Computable Models Spectra of Ehrenfeucht Theories

This study is devoted to the class of Ehrenfeucht theories, this class of theories has been well studied and has attracted considerable attention.

Definition 1. *A theory is an **Ehrenfeucht theory** if it has finitely (greater then one) many models.*

Vaught proved that no complete theory has exactly two models. On the other hand, for each $n \geqslant 3$ there exists a theory with exactly n models. For example, the theory of the model $< Q; \leqslant; c_0, c_1, \ldots >$, where $< Q; \leqslant >$ is the natural ordering of rationals and $c_0 < c_1 < \ldots$, has exactly three models. This example can be generalized to give theories with exactly $n \geqslant 4$ models. Inspired by

Theorem 2. [3, 4] *Let T be an uncountably categorical theory. Then T is decidable if and only if T has a decidable model if and only if all models of T have decidable presentations.*

Nerode posed the following question: If an Ehrenfeucht theory T is decidable then do all models of T have strong constructivizations? It turns out that models of decidable Ehrenfeucht theories are not as well behaved as decidable uncountably or decidable countably categorical theories. For instance, the following theorem is true.

Theorem 3. [8] *For each $n \geqslant 3$ there exists a decidable Ehrenfeucht theory T_0 that admits elimination of quantifiers, has exactly n models and exactly one model with a strong constructivization.*

In relation to this theorem we make the following comments. First of all we note that the prime model of any decidable Ehrenfeucht theory must have a strong constructivization. This follows from an effective version of the Omitting Types Theorem for decidable theories [7] which is not discussed in this paper. Hence the strongly constructive model of the theory T_0 in the theorem above is a prime model. Secondly, the reason that not all models of T_0 have strong constructivizations is that T_0 has a noncomputable type. Based on this Morley asked the following question that has become known as Morley's problem:

Question 4. *If all types of an Ehrenfeucht theory T are computable then do all models of T have strong constructivizations?*

This is an open problem which has been attempted by many with no success. Ash and Millar obtained several interesting results in the study of this question. One of the results is the following.

Definition 5. *An Ehrenfeucht theory T is **persistently Ehrenfeucht** if any complete extension of T with finitely many new constants is also an Ehrenfeucht theory.*

Theorem 6. [1] *If T is persistently Ehrenfeucht all of whose types are arithmetical then all models of T have arithmetic presentations.*

In relation to this theorem and Morley's problem, it is interesting to note that the following question, asked by Goncharov and Millar, is still open:

Question 7. *If T is an arithmetic Ehrenfeucht theory whose types are arithmetical, do then all models of T have arithmetic presentations?*

We now briefly discuss the problem of existence of constructive models of Ehrenfeucht theories. As for categorical theories, there has not been much research about finding constructive models for (undecidable) Ehrenfeucht theories. We note that the results in finding constructive models for undecidable Ehrenfeucht theories can be quite different from those about decidable Ehrenfeucht theories. We give an example. If all types of an Ehrenfeucht theory T are computable then T must have at least three strongly constructive models (a proof of this can, for example, be found in [5]). Therefore for any decidable Ehrenfeucht theory T with exactly three models, the saturated model of T has a strong constructivization if and only if all models of T have strong constructivizations. We also recall that the prime model of every decidable Ehrenfeucht theory has a strong constructivization. In contrast to this, in [6] the following theorem is proved:

Theorem 8. *There exists an Ehrenfeucht theory with exactly three models of which only the saturated one has a constructivization.*

We conclude with the following research proposal.

Problem 9. *Work towards characterizing strongly constructive or/and constructive models of Ehrenfeucht theories.*

It was a citation from [2]. And here there is one step on the path to the solution of 9.

Theorem 10. *There exists a theory T such that*

1. *T is Ehrenfeucht theory;*
2. *The prime model of T has computable presentation;*
3. *The saturated model of T has computable presentation;*
4. *There exists a model of T with no computable presentation.*

References

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