Ramsey classes of finite trees and SOP\(_2\)

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Outline

1. basic notions

2. indiscernibility as a tool

3. application to trees
Classification theory seeks to isolate properties that act as good dividing lines between more-complicated and less-complicated theories.

Often such a property is described by the presence of a formula encoding certain information.

In our discussion of trees $T$, nodes $\eta, \nu \in T$ will be written $\eta \perp \nu$ to signify that they are incomparable with respect to the partial tree order.

Typically $T$ will be $\omega^{<\omega}, 2^{<\omega}$.

In general $a, x$ stand for finite tuples $\bar{a}, \bar{x}$ of parameters/variables.
Here is such a dividing-line property.

**Definition**

A theory $T$ has *tree property-1* ($TP_1$) if there is a model $M \models T$, a formula $\varphi(x; y)$ and parameters $a_\eta \in |M|$ such that:

1. $\{ \varphi(x; a_\sigma|_n) : \sigma \in \omega^\omega \}$ is consistent ("branches are consistent"), and
2. $\{ \varphi(x; a_\eta) \land \varphi(x; a_\nu) \}$ is inconsistent, for $\eta, \nu \in \omega^{<\omega}, \eta \perp \nu$ ("incomparable nodes are inconsistent")

- A theory with $TP_1$ is on the more-complicated side of the dividing line provided by the property, $TP_1$.
- Naming a set $\omega^{<\omega}$ implies facts about this set that can be expressed in a first-order way. Can we isolate the relevant parts of the “theory” of this set?
TP₁ and SOP₂

- Here we name a second property, SOP₂ which is equivalent to TP₁ for theories:

**Definition**

A theory $T$ has **strong order property-₂ (SOP₂)** if there is a model $M \models T$, a formula $\varphi(x; y)$ and parameters $a_\eta \in |M|$ such that:

1. $\{\varphi(x; a_\sigma | n) : \sigma \in 2^\omega\}$ is consistent (“branches are consistent”), and
2. $\{\varphi(x; a_\eta) \land \varphi(x; a_\nu)\}$ is inconsistent, for $\eta, \nu \in 2^{<\omega}, \eta \perp \nu$ (“incomparable nodes are inconsistent”)

- There are many relations we could suggest to be basic relations on our tree: $\sqsubseteq$ (partial order), $\land$ (meet function), $<_{\text{lex}}$ (linear order extending $\sqsubseteq$).

- We need only look at $\sqsubseteq$-embeddings to transfer SOP₂ to TP₁; to obtain *trees* with the right partition properties, we may be required to take on more of the language.
what structure on $2^{<\omega}$ is relevant to SOP$_2$?

- We might feel we had isolated the relevant part of the “theory” of $2^{<\omega}$ if somehow $M = (2^{<\omega}, \leq)$ and $\varphi(x; y) = (x \leq y)$ gave the most canonical example of SOP$_2$. (This is not so.)

- The *strict order property (sOP)* is another dividing-line property that is known to be strictly stronger than SOP$_2$.

- A theory $T$ has the strict order property if there is a formula $\varphi(x; y)$ and parameters in some $M \models T$, $(a_i : i < \omega)$ such that the following implication holds strictly:

  $$\varphi(x, a_i) \Rightarrow \varphi(x; a_{i+1})$$

- $x \leq y$ witnesses the sOP in $2^{<\omega}$, so this can’t be our best example of SOP$_2$. 

An early effort to better understand the witnesses \((a_\eta : \eta \in 2^{<\omega})\) to SOP\(_2\) in a theory was to find an assumption of indiscernibility we could make, without loss of generality.

This approach was first pursued in [DS04] for SOP\(_2\); the following notion of \(I\)-indexed indiscernible is from [She90]:

**Definition**

Fix structures \(I, M\). An \(I\)-indexed indiscernible is a set of parameters from \(M\), \((b_i : i \in I)\) such that for all \(n < \omega\) and \(i_1, \ldots, i_n; j_1, \ldots, j_n\) from \(I\):

\[
\text{qftp}(i_1, \ldots, i_n; I) = \text{qftp}(j_1, \ldots, j_n; I) \Rightarrow \text{tp}(b_{i_1}, \ldots, b_{i_n}) = \text{tp}(b_{j_1}, \ldots, b_{j_n})
\]

- We say “quantifier-free type” in order to get a stronger notion of homogeneity.
We would like to assume parameters “in a certain configuration” are indiscernible, without loss of generality.

Definition
Fix a structure $I$ and parameters $\mathbf{I} := (a_i : i \in I)$ from some structure $M$. Define the $EM$-type of $\mathbf{I}$ to be:

$$EMtp(\mathbf{I})(\{x_i : i \in I\}) := \{\psi(x_{i_1}, \ldots, x_{i_n}) : n < \omega, \psi(x_1, \ldots, x_n) \in \mathcal{L}(M),$$

for all $j_1, \ldots, j_n$ from $I$ such that $qftp(j) = qftp(i),$

$$\models \psi(a_{j_1}, \ldots, a_{j_n})\}$$

The $I = (\omega, <)$ case of the above is referred to as $EM(\mathbf{I})$ in [TZ11]. We are careful not to confuse our terminology with $EM(I, \Phi)$ ([Bal09, She90]), which is a term that denotes a certain type of model. Note that $EMtp(\mathbf{I})$ derives a kind of profile/pattern/template from an $I$-indexed set of parameters, whether or not this set is indiscernible.
We want some terminology for the next development. Fix a structure $I$ (with some intended language.)

The *age*, $\text{age}(I)$, of a structure $I$ is the class of all finitely-generated substructures of $I$, closed under isomorphism.

Let $(C_A)$ be the substructures of $C$ isomorphic to $A$ (the "$A$-substructures of $C$.")

Say that a class $\mathcal{K}$ of finite structures is a *Ramsey class* if for all $A, B \in \mathcal{K}$ there is a $C \in \mathcal{K}$ such that given any 2-coloring $c : (C_A) \to \{0, 1\}$ there is a $B' \subseteq C$, $B' \cong B$, such that $c : (B'_A) \to \{i_0\}$, for some choice of $i_0 \in \{0, 1\}$.

It is equivalent to state the property for $k$-colorings, where $k < \omega$ is arbitrary $\geq 2$. 
Consider the property: for any $I$-indexed parameters $I = (a_i : i \in I)$ from sufficiently-saturated $M$ we may find $I$-indexed indiscernible $J = (b_s : s \in I) \models EMtp(I)$.

We may call the latter the *modeling property* for $I$-indexed indiscernibles.

**Theorem ([Sco12])**

For $I$ a structure in a finite relational language, where one basic relation $<$ linearly orders $I$, $I$-indexed indiscernibles have the modeling property just in case $age(I)$ is a Ramsey class.
The following generalization helps us deal with the case of $I = (2^{<\omega}, \subseteq, \wedge, <_{\text{lex}})$.

**Theorem**

*For uniformly locally finite $I$ in a finite language, where one basic relation $<$ linearly orders $I$, $I$-indexed indiscernibles have the modeling property just in case $\text{age}(I)$ is a Ramsey class.*

- A similar argument to one in [Sco12] shows that the modeling property implies the Ramsey class property for $\text{age}(I)$.
- This argument requires that we isolate the quantifier-free types by way of formulas, and we can still do this.
This direction is a little harder because there isn’t as obvious a correspondence between realizations of a quantifier-free type and substructures of $I$.

For $\bar{i} \models \eta(v_1, \ldots, v_n)$, a complete quantifier-free type (consistent with $v_1 < \ldots < v_n$), and $A = \langle \bar{i} \rangle$ the substructure generated by $\bar{i}$, let $\text{cl}(\bar{i})(x_1, \ldots, x_N)$ be the isomorphism-type of $A$ in $<\!-$increasing enumeration.

Let $x_{i_1}, \ldots, x_{i_n}$ be the indices at which $\bar{i}$ occurs in the increasing enumeration of $A$. Every copy of $A$ determines a unique copy of $\bar{i}$, and every copy of $\bar{i}$ in a structure $B$ occurs within a copy of $A$ in $B$.

Homogeneity for copies of $A$ implies homogeneity for $\bar{j} \models \eta$, as we shall see from the nature of a type-coloring:
For a finite structure $B$ of size $m$, let $p_B(x_1, \ldots, x_m)$ be the complete quantifier-free type of $B$ listed in $\prec$-increasing order.

**Definition**

Let $I$ be any structure. By a *type-coloring of tuples from $I$* we mean a $\chi$-coloring ($\chi$ a cardinal) \[ c : I^{<\omega} \to \chi \] with the property that for length-$m$ $\bar{b}, \bar{b}' \in I$ such that $c(\bar{b}) = c(\bar{b}')$, for any $n \leq m$ \[ c(\langle b_{i_1}, \ldots, b_{i_n} \rangle) = c(\langle b'_{i_1}, \ldots, b'_{i_n} \rangle) \]

If we let $\Delta(x_1, \ldots, x_n)$ be a finite set of formulas from $M$, then an $I$-indexed set in $M$, $(a_i : i \in I)$ comes equipped with a (finite) type-coloring by way of $c(\langle i_1, \ldots, i_n \rangle) = \text{tp}_\Delta(a_{i_1}, \ldots, a_{i_n}; M)$. 
in sum

- Given an $I$-indexed set of parameters $I = (a_i : i \in I)$, we have a type-coloring of tuples from $I$.

- Here is the “type of our indiscernible”:
  \[
  \Gamma(x_i : i \in I) = \{ \psi(x_{i1}, \ldots, x_{im}) \rightarrow \psi(x_{j1}, \ldots, x_{jm}) : \\
  \psi(x_1, \ldots, x_m) \in \mathcal{L}(M); \text{ qftp}(\vec{i}) = \text{ qftp}(\vec{j}); \; \vec{i}, \vec{j} \text{ from } I \}
  \]

- To find our $I$-indexed indiscernible $\models \text{ EMtp}(I)$, it suffices to satisfy a finite portion of the “type of our indiscernible” in $(a_i : i \in I)$, a portion indexed by a finite set $I_0 \subseteq I$ and mentioning a finite set of $\mathcal{L}(M)$-formulas $\Delta$.

- This amounts to, for given structures $A, B = \langle I_0 \rangle$, finding a homogeneous $B' \cong B$ in $I$ for the type-coloring above, as it applies to $A$-substructures of $I$.

- In general we must perform an induction on the $A_1, \ldots, A_n$ that are generated by tuples from $I_0$. 

• It would be good to develop a technology for countable languages.
• The non-locally finite case does not seem practicable, because partition properties often fail when we are searching for an infinite substructure $B$.
• For example, $\mathbb{Q} \not\rightarrow (\mathbb{Q})_2^{a_1 < a_2}$.
• Similarly for the random graph $\mathcal{R}$: $\mathcal{R} \not\rightarrow (\mathcal{R})_2^{a_1 R a_2}$. 
Thanks for your attention!
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