_{\lambda^+} \) hold for all singular \( \lambda \), in ZFC.
Main technical result
For every singular cardinal $\lambda$, there exists a function
\( rts : [\lambda^+]^2 \to [\lambda^+]^2 \) such that for every cofinal subsets \( X, Y \) of \( \lambda^+ \), there exists a cofinal subset \( Z \subseteq \lambda^+ \) such that \( rts[X \odot Y] \supseteq Z \odot Z \).

Remark: our proof builds heavily on previous arguments of Shelah, Todorčević, and most notably — Eiswerth.
Transforming Rectangles into Squares — in ZFC

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For every singular cardinal $\lambda$, there exists a function $rts : [\lambda^+]^2 \to [\lambda^+]^2$ such that for every cofinal subsets $X, Y$ of $\lambda^+$, there exists a cofinal subset $Z \subseteq \lambda^+$ such that $rts[X \ast Y] \supseteq Z \ast Z$.

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The definition of $rts$

- Fix a matrix of local clubs $\langle C_\alpha^i \mid \alpha < \lambda^+, i < \text{cf}(\lambda) \rangle$ that incorporates a club-guessing sequence/matrix.
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- Fix a matrix of local clubs \( \langle C^i_\alpha \mid \alpha < \lambda^+, i < \text{cf}(\lambda) \rangle \) that incorporates a club-guessing sequence/matrix.
- Adapt Shelah’s proof of \( \lambda^+ \not\rightarrow [\lambda^+; \lambda^+]_\text{cf}(\lambda) \), to get a function \( f : [\lambda^+]^2 \rightarrow <\omega \text{ cf}(\lambda) \times <\omega \text{ cf}(\lambda) \) with strong properties.
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For every singular cardinal $\lambda$, there exists a function $rts : [\lambda^+]^2 \rightarrow [\lambda^+]^2$ such that for every cofinal subsets $X, Y$ of $\lambda^+$, there exists a cofinal subset $Z \subseteq \lambda^+$ such that $rts[X \odot Y] \supseteq Z \odot Z$.

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- Adapt Shelah’s proof of $\lambda^+ \not\rightarrow [\lambda^+; \lambda^+]_{\text{cf}(\lambda)}^2$, to get a function $f : [\lambda^+]^2 \rightarrow <\omega \text{ cf}(\lambda) \times <\omega \text{ cf}(\lambda)$ with strong properties.
- Given $\alpha < \beta < \lambda^+$, consider $(\sigma, \eta) = f(\alpha, \beta);$
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For every singular cardinal \( \lambda \), there exists a function
\( rts : [\lambda^+]^2 \to [\lambda^+]^2 \) such that for every cofinal subsets \( X, Y \) of \( \lambda^+ \),
there exists a cofinal subset \( Z \subseteq \lambda^+ \) such that
\( rts[X \otimes Y] \supseteq Z \otimes Z \).

Remark: our proof builds heavily on previous arguments of Shelah,
Todorčević, and most notably — Eisworth.

The definition of \( rts \)

- Fix a matrix of local clubs \( \langle C^i_\alpha | \alpha < \lambda^+, i < \text{cf}(\lambda) \rangle \) that incorporates a club-guessing sequence/matrix.
- Adapt Shelah’s proof of \( \lambda^+ \not\to [\lambda^+ ; \lambda^+]_{\text{cf}(\lambda)}^2 \), to get a function
\( f : [\lambda^+]^2 \to <\omega \text{ cf}(\lambda) \times <\omega \text{ cf}(\lambda) \) with strong properties.
- Given \( \alpha < \beta < \lambda^+ \), consider \( (\sigma, \eta) = f(\alpha, \beta) \);
- Let \( \beta_0 := \beta \), and \( \beta_{n+1} := \min(C^{\sigma(n)}_{\beta_n} \setminus \alpha) \) for all \( n \in \text{dom}(\sigma) \);
Transforming Rectangles into Squares (Cont.)

The definition of \( rts \)

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- Fix a function \( f : [\lambda^+]^2 \rightarrow \langle \omega \text{ cf}(\lambda) \times \langle \omega \text{ cf}(\lambda) \rangle \) with strong coloring properties;
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- Let \( \gamma := \max\{\sup(C^{\sigma(n)}_{\beta_n} \cap \alpha) \mid n \in \text{dom}(\sigma)\}\).
The definition of $rts$

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- Let $\gamma := \max\{\sup(C^{\sigma(n)}_{\beta_n} \cap \alpha) \mid n \in \text{dom}(\sigma)\}$;
- Let $\alpha_0 := \alpha$, and $\alpha_{m+1} := \min(C^{\eta(m)}_{\alpha_m} \setminus \gamma + 1)$ for $m \in \text{dom}(\eta)$.
Transforming Rectangles into Squares (Cont.)

The definition of \( rts \)

- Fix a matrix of local clubs \( \langle C^i_\alpha \mid \alpha < \lambda^+, i < \text{cf}(\lambda) \rangle \) that incorporates a club-guessing sequence/matrix;
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- Given \( \alpha < \beta < \lambda^+ \), consider \( (\sigma, \eta) = f(\alpha, \beta) \);
- Let \( \beta_0 := \beta \), and \( \beta_{n+1} := \min(C^\sigma(n)_{\beta_n} \setminus \alpha) \) for all \( n \in \text{dom}(\sigma) \);
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- Let \( \alpha_0 := \alpha \), and \( \alpha_{m+1} := \min(C^\eta(m)_{\alpha_m} \setminus \gamma + 1) \) for \( m \in \text{dom}(\eta) \);
- Put \( rts(\alpha, \beta) := (\alpha_{\text{dom}(\eta)}, \beta_{\text{dom}(\sigma)}) \).
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- Let $\alpha_0 := \alpha$, and $\alpha_{m+1} := \min(C^{\eta(m)}_{\alpha_m} \setminus \gamma + 1)$ for $m \in \text{dom}(\eta)$
- Put $rts(\alpha, \beta) := (\alpha_{\text{dom}(\eta)}, \beta_{\text{dom}(\sigma)})$.

The definition of $rts$ is quite natural in this context, and so the main point is to verify that the definition does the job.
Why does $rts$ work

- For every cofinal subset $X \subseteq \lambda^+$, every ordinal $\delta < \lambda^+$, and every type $p$ in the language of the matrix-based walks, let $X_p(\delta) := \{ \alpha \in X \mid \text{the pair } (\delta, \alpha) \text{ realizes the type } p \}$;
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- Denote $S^X_p := \{ \delta < \lambda^+ \mid \sup(X_p(\delta)) = \sup(X) \}$;
- Use the fact that the chosen matrix incorporates club guessing to argue that for every cofinal subsets of $\lambda^+$, $X$ and $Y$, there exists a type $p$, for which $S^X_p \cap S^Y_p$ is stationary;
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- Use the fact that $f$ oscillates quite nicely on rectangles $X \ast Y$, so that it can produce sequences $(\sigma, \eta)$ with successful guidelines on which columns of the matrix to advise throughout the walks, and at which step of the walks to stop. This insures that the type $p$ gets realized quite frequently;
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- Use the fact that $f$ oscillates quite nicely on rectangles $X \star Y$, so that it can produce sequences $(\sigma, \eta)$ with successful guidelines on which columns of the matrix to advise throughout the walks, and at which step of the walks to stop. This insures that the type $p$ gets realized quite frequently;
- Conclude that $rts[X \star Y] \supseteq [S^X_p \cap S^Y_p \cap C]^2$ for the club $C$ of ordinals of the form $M \cap \lambda^+$, for elementary submodels $M \prec H_\chi$ of size $\lambda$, that contains all relevant objects.