The Solovay Hierarchy

Grigor Sargsyan

Department of Mathematics
Rutgers University

ASL
April 2, 2012
Madison, Wisconsin
The large cardinal phenomenon

The large cardinal hierarchy is a consistency strength hierarchy.
The large cardinal hierarchy is a consistency strength hierarchy. More precisely,

- The large cardinal axioms are linearly ordered according to their consistency strengths.
The large cardinal hierarchy is a consistency strength hierarchy. More precisely,

- The large cardinal axioms are linearly ordered according to their consistency strengths.
- The consistency of any mathematical theory can be reduced to that of the consistency of some large cardinal axiom.
The large cardinal phenomenon

The large cardinal hierarchy is a consistency strength hierarchy. More precisely,

- The large cardinal axioms are linearly ordered according to their consistency strengths.
- The consistency of any mathematical theory can be reduced to that of the consistency of some large cardinal axiom.
The use of large cardinal axioms

The large cardinal hierarchy have been used as the backbone of the following axiomatic systems.

- Forcing axioms such as *PFA*: used to solve problems in analysis, operator algebras, combinatorics and etc.
The use of large cardinal axioms

The large cardinal hierarchy have been used as the backbone of the following axiomatic systems.

- Forcing axioms such as $PFA$: used to solve problems in analysis, operator algebras, combinatorics and etc.
- Determinacy axioms such as $PD$ or $AD$: used to solve problems in analysis.
The use of large cardinal axioms

The large cardinal hierarchy have been used as the backbone of the following axiomatic systems.

- Forcing axioms such as \textit{PFA}: used to solve problems in analysis, operator algebras, combinatorics and etc.
- Determinacy axioms such as \textit{PD} or \textit{AD}: used to solve problems in analysis.
- Generic embeddings: used to solve many combinatorial problems.
Examples

(Spector, 1955) If all coanalytic sets have the perfect set property then there is an inner model with an inaccessible cardinal.
Reversing the large cardinal phenomenon

Examples

- (Spector, 1955) If all coanalytic sets have the perfect set property then there is an inner model with an inaccessible cardinal.

- (Woodin, 90s) If Projective Determinacy holds then for each $n$ there is an inner model with $n$ Woodin cardinals.

Remark
Without such reversals the large cardinal phenomenon has interesting but ultimately not an important content. However, such reversals have been established for a very small initial segment of the large cardinal hierarchy.
Reversing the large cardinal phenomenon

Examples

- (Spector, 1955) If all coanalytic sets have the perfect set property then there is an inner model with an inaccessible cardinal.

- (Woodin, 90s) If Projective Determinacy holds then for each $n$ there is an inner model with $n$ Woodin cardinals.

- (Steel, 2004) If $PFA$ holds then there is an inner model with infinitely many Woodin cardinals.

Remark

Without such reversals the large cardinal phenomenon has interesting but ultimately not an important content. However, such reversals have been established for a very small initial segment of the large cardinal hierarchy.
Reversing the large cardinal phenomenon

Examples

- (Spector, 1955) If all coanalytic sets have the perfect set property then there is an inner model with an inaccessible cardinal.
- (Woodin, 90s) If Projective Determinacy holds then for each $n$ there is an inner model with $n$ Woodin cardinals.
- (Steel, 2004) If PFA holds then there is an inner model with infinitely many Woodin cardinals.

Remark

- *Without such reversals the large cardinal phenomenon has interesting but ultimately not an important content.*
Examples

- (Spector, 1955) If all coanalytic sets have the perfect set property then there is an inner model with an inaccessible cardinal.
- (Woodin, 90s) If Projective Determinacy holds then for each $n$ there is an inner model with $n$ Woodin cardinals.
- (Steel, 2004) If $PFA$ holds then there is an inner model with infinitely many Woodin cardinals.

Remark

- *Without such reversals the large cardinal phenomenon has interesting but ultimately not an important content.*
- *However, such reversals have been established for a very small initial segment of the large cardinal hierarchy.*
Bad news or perhaps a good news

The current methods can only deal with large cardinals in the region of Woodin cardinals.
Bad news or perhaps a good news

The current methods can only deal with large cardinals in the region of Woodin cardinals.

**Theorem**

*Assume PFA. Then there is a model with a proper class of Woodin cardinals and strong cardinals.*
The current methods can only deal with large cardinals in the region of Woodin cardinals.

**Theorem**

Assume PFA. Then there is a model with a proper class of Woodin cardinals and strong cardinals.

**Remark**

*There is no known systematic way of getting reversals much beyond the large cardinal axiom of the theorem.*
Large cardinal axioms are reflection principles asserting the existence of elementary embeddings.
The large cardinal hierarchy

- Large cardinal axioms are reflection principles asserting the existence of elementary embeddings.
- A typical large cardinal axiom states that there is a nontrivial elementary embedding
  \[ j : V \rightarrow M \]
such that \( M \) is “close” to \( V \).

Theorem (Kunen)
There is no \( j : V \rightarrow V \) such that \( j \neq \text{id} \).
Large cardinal axioms are reflection principles asserting the existence of elementary embeddings.

A typical large cardinal axiom states that there is a nontrivial elementary embedding

\[ j : V \rightarrow M \]

such that \( M \) is “close” to \( V \). You are completely free to decide what “close” means here,
The large cardinal hierarchy

- Large cardinal axioms are reflection principles asserting the existence of elementary embeddings.
- A typical large cardinal axiom states that there is a nontrivial elementary embedding
  \[ j : V \to M \]
  such that \( M \) is “close” to \( V \). You are completely free to decide what “close” means here, but be careful.

**Theorem (Kunen)**

*There is no \( j : V \to V \) such that \( j \neq id \).*
Examples

- Given $j : V \rightarrow M$ such that $j \neq id$, $\text{crit}(j)$ is the least ordinal $\kappa$ such that $j(\kappa) > \kappa$. 

$\kappa$ is a measurable cardinal if there is $j : V \rightarrow M$ such that $\text{crit}(j) = \kappa$ and $M$ is closed under $\kappa$-sequence, i.e. for every $f : \kappa \rightarrow M$, $f \in M$.

$\kappa$ is a supercompact cardinal if for every $\lambda$ there is $j : V \rightarrow M$ such that $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $M$ is closed under $\lambda$-sequences.
Examples

- Given $j : V \rightarrow M$ such that $j \neq id$, $\text{crit}(j)$ is the least ordinal $\kappa$ such that $j(\kappa) > \kappa$.

- $\kappa$ is a measurable cardinal if there is $j : V \rightarrow M$ such that $\text{crit}(j) = \kappa$ and $M$ is closed under $\kappa$-sequence, i.e. for every $f : \kappa \rightarrow M$, $f \in M$. 
Examples

- Given $j : V \to M$ such that $j \neq id$, $\text{crit}(j)$ is the least ordinal $\kappa$ such that $j(\kappa) > \kappa$.

- $\kappa$ is a measurable cardinal if there is $j : V \to M$ such that $\text{crit}(j) = \kappa$ and $M$ is closed under $\kappa$-sequence, i.e. for every $f : \kappa \to M$, $f \in M$.

- $\kappa$ is a supercompact cardinal if for every $\lambda$ there is $j : V \to M$ such that $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $M$ is closed under $\lambda$-sequences.
Examples

1. measurable cardinals,
2. strong cardinals,
3. Woodin cardinals,
4. Shelah cardinals,
5. superstrong cardinals,
6. subcompact cardinals,
7. supercompact cardinals,
8. huge cardinals,
9. etc (look at Kanamori’s book).

Remark

Woodin cardinals are tiny when compared to superstrong cardinals which are tiny when compared to supercompact cardinals.
Examples

1. measurable cardinals,
2. strong cardinals,
3. Woodin cardinals,
4. Shelah cardinals,
5. superstrong cardinals,
6. subcompcact cardinals,
7. supercompact cardinals,
8. huge cardinals,
9. etc (look at Kanamori’s book).

Remark

Woodin cardinals are tiny when compared to superstrong cardinals which are tiny when compared to supercompact cardinals.
Conjecture (The PFA Conjecture)

PFA implies there is an inner model with

- a Woodin cardinal which is a limit of Woodin cardinals,
Conjecture (The PFA Conjecture)

*PFA implies there is an inner model with*

- a *Woodin cardinal which is a limit of Woodin cardinals,*
- a *superstrong cardinal,*
Conjecture (The PFA Conjecture)

PFA implies there is an inner model with

- a Woodin cardinal which is a limit of Woodin cardinals,
- a superstrong cardinal,
- a supercompact cardinal.
Conjecture (The PFA Conjecture)

PFA implies there is an inner model with

- a Woodin cardinal which is a limit of Woodin cardinals,
- a superstrong cardinal,
- a supercompact cardinal.

Remark

- Part 3 is the most important one as it will give an equiconsistency.
Conjecture (The PFA Conjecture)

PFA implies there is an inner model with

- a Woodin cardinal which is a limit of Woodin cardinals,
- a superstrong cardinal,
- a supercompact cardinal.

Remark

- Part 3 is the most important one as it will give an equiconsistency.
- While the conjecture has been open for a long time, it is only a test question.
The classical approach: the inner model problem

Problem (The Inner Model Problem)

*Construct canonical inner models with large cardinals.*
The classical approach: the inner model problem

Problem (The Inner Model Problem)

*Construct canonical inner models with large cardinals.*

Remark

- The canonical inner models are models that resemble $L$, such models are called mice.
The classical approach: the inner model problem

Problem (The Inner Model Problem)
*Construct canonical inner models with large cardinals.*

Remark
- *The canonical inner models are models that resemble L, such models are called mice.*
- *While the problem is open for almost all large cardinals that are significantly bigger than Woodin cardinals, the desired cardinal is the supercompact cardinal.*
The classical approach: the inner model problem

Problem (The Inner Model Problem)

Construct canonical inner models with large cardinals.

Remark

- The canonical inner models are models that resemble $L$, such models are called mice.
- While the problem is open for almost all large cardinals that are significantly bigger than Woodin cardinals, the desired cardinal is the supercompact cardinal.
- The goal is to develop tools for systematically constructing such canonical models with large cardinals while working under various theories extending ZFC.
The origin of the problem

Definition (Gödel)

- $L_0 = \emptyset$,
- $L_{\alpha+1} = \{ A \subseteq L_\alpha : A$ is definable over $(L_\alpha, \in)$ with parameters$\}$.
- for limit $\lambda$, $L_\lambda = \bigcup_{\alpha < \lambda} L_\alpha$.
- $L = \bigcup_{\alpha \in \text{Ord}} L_\alpha$.

Theorem (Scott, 1961)

Suppose there is a measurable cardinal. Then $V \neq L$. 
The origin of the problem

Definition (Gödel)
- $L_0 = \emptyset$,
- $L_{\alpha+1} = \{A \subseteq L_\alpha : A$ is definable over $(L_\alpha, \in)$ with parameters$\}$.
- for limit $\lambda$, $L_\lambda = \bigcup_{\alpha < \lambda} L_\alpha$.
- $L = \bigcup_{\alpha \in \text{Ord}} L_\alpha$.

Theorem (Scott, 1961)

Suppose there is a measurable cardinal. Then $V \neq L$. 
Fine structural analysis of $L$ shows that $L$ has astonishingly canonical structure.
The Solovay Hierarchy

Fine structural analysis of $L$ shows that $L$ has astonishingly canonical structure. Because of this, it is impossible not to ask whether large cardinals can coexist with such a canonical structure.
Fine structural analysis of $L$ shows that $L$ has astonishingly canonical structure.

Because of this, it is impossible not to ask whether large cardinals can coexist with such a canonical structure.

This is exactly the content of the inner model problem.
• Fine structural analysis of $L$ shows that $L$ has astonishingly canonical structure.
• Because of this, it is impossible not to ask whether large cardinals can coexist with such a canonical structure.
• This is exactly the content of the inner model problem.
• But what are these canonical models?
The idea.

Remark

- *All large cardinals can be defined in terms of the existence of ultrafilters or extenders.*
The idea.

Remark

- All large cardinals can be defined in terms of the existence of ultrafilters or extenders.
- An extender $E$ is a coherent family of ultrafilters. It is best to think of them as just ultrafilters that code bigger embeddings than usual ultrafilters.
Remark

- All large cardinals can be defined in terms of the existence of ultrafilters or extenders.

- An extender $E$ is a coherent family of ultrafilters. It is best to think of them as just ultrafilters that code bigger embeddings than usual ultrafilters.

- Since all large cardinals can be defined via extenders, it is natural to look for canonical models with large cardinals among the models of the form $L[\vec{E}]$ where $\vec{E}$ is a sequence of extenders.
The model $L[A]$

Definition (Gödel)

- $L_0[A] = \emptyset$,
- $L_{\alpha+1}[A] = \{ B \subseteq L_\alpha[A] : B \text{ is definable over } (L_\alpha[A], \in, A \cap L_\alpha[A]) \text{ with parameters } \}$,
- $L_\lambda[A] = \bigcup_{\alpha < \lambda} L_\alpha[A]$,
- $L[A] = \bigcup_{\alpha \in \text{Ord}} L_\alpha[A]$. 

Remark $L[A]$ may not be canonical, it depends on $A$. The idea is to consider $L[\vec{E}]$ where $\vec{E}$ is a sequence of extenders and show that it is “canonical” and has large cardinals.
The model $L[A]$  

**Definition (Gödel)**  
- $L_0[A] = \emptyset$,  
- $L_{\alpha+1}[A] = \{ B \subseteq L_\alpha[A] : B \text{ is definable over } (L_\alpha[A], \in, A \cap L_\alpha[A]) \text{ with parameters } \}$.  
- $L_\lambda[A] = \bigcup_{\alpha < \lambda} L_\alpha[A]$,  
- $L[A] = \bigcup_{\alpha \in \text{Ord}} L_\alpha[A]$.  

**Remark**  
- $L[A]$ may not be canonical, it depends on $A$.  

The Solovay Hierarchy
The model $L[A]$

Definition (Gödel)

- $L_0[A] = \emptyset$,
- $L_{\alpha+1}[A] = \{ B \subseteq L_\alpha[A] : B \text{ is definable over } (L_\alpha[A], \in, A \cap L_\alpha[A]) \text{ with parameters } \}$.
- $L_\lambda[A] = \bigcup_{\alpha < \lambda} L_\alpha[A]$,
- $L[A] = \bigcup_{\alpha \in \text{Ord}} L_\alpha[A]$.

Remark

- $L[A]$ may not be canonical, it depends on $A$.
- The idea is to consider $L[\vec{E}]$ where $\vec{E}$ is a sequence of extenders and show that it is “canonical” and has large cardinals.
Premice and mice

Definition

- A premouse is a structure of the form $L_\alpha[\vec{E}]$ where $\vec{E}$ is a sequence of extenders.
Premice and mice

Definition

- A premouse is a structure of the form $L_\alpha[\vec{E}]$ where $\vec{E}$ is a sequence of extenders.
- A mouse is an iterable premouse.
Premice and mice

Definition

- A premouse is a structure of the form $L_\alpha[\vec{E}]$ where $\vec{E}$ is a sequence of extenders.
- A mouse is an iterable premouse.
- Iterability is a fancy way of saying that all the ways of taking ultrapowers and direct limits produce well-founded models. More precisely, look at the picture.
Summary and remarks.

The iteration game on a premouse $\mathcal{M} = L_\alpha[\vec{E}]$ is the game where two plays keep producing ultrapowers and direct limits.
Summary and remarks.

1. The iteration game on a premouse $\mathcal{M} = L_\alpha[\vec{E}]$ is the game where two plays keep producing ultrapowers and direct limits.

2. An iteration strategy for $\mathcal{M} = L_\alpha[\vec{E}]$ is a winning strategy for $II$ in the iteration game on $\mathcal{M}$.
The Solovay Hierarchy

Summary and remarks.

1. The iteration game on a premouse $\mathcal{M} = L_\alpha [\vec{E}]$ is the game where two plays keep producing ultrapowers and direct limits.

2. An iteration strategy for $\mathcal{M} = L_\alpha [\vec{E}]$ is a winning strategy for $II$ in the iteration game on $\mathcal{M}$.

3. In general, to have a good theory of mice, $\omega_1 + 1$-iterability is all that is needed.
Summary and remarks.

1. The iteration game on a premouse $\mathcal{M} = L_\alpha[\vec{E}]$ is the game where two players keep producing ultrapowers and direct limits.

2. An iteration strategy for $\mathcal{M} = L_\alpha[\vec{E}]$ is a winning strategy for II in the iteration game on $\mathcal{M}$.

3. In general, to have a good theory of mice, $\omega_1 + 1$-iterability is all that is needed.

4. Notice that it must be hard to construct such strategies as there are trees of height $\omega_1$ with no branch.
The inner model problem revisited.

Problem (The inner model problem)

Construct mice with large cardinals.
Mice are canonical: comparison

Definition

Given two mice $\mathcal{M}$ and $\mathcal{N}$, write $\mathcal{M} \preceq \mathcal{N}$ if $\mathcal{M} = L_\alpha[\vec{E}]$, $\mathcal{N} = L_\beta[\vec{F}]$, $\alpha \leq \beta$ and $\vec{E} = \vec{F} \upharpoonright \alpha$. 
Mice are canonical: comparison

Definition

1. Given two mice $\mathcal{M}$ and $\mathcal{N}$, write $\mathcal{M} \sqsubseteq \mathcal{N}$ if $\mathcal{M} = L_\alpha[\vec{E}]$, $\mathcal{N} = L_\beta[\vec{F}]$, $\alpha \leq \beta$ and $\vec{E} = \vec{F} \upharpoonright \alpha$.

2. Comparison is the statement: Given two mice $\mathcal{M}$ and $\mathcal{N}$ with iteration strategies $\Sigma$ and $\Lambda$, there are a $\Sigma$-iterate $\mathcal{P}$ of $\mathcal{M}$ and a $\Lambda$-iterate $\mathcal{Q}$ of $\mathcal{N}$ such that either $\mathcal{P} \sqsubseteq \mathcal{Q}$ or $\mathcal{Q} \sqsubseteq \mathcal{P}$. 
Mice are canonical: comparison

Definition

- Given two mice $\mathcal{M}$ and $\mathcal{N}$, write $\mathcal{M} \trianglelefteq \mathcal{N}$ if $\mathcal{M} = L_\alpha[\vec{E}]$, $\mathcal{N} = L_\beta[\vec{F}]$, $\alpha \leq \beta$ and $\vec{E} = \vec{F} \upharpoonright \alpha$.

- Comparison is the statement: Given two mice $\mathcal{M}$ and $\mathcal{N}$ with iteration strategies $\Sigma$ and $\Lambda$, there are a $\Sigma$-iterate $\mathcal{P}$ of $\mathcal{M}$ and a $\Lambda$-iterate $\mathcal{Q}$ of $\mathcal{N}$ such that either $\mathcal{P} \trianglelefteq \mathcal{Q}$ or $\mathcal{Q} \trianglelefteq \mathcal{P}$.

Theorem (Mitchell-Steel, early 90s)

Comparison holds.
Mice are canonical: comparison

Definition

- Given two mice $\mathcal{M}$ and $\mathcal{N}$, write $\mathcal{M} \preceq \mathcal{N}$ if $\mathcal{M} = L_\alpha[\vec{E}]$, $\mathcal{N} = L_\beta[\vec{F}]$, $\alpha \leq \beta$ and $\vec{E} = \vec{F} \upharpoonright \alpha$.

- Comparison is the statement: Given two mice $\mathcal{M}$ and $\mathcal{N}$ with iteration strategies $\Sigma$ and $\Lambda$, there are a $\Sigma$-iterate $P$ of $\mathcal{M}$ and a $\Lambda$-iterate $Q$ of $\mathcal{N}$ such that either $P \preceq Q$ or $Q \preceq P$.

Theorem (Mitchell-Steel, early 90s)

Comparison holds.
An important corollary

Given a premouse $\mathcal{M}$, let $\leq^\mathcal{M}$ be the constructibility order of $\mathcal{M}$.
An important corollary

Given a premouse $\mathcal{M}$, let $\leq^\mathcal{M}$ be the constructibility order of $\mathcal{M}$.

**Corollary**

*If $\mathcal{M}$ and $\mathcal{N}$ are two mice then $\mathbb{R}^2 \cap \leq^\mathcal{M}$ is compatible with $\mathbb{R}^2 \cap \leq^\mathcal{N}$.***

**Remark**

*Hence, mice can only have canonical reals.*
Remark

- To develop tools for establishing reversals it is enough to develop tools for solving the inner model problem.
Remark

- To develop tools for establishing reversals it is enough to develop tools for solving the inner model problem.
- There is a recent approach that goes through descriptive set theory.
Remark

- To develop tools for establishing reversals it is enough to develop tools for solving the inner model problem.
- There is a recent approach that goes through descriptive set theory.
- The classical approach, via $K^c$-constructions, reduces to constructing canonical iteration strategies, or $\omega_1 + 1$-iteration strategies whose $\omega_1$ part is universally Baire. This approach, too, seems to lead to descriptive set theory.
The “new” approach, the core model induction

The idea is to analyze the models of the Solovay hierarchy, which is a determinacy hierarchy, and show that they contain complicated iteration strategies for mice with large cardinals.
The “new” approach, the core model induction

- The idea is to analyze the models of the Solovay hierarchy, which is a determinacy hierarchy, and show that they contain complicated iteration strategies for mice with large cardinals.
- As far as establishing reversals goes, the Solovay hierarchy becomes an intermediary.
The “new” approach, the core model induction

- The idea is to analyze the models of the Solovay hierarchy, which is a determinacy hierarchy, and show that they contain complicated iteration strategies for mice with large cardinals.
- As far as establishing reversals goes, the Solovay hierarchy becomes an intermediary.
- Key Point: For this to be successful, it is necessary to show that the Solovay hierarchy, just like the large cardinal hierarchy, is a consistency strength hierarchy that covers all the levels of the large cardinal hierarchy.
The “new” approach, the core model induction

- The idea is to analyze the models of the Solovay hierarchy, which is a determinacy hierarchy, and show that they contain complicated iteration strategies for mice with large cardinals.
- As far as establishing reversals goes, the Solovay hierarchy becomes an intermediary.
- Key Point: For this to be successful, it is necessary to show that the Solovay hierarchy, just like the large cardinal hierarchy, is a consistency strength hierarchy that covers all the levels of the large cardinal hierarchy. This has not yet been established.
The main problem of descriptive inner model theory

Problem (The main problem)

*Find determinacy theories that catch up with the large cardinal hierarchy.*

Remark

*The Solovay hierarchy is one candidate.*
The Solovay sequence

Assume $AD$. First

\[ \Theta = \sup\{ \alpha : \exists f (f \text{ is a surjection of } \mathbb{R} \text{ onto } \alpha) \}. \]
The Solovay sequence

Assume  $AD$.  First

$$\Theta = \sup\{\alpha : \exists f (f \text{ is a surjection of } \mathbb{R} \text{ onto } \alpha)\}.$$  

The Solovay sequence is a closed sequence of ordinals \langle \theta_\alpha : \alpha \leq \Omega \rangle defined as follows:

\[ \theta_0 = \sup\{\beta : \text{there is an } OD \text{ surjection } f : \mathcal{P}(\omega) \to \beta\}, \]
The Solovay sequence

Assume $AD$. First

$$\Theta = \sup\{ \alpha : \exists f (f \text{ is a surjection of } \mathbb{R} \text{ onto } \alpha) \}.$$ 

The Solovay sequence is a closed sequence of ordinals $\langle \theta_\alpha : \alpha \leq \Omega \rangle$ defined as follows:

1. $\theta_0 = \sup\{ \beta : \text{there is an } OD \text{ surjection } f : \mathcal{P}(\omega) \rightarrow \beta \}$,

2. if $\theta_\alpha < \Theta$ then
   $$\theta_{\alpha+1} = \sup\{ \beta : \text{there is an } OD \text{ surjection } f : \mathcal{P}(\theta_\alpha) \rightarrow \beta \},$$
The Solovay sequence

Assume $AD$. First

$$\Theta = \sup \{ \alpha : \exists f (f \text{ is a surjection of } \mathbb{R} \text{ onto } \alpha) \}. $$

The Solovay sequence is a closed sequence of ordinals $\langle \theta_\alpha : \alpha \leq \Omega \rangle$ defined as follows:

1. $\theta_0 = \sup \{ \beta : \text{there is an } OD \text{ surjection } f : \mathcal{P}(\omega) \to \beta \}$,
2. if $\theta_\alpha < \Theta$ then
   $$\theta_{\alpha+1} = \sup \{ \beta : \text{there is an } OD \text{ surjection } f : \mathcal{P}(\theta_\alpha) \to \beta \},$$
3. $\theta_\lambda = \sup_{\alpha < \lambda} \theta_\alpha$. 
The Solovay sequence

Assume $AD$. First

$$\Theta = \sup\{\alpha : \exists f (f \text{ is a surjection of } \mathbb{R} \text{ onto } \alpha)\}.$$

The Solovay sequence is a closed sequence of ordinals $\langle \theta_\alpha : \alpha \leq \Omega \rangle$ defined as follows:

1. $\theta_0 = \sup\{\beta : \text{there is an OD surjection } f : \mathcal{P}(\omega) \to \beta\}$,
2. if $\theta_\alpha < \Theta$ then
   $$\theta_{\alpha+1} = \sup\{\beta : \text{there is an OD surjection } f : \mathcal{P}(\theta_\alpha) \to \beta\},$$
3. $\theta_\lambda = \sup_{\alpha < \lambda} \theta_\alpha$.
4. $\Theta = \theta_\Omega$. 
The Solovay hierarchy

$AD^+$ is an axiomatic system extending $AD$. 
The Solovay hierarchy

$AD^+$ is an axiomatic system extending $AD$. The axioms of the Solovay hierarchy are

$$AD^+ + \theta_0 = \Theta <_{con} AD^+ + \theta_1 = \Theta <_{con} \ldots <_{con} AD^+ + \theta_{\omega_1} = \Theta <_{con} \ldots$$
The Solovay hierarchy

$AD^+$ is an axiomatic system extending $AD$. The axioms of the Solovay hierarchy are

$$AD^+ + \theta_0 = \Theta <_{con} AD^+ + \theta_1 = \Theta <_{con} ... <_{con} AD^+ + \theta_{\omega_1} = \Theta <_{con} ...$$

Theorem (Martin, Woodin, 80s)

Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. Then $AD_{\mathbb{R}}$ implies that $\Theta = \theta_\Omega$ for some limit ordinal $\Omega$. 


Some important axioms from the hierarchy

HOD is the class of hereditarily ordinal definable sets. It satisfies \textit{ZFC}.

**Examples**

- $AD_R + \text{“$\Theta$ is regular”}$.
- $AD_R + \text{“$\Theta$ is Mahlo in HOD”}$.
- $AD_R + \text{“$\Theta$ is weakly compact in HOD”}$.
- $AD_R + \text{“$\Theta$ is measurable”}$.
- $AD_R + \text{“$\Theta$ is Mahlo”}$.
A set of reals is called $\kappa$-Suslin if there is a tree $T \subseteq \bigcup_{n<\omega} \omega^n \times \kappa^n$ such that

$$A = \{x \in \omega^\omega : \exists f \in \kappa^\omega ((x, f) \text{ is a branch of } T)\}.$$
A set of reals is called $\kappa$-Suslin if there is a tree $T \subseteq \bigcup_{n<\omega} \omega^n \times \kappa^n$ such that
\[ A = \{ x \in \omega^\omega : \exists f \in \kappa^\omega ((x, f) \text{ is a branch of } T) \}. \]

$\kappa$ is called a Suslin cardinal if there is $A \subseteq \mathbb{R}$ such that $A$ is $\kappa$-Suslin but not $\lambda$-Suslin for all $\lambda < \kappa$. 

More important axioms
A set of reals is called \( \kappa \)-Suslin if there is a tree \( T \subseteq \bigcup_{n<\omega} \omega^n \times \kappa^n \) such that
\[
A = \{ x \in \omega^\omega : \exists f \in \kappa^\omega ((x, f) \text{ is a branch of } T) \}.
\]
\( \kappa \) is called a Suslin cardinal if there is \( A \subseteq \mathbb{R} \) such that \( A \) is \( \kappa \)-Suslin but not \( \lambda \)-Suslin for all \( \lambda < \kappa \).

(LST) \( AD^+ + \Theta = \theta_{\alpha+1} + \text{“} \theta_\alpha \text{ is the largest Suslin cardinal”} \).
More important axioms

- A set of reals is called \( \kappa \)-Suslin if there is a tree
  \( T \subseteq \bigcup_{n<\omega} \omega^n \times \kappa^n \) such that
  \[
  A = \{ x \in \omega^\omega : \exists f \in \kappa^\omega ((x, f) \text{ is a branch of } T) \}.
  \]

- \( \kappa \) is called a Suslin cardinal if there is \( A \subseteq \mathbb{R} \) such that \( A \) is \( \kappa \)-Suslin but not \( \lambda \)-Suslin for all \( \lambda < \kappa \).

- (LST) \( AD^+ + \Theta = \theta_{\alpha+1} + \text{“} \theta_\alpha \text{ is the largest Suslin cardinal”} \).

- Let \( \phi \) be a large cardinal axiom. Then let
  \[
  S_\phi = \text{def } LST + V^{\text{HOD}} \models \exists \kappa \phi[\kappa].
  \]
The main conjecture

Conjecture (The Main Conjecture)

For each $\phi$, $S_\phi$ is consistent relative to some large cardinal.

Remark

- The conjecture should be viewed as an approach to the main problem.
The main conjecture

Conjecture (The Main Conjecture)

For each $\phi$, $S_\phi$ is consistent relative to some large cardinal.

Remark

- The conjecture should be viewed as an approach to the main problem.
- It might be argued that $S_\phi$ is a superficial way of making the Solovay hierarchy more powerful.
Conjecture (The Main Conjecture)

For each $\phi$, $S_\phi$ is consistent relative to some large cardinal.

Remark

- The conjecture should be viewed as an approach to the main problem.
- It might be argued that $S_\phi$ is a superficial way of making the Solovay hierarchy more powerful.
- (Woodin) Under AD, if $\theta_{\alpha+1}$ exists then it is Woodin in HOD.
The main conjecture

Conjecture (The Main Conjecture)
For each $\phi$, $S_\phi$ is consistent relative to some large cardinal.

Remark
- The conjecture should be viewed as an approach to the main problem.
- It might be argued that $S_\phi$ is a superficial way of making the Solovay hierarchy more powerful.
- (Woodin) Under AD, if $\theta_{\alpha+1}$ exists then it is Woodin in HOD.
- One arrives at these axioms by analyzing HOD: Under AD$^+$, HOD is a some kind of mouse, a hod mouse, a structure constructed from a sequence of extenders and strategies.
The main conjecture

Conjecture (The Main Conjecture)

For each $\phi$, $S_\phi$ is consistent relative to some large cardinal.

Remark

- The conjecture should be viewed as an approach to the main problem.
- It might be argued that $S_\phi$ is a superficial way of making the Solovay hierarchy more powerful.
- (Woodin) Under AD, if $\theta_{\alpha+1}$ exists then it is Woodin in $\text{HOD}$.
- One arrives at these axioms by analyzing HOD: Under AD$^+$, HOD is a some kind of mouse, a hod mouse, a structure constructed from a sequence of extenders and strategies. The analysis implies that we ought to consider such axioms.
The consistency of the axioms.

**Theorem (2008)**

*Suppose there is a Woodin cardinal which is a limit of Woodin cardinals. Then there is an inner model $M$ such that $\mathbb{R} \subseteq M$ and $M \models AD_{\mathbb{R}} + \text{"$\Theta$ is measurable"}$.***
The Solovay Hierarchy

The consistency of the axioms.

Theorem (2008)

Suppose there is a Woodin cardinal which is a limit of Woodin cardinals. Then there is an inner model $M$ such that $\mathbb{R} \subseteq M$ and $M \models AD_{\mathbb{R}} + \text{“}\Theta \text{ is measurable”}.$

Remark

- Many similar axioms from the Solovay hierarchy can be shown to be consistent relative to some large cardinal axiom. In particular, many approximations of LST have been shown to be consistent relative to large cardinals.
The consistency of the axioms.

Theorem (2008)

Suppose there is a Woodin cardinal which is a limit of Woodin cardinals. Then there is an inner model $M$ such that $\mathbb{R} \subseteq M$ and $M \models AD_\mathbb{R} + \text{"}\Theta \text{ is measurable"}$. 

Remark

- Many similar axioms from the Solovay hierarchy can be shown to be consistent relative to some large cardinal axiom. In particular, many approximations of LST have been shown to be consistent relative to large cardinals.
- However, LST itself is somewhat mysterious, perhaps for a good reason.
Examples of reversals using the Solovay hierarchy

Theorem (2010)

Assume PFA. Then there is an inner model $M$ such that $\mathbb{R} \subseteq M$ and $M \models AD_\mathbb{R} + \text{“$\Theta$ is regular”}$. 
Examples of reversals using the Solovay hierarchy

Theorem (2010)
Assume PFA. Then there is an inner model $M$ such that $\mathbb{R} \subseteq M$ and $M \models AD_{\mathbb{R}} + \text{"$\Theta$ is regular"}$. 

Theorem (Steel, 2008)
Assume $AD_{\mathbb{R}} + \text{"$\Theta$ is regular"}$. Then there is an inner model of $\text{ZFC} + \text{" there is a proper class of Woodin cardinals and strong cardinals"}$. 
Examples of reversals using the Solovay hierarchy

Theorem (2010)
Assume PFA. Then there is an inner model $M$ such that $\mathbb{R} \subseteq M$ and $M \models AD_{\mathbb{R}} + \text{“$\Theta$ is regular”}$. 

Theorem (Steel, 2008)
Assume $AD_{\mathbb{R}} + \text{“$\Theta$ is regular”}$. Then there is an inner model of $ZFC + \text{“there is a proper class of Woodin cardinals and strong cardinals”}$. 

Corollary
Assume PFA. Then there is an inner model of $ZFC + \text{“there is a proper class of Woodin cardinals and strong cardinals”}$. 
Evidence that descriptive set theoretic axioms do get stronger and stronger

Theorem (Woodin, 90s)
Assume $AD_{\mathbb{R}} + \text{"}\Theta \text{ is regular}."

Evidence that descriptive set theoretic axioms do get stronger and stronger

Theorem (Woodin, 90s)

Assume $AD_{\mathbb{R}} + \text{"\(\Theta\) is regular"}$.  

1. There is a partial ordering $\mathbb{P}$, such that $MM(c) \text{ holds in } V^\mathbb{P}$. 

Remark

The usual forcing methods require at least a supercompact cardinal to force either of the conclusions and both of these conclusions have a significant large cardinal strength and are probably equiconsistent with $AD_{\mathbb{R}} + \text{"\(\Theta\) is regular"}$. 

Evidence that descriptive set theoretic axioms do get stronger and stronger

Theorem (Woodin, 90s)
Assume $AD_{\mathbb{R}} + \text{“}\Theta \text{ is regular”}$.

1. There is a partial ordering $\mathbb{P}$, such that $MM(c)$ holds in $V^\mathbb{P}$.
2. There is a partial ordering $\mathbb{P}$ which forces $\text{CH} + \text{ there is an } \omega_1\text{-dense ideal on } \omega_1$.

Remark
The usual forcing methods require at least a supercompact cardinal to force either of the conclusions and both of these conclusions have a significant large cardinal strength and are probably equiconsistent with $AD_{\mathbb{R}} + \text{“}\Theta \text{ is regular”}$. 
Evidence that descriptive set theoretic axioms do get stronger and stronger

Theorem (Woodin, 90s)
Assume $AD_R + \ "\Theta \ is \ regular\"$.

1. There is a partial ordering $P$, such that $MM(c)$ holds in $V^P$.
2. There is a partial ordering $P$ which forces $CH +$ there is an $\omega_1$-dense ideal on $\omega_1$.

Remark
The usual forcing methods require at least a supercompact cardinal to force either of the conclusions and both of these conclusions have a significant large cardinal strength and are probably equiconsistent with $AD_R + \ "\Theta \ is \ regular\"$. 
Forcing failure of square

Theorem (Caicedo, Larson, S., Schindler, Steel, Zeman, 2011)

Assume $\text{AD}_R$. Suppose the set

$$\{\kappa < \Theta : \kappa \text{ is regular in HOD and } \text{cf} (\kappa) = \omega_1 \}$$

is stationary in $\Theta$. Then there is a partial ordering $\mathbb{P}$ such that

$$V^\mathbb{P} \models \text{MM}(c) + \neg \Box (\omega_2) + \neg \Box \omega_2.$$
Forcing failure of square

Theorem (Caicedo, Larson, S., Schindler, Steel, Zeman, 2011)

Assume $\text{AD}_{\mathbb{R}}$. Suppose the set

$$\{\kappa < \Theta : \kappa \text{ is regular in } \text{HOD} \text{ and } \text{cf}(\kappa) = \omega_1\}$$

is stationary in $\Theta$. Then there is a partial ordering $\mathbb{P}$ such that

$$V^\mathbb{P} \models \text{MM}(c) + \neg \square(\omega_2) + \neg \square_\omega \omega_2.$$

Remark

*To force just $\neg \square(\omega_2) + \neg \square_\omega \omega_2$ via conventional techniques one needs at least a subcompact cardinal which is much stronger than superstrong cardinals.*
An alternative way of solving the main open problem is the following.

Problem

Find a determinacy theory $T$ such that the following hold.

1. $T$ implies that there is a poset $P$ such that $P$ forces $\text{ZFC} + \neg \square(\omega_3) + \neg \square\omega_3$.

Remark

Letting $T$ be as above, there is a good evidence that it will have to be stronger than a superstrong cardinal.
An alternative way of solving the main open problem is the following.

Problem

*Find a determinacy theory $T$ such that the following hold.*

1. $T$ implies that there is a poset $\mathbb{P}$ such that $\mathbb{P}$ forces $\text{ZFC} + \neg \Box(\omega_3) + \neg \Box \omega_3$.
2. $T$ implies that there is a poset $\mathbb{P}$ such that $\mathbb{P}$ forces $\text{ZFC} + \forall \kappa \geq \omega_2 \neg \Box(\kappa)$.
An alternative way of solving the main open problem is the following.

**Problem**

*Find a determinacy theory $T$ such that the following hold.*

1. **$T$ implies that there is a poset $P$ such that $P$ forces**
   $\text{ZFC} + \neg \square(\omega_3) + \neg \square_{\omega_3}$. 

2. **$T$ implies that there is a poset $P$ such that $P$ forces**
   $\text{ZFC} + \forall \kappa \geq \omega_2 \neg \square(\kappa)$. 

3. **$T$ implies that there is a poset $P$ such that $P$ forces**
   $\text{ZFC} + \text{PFA}$. 

**Remark**

Letting $T$ be as above, there is a good evidence that it will have to be stronger than a superstrong cardinal.
An alternative way of solving the main open problem is the following.

**Problem**

*Find a determinacy theory $T$ such that the following hold.*

1. **$T$ implies that there is a poset $\mathbb{P}$ such that $\mathbb{P}$ forces**
   
   $\text{ZFC} + \neg \square(\omega_3) + \neg \square \omega_3$.

2. **$T$ implies that there is a poset $\mathbb{P}$ such that $\mathbb{P}$ forces**
   
   $\text{ZFC} + \forall \kappa \geq \omega_2 \neg \square(\kappa)$.

3. **$T$ implies that there is a poset $\mathbb{P}$ such that $\mathbb{P}$ forces**
   
   $\text{ZFC} + \text{PFA}$.

**Remark**

*Letting $T$ be as above, there is a good evidence that it will have to be stronger than a superstrong cardinal.*
While the future is uncertain, it is definitely going to be green.
While the future is uncertain, it is definitely going to be green.

HOD of models of determinacy has emerged as a key not-well understood object and understanding it will shed light on many mysteries.
While the future is uncertain, it is definitely going to be green.

HOD of models of determinacy has emerged as a key not-well understood object and understanding it will shed light on many mysteries.

The analysis of HOD might just as well lead to, via Woodin’s axiom, the theory of ultimate $L$ or rather, the ultimate foundation appropriate for studying all of mathematics without any bias towards a particular theory.
While the future is uncertain, it is definitely going to be green.

HOD of models of determinacy has emerged as a key not-well understood object and understanding it will shed light on many mysteries.

The analysis of HOD might just as well lead to, via Woodin’s axiom, the theory of ultimate $L$ or rather, the ultimate foundation appropriate for studying all of mathematics without any bias towards a particular theory.

For now, however, we can only say: to be continued.