A DECOMPOSITION OF THE ROGERS SEMILATTICE OF A FAMILY OF D.C.E. SETS

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Abstract. Khutoretskii's Theorem states that the Rogers semilattice of any family of c.e. sets has either at most one or infinitely many elements. A lemma in the inductive step of the proof shows that no Rogers semilattice can be partitioned into a principal ideal and a principal filter.

We show that such a partitioning is possible for some family of d.c.e. sets. In fact, we construct a family of c.e. sets which, when viewed as a family of d.c.e. sets, has (up to equivalence) exactly two computable Friedberg numberings μ and ν, and μ reduces to any computable numbering not equivalent to ν.

The question of whether the full statement of Khutoretskii's Theorem fails for families of d.c.e. sets remains open.

1. Introduction and Main Theorem

The theory of numberings grew out of Gödel's idea to code countable families of objects by integers in such a way that the object could be effectively "recovered" and studied from its index, or Gödel number. The theory was further developed by Kleene, Kolmogorov, Uspenskii, Friedberg, Rogers, and then mainly by the Novosibirsk school of algebra and logic under the direction of Mal'cev and Ershov. The focus shifted soon to a more general study of how the set of integers could be used to describe families of "constructive" objects (formulas, sets, functions, etc.).

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A *numbering* (or *numeration*[1]) is simply a *surjective* map $\nu : \omega \to S$ from the set of integers onto a nonempty set S of objects one wants to study from a constructive point of view. Two numberings $\mu, \nu : \omega \to S$ can be compared by defining $\mu \le \nu$ (μ is *reducible to* ν) if there is a computable function f such that $\mu = \nu \circ f$, i.e., if given any μ-index of any element of S, we can effectively compute a ν-index for it. Two numberings μ and ν are *equivalent* (written $\mu \equiv \nu$) if $\mu \le \nu$ and $\nu \le \mu$. If we denote the family of all numberings of a set S by $\mathrm{Num}(S)$, then $\mathrm{Num}(S)/\equiv$ forms an upper semilattice under the partial ordering induced by $\le$, where the join of two numberings μ and ν is defined by $(\mu \oplus \nu)(2n) = \mu(n)$ and $(\mu \oplus \nu)(2n+1) = \nu(n)$.

In this paper, we will concentrate on numberings of families $S \subseteq \Sigma^i_n$ of sets in one of the following hierarchies: Kleene's arithmetical hierarchy Σ^0_n, Kleene's analytic hierarchy Σ^1_n, and Ershov's difference hierarchy Σ^{-1}_n. Recall here that a set $A \subseteq \omega$ is in Ershov's hierarchy class Σ^{-1}_n if A is *n-computably enumerable* (*n-c. e.*), i.e., if $A = \lim_s A_s$ for a uniformly computable sequence of sets A_s such that $A_0 = \emptyset$ and for each x, there are at most n many s such that $A_s(x) \ne A_{s+1}(x)$. Recall also that a set A is a *difference of computably enumerable sets* (*d.c.e.*) if A is 2-c.e., i.e., if A is of the form $A_0 - A_1$ for computably enumerable sets A_0 and A_1.

We will now call a numbering ν of a family $S \subseteq \Sigma^i_n$ *computable* if the relation "$x \in \nu(n)$" is Σ^i_n, or equivalently, if the sequence $\{\nu(n)\}_{n\in\omega}$ of sets is uniformly Σ^i_n. If we denote the family of all computable numberings of a set S by $\mathrm{Com}(S)$, then $\mathrm{Com}(S)/\equiv$ forms an ideal of $\mathrm{Num}(S)/\equiv$. The upper semilattice $\mathrm{Com}(S)/\equiv$ of all computable numberings $S \subseteq \Sigma^i_n$ modulo reducibility is usually called the *Rogers semilattice* of S and denoted by $\mathcal{R}^i_n(S)$.

(For more background on the theory of numberings, see Ershov's German 3-part monograph [Er73, Er75, Er77], Ershov's Russian textbook [Er77a], or Ershov's survey article [Er99], as well as the survey article by Badaev and Goncharov [BG00].)

The Rogers semilattices $\mathcal{R}^0_1(S)$ of families of computably enumerable sets have been studied extensively, especially by the Novosibirsk school of algebra and logic. One of the main results about these semilattices is

Theorem 1 (Khutoretskii's Theorem [Kh71]). *Let S be a family of computably enumerable sets.*

(1) If $\mu \not\le \nu$ are computable numberings of S then there is a computable numbering π of S with $\pi \not\le \nu$ and $\mu \not\le \pi \oplus \nu$.

[1] At the 1999 Mal'cev Meeting, Ershov suggested the use of the word "numbering" instead of "numeration". We choose to follow his suggestion here.

(2) *If the Rogers semilattice $\mathcal{R}^0_1(S)$ of S contains more than one element, then it is infinite.*

The proof of part (1) of Khutoretskii's Theorem is a finite-injury priority argument; part (2) follows from part (1) by induction.

A long-standing open question (see, e. g., Badaev and Goncharov [BG00]) is whether Khutoretskii's result also holds for other classes of Ershov's hierarchy. Our main result is that part (1) of Khutoretskii's Theorem fails for some family of d.c.e. sets:

Main Theorem. *There is a family $\mathcal{F}$ of d.c.e. sets, and there are computable numberings μ and ν of the family $\mathcal{F}$ such that $\mu \not\leq \nu$ and such that for any computable numbering π of $\mathcal{F}$, either $\mu \leq \pi$ or $\pi \leq \nu$. In addition, we can ensure the following:*

- *$\mathcal{F}$ is a family of c.e. sets and ν is a computable numbering of $\mathcal{F}$ as a family of c.e. sets;*
- *both μ and ν can be made Friedberg and thus minimal numberings; and so*
- *any computable numbering π of $\mathcal{F}$ satisfies $\pi \equiv \nu$ or $\mu \leq \pi$.*

The rest of this paper is devoted to the proof of our Main Theorem, which is an infinite-injury priority argument using a tree of strategies.

2. The Intuition for the Proof of the Main Theorem

We need to build two computable numberings μ and ν of a family $\mathcal{F}$ of d.c.e. sets. For an arbitrary numbering π of a family of d.c.e. sets, we denote by $\mathcal{F}_\pi$ the family of d.c.e. sets enumerated by π. We now need to meet, for all computable numberings π of a family of d.c.e. sets, and for all computable functions f, the following

Requirements:

$$\mathcal{G} : \mathcal{F} = \mathcal{F}_\mu = \mathcal{F}_\nu$$
$$\mathcal{R}_\pi : \mathcal{F}_\pi = \mathcal{F} \implies \exists u\,(\pi = \nu \circ u) \text{ or } \exists v\,(\mu = \pi \circ v)$$
$$\mathcal{S}_f : \mu \neq \nu \circ f$$

where u and v are computable functions built by us, i.e., by individual strategies on the tree of strategies T.

Strategy for $\mathcal{G}$: We meet this global requirement $\mathcal{G}$ by (implicitly) defining a $\mathbf{0}'$-computable permutation g with $\mathbf{0}'$-computable inverse function such that $\mu = \nu \circ g$, and ensuring that these reductions are indeed correct. We build g by one-to-one computable approximations. As long as we ensure that we redefine $g(m)$ at most finitely often for each m, and that for every n, eventually $n = g(m)$ for some m, g will be a correct $\mathbf{0}'$-computable permutation. Indeed, we build a Friedberg numbering μ and consider ν as a $\mathbf{0}'$-computable copy of μ.

More precisely, we define a computable partition $\{A_i\}_{i\in\omega}$ of ω into infinite computable subsets $A_i = \{a(i,0) < a(i,1) < a(i,2) < \dots\}$, trying to mark each μ-set and its ν-copy by a pair of "active markers" $a(i',2j)$ and $a(i',2j+1)$ and possibly several "locking markers" $a(i,j')$ such that

(1) for each i and each stage s, there is at most one j such that markers $a(i,2j)$ and $a(i,2j+1)$ are active at stage s;
(2) once a marker $a(i,k)$ is no longer active at a stage, it will never again be active, but it may become a locking marker, and it will then remain a locking marker until (if ever) the strategy is initialized;
(3) for each μ-set and for each stage s, there is at most one pair of active markers $a(i,2j)$ and $a(i,2j+1)$ such that, at stage s, that set is the only set in our family $\mathcal{F}$ containing the markers $a(i,2j)$ and $a(i,2j+1)$; in addition, there may be several locking markers $a(i',j')$ (where i and all i' are pairwise distinct) such that, at stage s, that set is the only set in $\mathcal{F}$ from which $a(i',j')$ has been extracted; and
(4) each μ-set (and hence its ν-copy) will eventually be "marked" by a permanent pair of active markers $a(i,2j)$ and $a(i,2j+1)$.

We now build this reduction g by computable approximations, by matching each μ-set containing a pair of active markers $a(i,2j)$ and $a(i,2j+1)$ with the ν-set containing the same pair of markers. (The locking markers will aid us in ensuring that our v-reductions work.) As long as we ensure that we redefine g and g^{-1} at most finitely often at each argument, g will be a correct $\mathbf{0}'$-computable permutation between μ and ν, ensuring the satisfaction of the $\mathcal{G}$-requirement. (In the full construction later on, we will actually use a separate set of markers for each $\mathcal{R}_\pi$-strategy since the markers will also be used to ensure that the family enumerated by π does not equal our family $\mathcal{F}$ in case we cannot satisfy the requirement $\mathcal{R}_\pi$ via a reduction u or v.)

Strategy for $\mathcal{R}_\pi$ in isolation: For this, we simply build the computable reduction u, defining $u(p)$ for larger and larger p. An $\mathcal{R}_\pi$-strategy will have three possible *outcomes*: finite (since $\mathcal{F}_\pi \neq \mathcal{F}_\nu$), u (if the $\mathcal{R}_\pi$-strategy is able to make u total and correct), and v (if the $\mathcal{R}_\pi$-strategy is forced to make u undefined infinitely often, and to make v total and correct, as explained later on).

More precisely, an $\mathcal{R}_\pi$-strategy in isolation simply, for each index p,

(1) waits for a pair of active markers $a(i, 2j)$ and $a(i, 2j + 1)$ to appear in $\pi(p)$ and
(2) sets $u(p) = n$ for some index n such that $\nu(n)$ contains $a(i, 2j)$ and $a(i, 2j + 1)$.
(3) If later an active marker $a(i, k)$ (for some $k \in \{2j, 2j + 1\}$) ever leaves $\pi(p)$, then "kill" the numbering π by enumerating $a(i, k)$ into all μ-sets and their ν-copies.

Note that this last action will force $\mathcal{F} \neq \mathcal{F}_\pi$ since $a(i, k)$ is not in $\pi(p)$ for this p but is in all μ- and ν-sets. (It is because of this possibility that each $\mathcal{R}$-strategy needs to use its own set of markers later on, since this action by one $\mathcal{R}$-strategy cannot be allowed to interfere with the markers of another $\mathcal{R}$-strategy.) Similarly, if the $\mathcal{R}_\pi$-strategy finds an index p such that $\pi(p)$ does not contain active markers $a(i, 2j)$ and $a(i, 2j + 1)$, then again clearly $\mathcal{F} \neq \mathcal{F}_\pi$.

Strategy for $\mathcal{S}_f$ in isolation: This is simply a finite-injury strategy:

(1) Pick a fresh index m.
(2) Wait until (if ever) $f(m)$ is defined, say $f(m) = n$.
(3) If $g(m) \neq n$ then $\mu(m)$ and $\nu(n)$ are marked by different pairs of active markers, so ensure $g(m) \neq n$ to remain true from now on.
(4) If $g(m) = n$ then $\mu(m)$ and $\nu(n)$ are both marked by the same pair of active markers $a(i, 2j)$ and $a(i, 2j + 1)$, say. In this case:
    (a) choose fresh markers $a(i', 2j')$ and $a(i', 2j' + 1)$ (for $i' \neq i$) and $a(i, 2j_0)$ and $a(i, 2j_0 + 1)$ (for $j_0 \neq j$), say,
    (b) enumerate $a(i', 2j')$ and $a(i', 2j' + 1)$ into $\mu(m)$ and extract $a(i, 2j + 1)$ from $\mu(m)$,
    (c) declare $a(i, 2j)$ and $a(i, 2j + 1)$ inactive,
    (d) enumerate $a(i, 2j_0)$ and $a(i, 2j_0 + 1)$ into $\nu(n)$,
    (e) call $a(i', 2j')$ and $a(i', 2j' + 1)$ active markers for $\mu(m)$, and
    (f) call $a(i, 2j_0)$ and $a(i, 2j_0 + 1)$ active markers for $\nu(n)$.

    (Some "maintenance" action will have to ensure that sets $\nu(n')$ (a copy of the new $\mu(m)$) and $\mu(m')$ (a copy of the new $\nu(n)$) are created, for fresh indices n' and m'. The active markers are also declared to be active in these copies. For the sake of the $\mathcal{G}$-requirement, redefine $g(m) = n'$ and $g(m') = n$.)

In either case, we will have ensured that $\mu(m) \neq \nu(f(m))$, satisfying the $\mathcal{S}_f$-requirement finitarily.

Even though the $\mathcal{S}_f$-strategies are finitary, their effect on the rest of the construction will be infinite since a lower-priority $\mathcal{R}_\pi$-strategy may encounter infinitely many indices p where $\pi(p)$ is one of the finitely many sets handled by a higher-priority $\mathcal{S}_f$-strategy.

Starting on the analysis of the interaction between various requirements, we now begin by considering

$\mathcal{S}_f$-strategies below one higher-priority $\mathcal{R}_\pi$-requirement: The problem with the $\mathcal{S}_f$-strategy described above is that possibly $n = u(p)$ for some p since $f(m)$ may become defined later than $u(p)$. Note that an $\mathcal{R}_\pi$-strategy marks $\mu(m)$ and $\nu(n)$ with active markers $a(i, 2j)$ and $a(i, 2j + 1)$. We must prevent active markers $a(i, 2j)$ and $a(i, 2j + 1)$ from being enumerated into any set different from $\mu(m)$ and $\nu(n)$. If one of the active markers $a(i, 2j)$ and $a(i, 2j + 1)$ leaves $\pi(p)$ at any stage, then we can "kill" the numbering π by declaring $a(i, 2j)$ and $a(i, 2j + 1)$ inactive and enumerating them into all μ-sets (and their copies). Thus, the active markers $a(i, 2j)$ and $a(i, 2j + 1)$ separate – and this is their main feature – the sets $\mu(m)$ and $\nu(n)$ from all the other sets of $\mathcal{F}$.

Hence, when an $\mathcal{S}_f$-strategy below the u-outcome of one higher-priority $\mathcal{R}_\pi$-strategy marks $\mu(m)$ by the absence of a marker $a(i, 2j+1)$ and replaces the markers by new active markers $a(i', 2j')$ and $a(i', 2j' + 1)$, then the set $\pi(p)$ has two possibilities: Either the old marker $a(i, 2j+1)$ remains in $\pi(p)$ (and so necessarily $a(i, 2j_0)$ and $a(i, 2j_0+1)$ both enter $\pi(p)$ since otherwise $\pi(p) \notin \mathcal{F}$), in which case u is still correct and there is no conflict; or $a(i', 2j')$ and $a(i', 2j'+1)$ enter $\pi(p)$ (and so necessarily $a(i, 2j+1)$ leaves $\pi(p)$ since otherwise again $\pi(p) \notin \mathcal{F}$), in which case u is now wrong on argument p. If we decided to keep u correct, we would have to abandon the satisfaction of the $\mathcal{S}_f$-requirement and make $\nu(n)$ equal to $\pi(p)$ and thus also $\mu(m)$, gaining nothing. Instead, we

(1) declare $a(i, 2j)$ inactive,
(2) call $a(i', 2j')$ and $a(i', 2j' + 1)$ active markers for both $\mu(m)$ and $\nu(g(m))$,
(3) call $a(i, 2j_0)$ and $a(i, 2j_0 + 1)$ active markers for both $\nu(n)$ and $\mu(g^{-1}(n))$,
(4) cancel u by making it totally undefined,
(5) start building the reduction v by setting $v(m) = p$ (since both $\mu(m)$ and $\pi(p)$, provided that $\mathcal{F}_\pi = \mathcal{F}$, are now (and will remain to be) the only set in our family $\mathcal{F}$ which contain $a(i, 2j)$ but not the old marker $a(i, 2j + 1)$),
(6) call the old marker $a(i, 2j + 1)$ the *locking marker* for $\mu(m)$ (note that $a(i, 2j + 1)$ has never left $\nu(g(m))$ since it has not yet been enumerated into $\nu(g(m))$!),

(7) add m to the stream S of witnesses to be used by $\mathcal{S}$-strategies assuming the v-outcome of the $\mathcal{R}$-strategy, and
(8) for each $m' < m$ for which $v(m')$ has not yet been defined, wait for an index p' such that $\mu(m')$ and $\pi(p')$ are marked by the same marker, and then set $v(m') = p'$.

Now, since $\pi(p)$ cannot contain the locking marker $a(i, 2j + 1)$ at any future stage, if $\mathcal{F} = \mathcal{F}_\pi$, and since the marker $a(i, 2j)$ is now inactive forever (and, therefore, cannot leave any set, and in particular not the set $\mu(m)$), necessarily $\pi(p) = \mu(m)$ no matter what further changes we make to $\mu(m)$, if we ensure that, for every $m' \neq m$, either $a(i, 2j) \notin \mu(m')$ or $a(i, 2j + 1) \in \mu(m')$. There are now two possibilities for the $\mathcal{R}_\pi$-strategy (still assuming that $\mathcal{F} = \mathcal{F}_\pi$): Either its reduction u is canceled at most finitely often, and so u succeeds after finitely much injury; or u is canceled infinitely often, v is made total since the action is performed for an increasing infinite sequence of indices m, each time by some $\mathcal{S}$-strategy below the u-outcome of the $\mathcal{R}_\pi$-strategy, and thus the reduction v succeeds.

The situation of an $\mathcal{S}_f$-strategy which assumes that a higher-priority $\mathcal{R}_\pi$-strategy succeeds via v is now very similar to the $\mathcal{S}_f$-strategy in isolation except that in this case, the $\mathcal{S}_f$-strategy can only use a μ-index m from the stream S of μ-indices below the v-outcome of the $\mathcal{R}_\pi$-strategy. The point is that this $\mathcal{S}_f$-strategy does not have to deal with the $\mathcal{R}_\pi$-strategy wanting to control $\nu(f(m))$ since u will always be completely undefined whenever this $\mathcal{S}_f$-strategy acts. For each such index m, both $\mu(m)$ and $\pi(v(m))$ are marked by the absence of some locking marker for $\mu(m)$.

Thus, we proceed as follows:

(1) Pick a fresh index m from the stream S of μ-indices below the v-outcome of the $\mathcal{R}_\pi$-strategy. Both $\mu(m)$ and $\pi(v(m))$ are marked by the absence of a locking marker $a(\hat{\imath}, \hat{\jmath})$, say.
(2) Wait until (if ever) $f(m)$ is defined, say $f(m) = n$.
(3) If $g(m) \neq n$ then $\mu(m)$ and $\nu(n)$ are marked by different pairs of active markers, so ensure $g(m) \neq n$ to remain true from now on.
(4) If $g(m) = n$ then $\mu(m)$ and $\nu(n)$ are both marked by the same pair of active markers $a(i, 2j)$ and $a(i, 2j+1)$, say. In this case:
    (a) choose fresh markers $a(i', 2j')$ and $a(i', 2j' + 1)$ (for $i' \neq i$) and $a(i, 2j_0)$ and $a(i, 2j_0 + 1)$ (for $j_0 \neq j$), say,
    (b) enumerate $a(i', 2j')$ and $a(i', 2j' + 1)$ into $\mu(m)$,
    (c) enumerate $a(i, 2j_0)$ and $a(i, 2j_0 + 1)$ into $\nu(n)$,
    (d) enumerate the locking marker $a(\hat{\imath}, \hat{\jmath})$ into $\nu(n)$,

(e) wait for the markers $a(i', 2j')$ and $a(i', 2j' + 1)$ to appear in $\pi(v(m))$, and after that
(f) declare the markers $a(i, 2j)$ and $a(i, 2j + 1)$ inactive,
(g) call $a(i', 2j')$ and $a(i', 2j' + 1)$ active markers for $\mu(m)$, and
(h) call $a(i, 2j_0)$ and $a(i, 2j_0 + 1)$ active markers for $\nu(n)$.

(5) Proceed to the "maintenance" action as described above in the $\mathcal{S}_f$-strategy in isolation.

Before starting the strategy, the old active markers $a(i, 2j)$ and $a(i, 2j + 1)$ are contained only in two equal sets of the family $\mathcal{F}$, namely, in $\mu(m)$ and $\nu(n)$, while the locking marker $a(\hat{\imath}, \hat{\jmath})$ has left both $\mu(m)$ and $\pi(v(m))$. During diagonalization by the $\mathcal{S}_f$-strategy, $\mu(m)$ and $\nu(n)$ became different, both contain still active markers $a(i, 2j)$ and $a(i, 2j + 1)$, but the locking marker $a(\hat{\imath}, \hat{\jmath})$ is now in $\nu(n)$. Therefore, $\pi(v(m))$ must equal $\mu(m)$, since otherwise $\mathcal{F}_\pi \neq \mathcal{F}$. Thus, the wait in (4e) will always end provided $\mathcal{F}_\pi = \mathcal{F}$.

After the strategy has finished, new active markers $a(i', 2j')$ and $a(i', 2j' + 1)$ are contained only in $\mu(m) = \nu(g(n))$, and we have that the locking marker $a(\hat{\imath}, \hat{\jmath})$ has left both $\mu(m)$ and $\pi(v(m))$. Therefore, we can use the μ-index m in diagonalization by another $\mathcal{S}_f$-strategy later on.

We next consider

$\mathcal{S}_f$-strategies below several higher-priority $\mathcal{R}_\pi$-requirements: This essentially just requires the usual nesting: Such an $\mathcal{S}$-strategy may only use numbers from the stream generated by the $\mathcal{R}$-strategy above it of lowest priority. In addition, since any $\mathcal{R}$-strategy may decide to "kill" the numbering π it works with by enumerating a marker into all μ- and ν-sets, we have to have each $\mathcal{R}$-strategy work with a disjoint set of different markers. In the end, each μ- and ν-set will be marked separately by each $\mathcal{R}$-strategy with infinite outcome u or v along the true path, and thus each μ- and ν-set will differ from any other μ- and ν-set at infinitely many numbers.

3. The Full Construction for the Main Theorem

We are now ready to describe the full construction. We assume the reader is familiar with priority arguments using a tree of strategies (see, e.g., Soare [So87, Ch. XIV] or Lempp [LeLN] for background).

The tree of strategies, the streams of μ-indices, the sets of α-markers, the links, and the locking α-markers: The construction will take place on a *tree of strategies* $T \subset \Lambda^{<\omega}$ where

$$\Lambda = \{v <_\Lambda u <_\Lambda \mathit{fin}\}$$

is the (ordered) *set of outcomes* of our strategies. (An $\mathcal{R}$-strategy can have all three outcomes, an $\mathcal{S}$-strategy can only have outcome *fin*.) We effectively order all $\mathcal{R}_\pi$- and $\mathcal{S}_f$-requirements of order type ω and assign the eth requirement in this list to all nodes $\xi \in T$ of length e, where each $\mathcal{R}$-strategy $\alpha \in T$ has as its immediate successors all of $\alpha^\frown\langle v\rangle$, $\alpha^\frown\langle u\rangle$, and $\alpha^\frown\langle fin\rangle$, whereas each $\mathcal{S}$-strategy $\beta \in T$ has as its only immediate successor $\beta^\frown\langle fin\rangle$. To make the description of the construction easier, we will assume the following

Convention 2. Using Kleene's Fixed-Point Theorem (cf. Soare [So87, p. 36]), we will assume that the highest-priority requirement in our list is an $\mathcal{R}_\pi$-requirement with a numbering π with $\mathcal{F}_\pi = \mathcal{F}_\nu$; in fact, we can and will assume that there is a fixed number s_0 such that any change at $\nu(n)(x)$ results in a corresponding change at $\pi(n)(x)$ exactly s_0 many stages later. (Thus we will later have that the node $\langle u\rangle$ is an initial segment of the true path of the construction. In fact, we may assume that only strategies comparable with $\langle u\rangle$ will ever be eligible to act.)

Each strategy $\xi \in T$ will work with a *stream* S_ξ of μ-indices, defined by induction on $|\xi|$. If ξ is an $\mathcal{S}$-strategy, then ξ will be restricted to choosing diagonalization witnesses m from its stream. (The stream will have no influence on the action of an $\mathcal{R}$-strategy, except to allow the recursive definition of S_ξ to continue.) At any stage s, the stream $S_{\langle\rangle}$ of the root $\langle\rangle$ of the tree will be the set $(0, s]$. The stream $S_{\xi^\frown\langle fin\rangle}$ will always be the set $S_\xi \cap (s_0, s]$ where s_0 is the least stage $\leq s$ at which $\xi^\frown\langle fin\rangle$ was eligible to act since its most recent initialization. Similarly, for an $\mathcal{R}$-strategy ξ, the stream $S_{\xi^\frown\langle u\rangle}$ will be the set $S_\xi \cap (s_0, s]$ where s_0 is the least stage $\leq s$ at which $\xi^\frown\langle u\rangle$ was eligible to act since its most recent initialization. On the other hand, the stream $S_{\xi^\frown\langle v\rangle}$ for an $\mathcal{R}$-strategy ξ is more complicated and will be defined dynamically during the construction. The only streams which are intervals will turn out to be those associated with strategies $\xi \in T$ such that there is no $\mathcal{R}$-strategy α with $\alpha^\frown\langle v\rangle \subseteq \xi$. Note, however, that, for every $\mathcal{R}$-strategy α, only some elements of a stream S_α will be enumerated into the stream $S_{\alpha^\frown\langle v\rangle}$. Note that $\alpha \subseteq \beta$ always implies $S_\alpha \supseteq S_\beta$.

Each $\mathcal{R}_\pi$-strategy $\alpha \in T$ is associated with infinitely many sets of "α-markers"

$$\begin{aligned} A^{\alpha,s} &= \{a(\alpha, s, i, j) \mid i \in \omega, j < 2\} \\ &= \{a(\alpha, s, 0, 0) < a(\alpha, s, 0, 1) < a(\alpha, s, 1, 0) < a(\alpha, s, 1, 1) < \dots\} \end{aligned}$$

such that these sets computably partition ω:

$$\omega = \bigsqcup_{\substack{\alpha\ \mathcal{R}\text{-strategy} \\ s\in\omega}} A^{\alpha,s}.$$

The intuition here is that α will use pairs $a(\alpha,s,i,0)$, $a(\alpha,s,i,1)$ of "α-markers" from the sets $A^{\alpha,s}$ (for various $i\in\omega$, where s is the first stage at which α was eligible to act since its most recent initialization) in order to mark μ-sets and their ν-copies and so that they can be matched up with π-sets. Note that each α has to use a separate set of such α-markers since we cannot afford for a higher-priority $\mathcal{R}$-strategy to lose control over its α-markers when a lower-priority $\mathcal{R}_\pi$-strategy "kills" its numbering π by enumerating one of its α-markers into *all* sets of our family $\mathcal{F}$ after this α-marker has left a π-set, thus ensuring that $\mathcal{F}\neq\mathcal{F}_\pi$.

Recall that g is the $\mathbf{0}'$-computable function used to satisfy the global $\mathcal{G}$-requirement. When an $\mathcal{S}_f$-strategy β works with μ-index m and finds $f(m)=g(m)$, we may have to destroy the reduction u of some numbering π to the numbering ν which has already been created by an $\mathcal{R}_\pi$-strategy α. We will use the set $P\subseteq\mathrm{dom}(u)$ of the π-indices p with $u(p)=g(m)$ and create a *u-link* from the $\mathcal{S}_f$-strategy β to the $\mathcal{R}_\pi$-strategy α with *parameter* m and this set of π-indices P. The collection of all u-links at stage s may be considered as a partial function $\mathrm{ulink}(\beta,\alpha)$ with values of type $\langle m,P\rangle$. We will use the notation $\mathrm{ulink}(\beta,\alpha)_i$ for ith coordinate of the pair $\langle m,P\rangle$ (where $i=1,2$).

In our construction, we give preference to the reduction v built by any $\mathcal{R}_\pi$-strategy α over its reduction u. We also try to keep v correct at any argument m once $v(m)$ has been defined at some stage of the construction. Here, we can encounter conflicts with a higher-priority $\mathcal{R}_\pi$-strategy α (which wants to ensure the correctness of $\mu=\pi\circ v$ via its function v) and $\mathcal{S}_f$-strategies trying to diagonalize $\mu=\nu\circ f$ for computable functions f. We will resolve this conflict by using a *locking α-marker*. If $v(m)$ is defined at a stage s, then the locking α-marker for $\mu(m)$ at this stage is an α-marker $a(\alpha,s,i,1)$ which has been enumerated into, and has later left, both $\mu(m)$ and $\pi(v(m))$ before stage s. We will use the locking α-marker $a(\alpha,s,i,1)$ to ensure the inequality $\mu(m)\neq\nu(f(m))$ as well as the equality $\mu(m)=\pi(v(m))$ (provided that $\mathcal{F}_\pi=\mathcal{F}$). We will create a *$v$-link* from the $\mathcal{S}_f$-strategy β to the $\mathcal{R}_\pi$-strategy α to distinguish the μ-index m (as the link's only parameter) after f is diagonalized at m with $v(m)$ defined. All v-links at stage s may be considered as a partial function $\mathrm{vlink}(\beta,\alpha)$ with values among μ-indices. We will refer to the value $\text{vlink}(\beta, \alpha)$ as the *parameter* of the v-link from the strategy β to the strategy α.

The construction stage by stage: At stage 0, all α-markers are inactive, all μ- and ν-sets as well as all streams and auxiliary sets are empty, all functions are undefined, and there is no link. But for every $\mathcal{R}_\pi$-strategy α, we let the parameter q_α be equal to 0 at stage 0. This parameter q_α denotes, at all stages of our construction, the least number q for which $u_\alpha(q)$ is currently undefined.

We will inductively show that the following *properties of the construction* hold at the end of each stage s:

(1) For every m, $g(m)$ is defined at stage s if and only if $\mu(m)$ contains at least one active α-marker for some α.
(2) The domain of the function g coincides with its range and is an initial segment of ω of length not less than s. g is a permutation on this initial segment.
(3) For every $m \in \text{dom}(g)$, the sets $\mu(m)$ and $\nu(g(m))$ are identical at stage s; in particular, their active α-markers are the same for all α.
(4) For every $\mathcal{R}$-strategy α and each set $\mu(m)$, the number of active α-markers for $\mu(m)$ is either 0 or 2. In the latter case, these α-markers are of the form $a(\alpha, s', i, 0)$ and $a(\alpha, s', i, 1)$ for some i and s'. At stage s, $\mu(m)$ may contain either one or both active α-markers for $\mu(m)$, if any.
(5) For every $\mathcal{R}$-strategy α and every μ-index m, if $\mu(m)$ contains exactly one active α-marker, then this α-marker is of the form $a(\alpha, s', i, 0)$.
(6) An active α-marker $a(\alpha, s', i, 0)$ can be contained in either one or two μ-sets. In contrast, an active α-marker $a(\alpha, s', i, 1)$ can only be contained in a unique μ-set.
(7) An active α-marker $a(\alpha, s', i, 0)$ is contained in the sets $\mu(m)$ and $\mu(m')$ with $m < m'$ at stage s if and only if, for some $\mathcal{S}$-strategy β, either $\text{ulink}(\beta, \alpha)_1 = m$ or $\text{vlink}(\beta, \alpha) = m$. In that latter case, the active α-marker $a(alpha, s', i, 1)$ is contained only in the set $\mu(m')$.
(8) For all $\mathcal{S}$-strategies β_1 and β_2 and every $\mathcal{R}_\pi$-strategy α, if $\beta_1 \neq \beta_2$ then

$$\text{ulink}(\beta_1, \alpha)_1 \neq \text{ulink}(\beta_2, \alpha)_1,$$
$$\text{ulink}(\beta_1, \alpha)_2 \cap \text{ulink}(\beta_2, \alpha)_2 = \emptyset,$$
$$\text{ulink}(\beta_1, \alpha)_1 \neq \text{vlink}(\beta_2, \alpha).$$

(9) For all strategies α_1 and α_2, if $\alpha_1 \subseteq \alpha_2$ then the stream S_{α_2} is a subset of the stream S_{α_1}.
(10) For every m and every $\mathcal{R}$-strategy α, if $m \in S_{\alpha\frown\langle v\rangle}$ then $v_\alpha(m)$ is defined and there exists a locking α-marker for $\mu(m)$ which has already left $\mu(m)$.

Each stage $s > 0$ consists of substages $t \leq s$ (where stage s may end before reaching substage s). All parameters will remain defined the same way as at the previous stage unless explicitly redefined. At substage t of stage s, a strategy $\xi \in T$ of length t (determined at substage $t-1$ if $t > 0$) will be *eligible to act* and proceed as described below. (If ξ has already *stopped* at a previous stage and not been *initialized* since then, then ξ immediately *ends the substage*, taking outcome *fin*.) If $t < s$ and ξ has a current outcome o, then the strategy $\xi\frown\langle o\rangle$ will be eligible to act at substage $t+1$; otherwise, we *end the stage* as described at the end of the construction.

The full $\mathcal{R}_\pi$-strategy: An $\mathcal{R}_\pi$-strategy α builds a computable reduction $u = u_\alpha$ and a computable reduction $v = v_\alpha$ (both local to α). If α can ensure that $\mathcal{F}_\pi \neq \mathcal{F}$, then α can make both u and v finite and have *fin* as its true outcome. Otherwise, α will either make u total (and eventually not make u completely undefined any more), under the true outcome u; or α will make u completely undefined infinitely often and make v total, under the true outcome v.

The $\mathcal{R}_\pi$-strategy α now proceeds as follows (where s' is the least stage $\leq s$ at which α has been eligible to act since α's most recent initialization):

(1) If there exist a μ-index m, a π-index p, and a number i such that $a(\alpha, s', i, 0)$ and $a(\alpha, s', i, 1)$ are active α-markers for $\mu(m)$, one of them, denoted by $a(\alpha, s', i, j)$, has left the set $\pi(p)$, and at least one of the following mutually exclusive clauses holds:

(a)
- $j = 0$ or $j = 1$,
- for all β, $\mathrm{ulink}(\beta, \alpha)_1 \neq m$ and $\mathrm{vlink}(\beta, \alpha) \neq m$, and
- $u(p) = g(m)$ or $v(m) = p$,

or

(b)
- $j = 0$,
- $\mathrm{ulink}(\beta, \alpha)_1 = m$ for some β, and
- $u(p) = g(m)$,

or

(c)
- $j = 0$,
- $\mathrm{vlink}(\beta, \alpha) = m$ for some β, and
- $v(m) = p$,

then

- pick the least such m and the corresponding active α-marker $a(\alpha, s', i, j)$ with the least j,
- declare the α-markers $a(\alpha, s', i, 0)$ and $a(\alpha, s', i, 1)$ to be no longer active for either $\mu(m)$ or $\nu(g(m))$,
- let both the functions u and v be completely undefined,
- enumerate $a(\alpha, s', i, j)$ into all μ- and ν-sets (including those μ- and ν-sets which do not yet have active α-markers for any α),
- *stop*, and *end the stage*. (The current outcome of the strategy α is not defined since we *end* stage s. At all future stages, we will immediately let the strategy α have current outcome *fin* since we have now ensured $\mathcal{F} \neq \mathcal{F}_\pi$.)

Otherwise, proceed to Step 2.

(2) If there exist a μ-index m, an $\mathcal{R}$-strategy $\alpha' \supseteq \alpha^\frown\langle v\rangle$, and a number i such that $a(\alpha, s', i, 1)$ is the locking marker for the set $\pi(v(m))$, and $a(\alpha, s', i, 1)$ are active α-markers for $\mu(m)$, one of them, denoted by $a(\alpha, s', i, j)$, has left the set $\pi(p)$, and at least one of the following mutually exclusive clauses holds:

(3) Check whether, for all $\mathcal{S}$-strategies β, neither $\mathrm{ulink}(\beta, \alpha)$ nor $\mathrm{vlink}(\beta, \alpha)$ is defined. If so, then proceed to Step 5; otherwise pick the least μ-index m which is the parameter of some u- or v-link. By Property 8, this parameter must be associated with a unique link. If $m = \mathrm{vlink}(\beta, \alpha)$ for some β, then proceed to Step 4. If $m = \mathrm{ulink}(\beta, \alpha)_1$ for some β, then proceed to Step 6.

(4) Then $\mathrm{vlink}(\beta, \alpha)$ was set equal to m by some $\mathcal{S}_f$-strategy β at a stage s_0. By Property 7, $\mu(m)$ contains only the active α-marker $a(\alpha, s', i, 0)$ but not the α-marker $a(\alpha, s', i, 1)$. By Properties 6 and 7, there exists a unique set $\mu(m')$ with $m' > m$ which also contains the same active α-marker $a(\alpha, s', i, 0)$. Furthermore, by the choice of m as a witness for the strategy β and Properties 9 and 10, $\mu(m)$ has a locking α-marker $a(\alpha, s', \hat{\imath}, 1)$, say, and $v(m)$ is defined. Let $a(\alpha, s', i'', 0)$ and $a(\alpha, s', i'', 1)$ be the fresh α-markers which have been enumerated into $\mu(m)$ but are not yet active. Let $a(\alpha, s', i', 0)$ and $a(\alpha, s', i', 1)$ be the fresh α-markers which have been enumerated into $\mu(m')$ but are not yet active.

Wait for the α-markers $a(\alpha, s', i'', 0)$ and $a(\alpha, s', i'', 1)$ to be enumerated into $\pi(v(m))$. (Note that if $\mathcal{F}_\pi = \mathcal{F}$, then this wait will eventually be over since, by Properties 5–7, $\pi(v(m))$ must equal $\mu(m)$ as $\mu(m')$ contains the locking α-marker $a(\alpha, s', \hat{\imath}, 1)$ while $a(\alpha, s', \hat{\imath}, 1)$ has already left $\pi(v(m))$.)

Strategy α always ends the substage with current outcome *fin* while waiting for this. Once the wait is over, we

(a) declare the α-markers $a(\alpha, s', i, 0)$ and $a(\alpha, s', i, 1)$ to be no longer active α-markers,
(b) call $a(\alpha, s', i'', 0)$ and $a(\alpha, s', i'', 1)$ the active α-markers for $\mu(m)$,
(c) call $a(\alpha, s', i', 0)$ and $a(\alpha, s', i', 1)$ the active α-markers for $\mu(m')$,
(d) *cancel the v-link* from β to α by letting $\text{vlink}(\beta, \alpha)$ be undefined, and
(e) proceed to Step 3.

(5) Search for a pair of active α-markers $a(\alpha, s', \tilde{\imath}, 0)$ and $a(\alpha, s', \tilde{\imath}, 1)$ contained in $\pi(q)$ for the current value of $q = q_\alpha$. If there is no such pair of α-markers, then *end the substage* with current outcome *fin*. Otherwise,
- fix an index $\tilde{m}$ such that $\mu(\tilde{m})$ contains both $a(\alpha, s', \tilde{\imath}, 0)$ and $a(\alpha, s', \tilde{\imath}, 1)$,
- set $u(q) = g(\tilde{m})$,
- increment the parameter q by $+1$, and
- *end the substage* with current outcome u.

(6) Let $\text{ulink}(\beta, \alpha)_2 = P$. (Intuitively, $\text{ulink}(\beta, \alpha) = \langle m, P\rangle$ means that β wants α to check, for each $p \in P$, whether the set $\pi(p)$ will behave like $\mu(m)$ or like $\nu(u(p))$.) Let $a(\alpha, s', i, 0)$ and $a(\alpha, s', i, 1)$ be the active α-markers for $\mu(m)$. (By Property 7, only one of these two α-markers, namely, $a(\alpha, s', i, 0)$, is contained in $\mu(m)$.)

For each π-index $p \in P$, check whether
(a) both $a(\alpha, s', i', 0)$ and $a(\alpha, s', i', 1)$ have entered $\pi(p)$ (here $a(\alpha, s', i', 0)$ and $a(\alpha, s', i', 1)$ are the fresh α-markers which have been enumerated into $\nu(u(p))$ but are not yet active), or
(b) both $a(\alpha, s', i'', 0)$ and $a(\alpha, s', i'', 1)$ have entered $\pi(p)$, and $a(\alpha, s', i, 1)$ has left $\pi(p)$ (here $a(\alpha, s', i'', 0)$ and $a(\alpha, s', i'', 1)$ are the fresh α-markers which have been enumerated into $\mu(m)$ but are not yet active).

(Intuitively, if neither 6a nor 6b applies, then the families $\mathcal{F}_\pi$ and $\mathcal{F}$ appear to be different. If 6a applies, then the set $\pi(p)$ behaves like $\nu(u(p))$ and u is still correct. If 6b applies, then the set $\pi(p)$ behaves like the set $\mu(m)$, which we just ensured to be different from $\nu(u(p))$, so u is incorrect and we need to extend the definition of v.)

If for some $p \in P$, neither clause 6a nor clause 6b applies, then *end the substage* with current outcome *fin*. Otherwise (when, for each $p \in P$, either 6a or 6b applies), *cancel the u-link* from β to α by performing the following actions:

(i) set $\operatorname{ulink}(\beta, \alpha)$ to be undefined,

(ii) fix the μ-index $m' > m$ such that $a(\alpha, s', i, 0)$ is also contained in $\mu(m')$ (this μ-index m' is unique by Property 7),

(iii) call $a(\alpha, s', i', 0)$ and $a(\alpha, s', i', 1)$ the new *active α-markers* for both $\mu(m')$ and $\nu(g(m'))$ (these α-markers $a(\alpha, s', i', 0)$ and $a(\alpha, s', i', 1)$ have been enumerated into both $\mu(m')$ and $\nu(g(m'))$ but have been inactive so far),

(iv) call $a(\alpha, s', i'', 0)$, $a(\alpha, s', i'', 1)$ the new *active α-markers* for both $\mu(m)$ and $\nu(g(m))$ (these α-markers $a(\alpha, s', i'', 0)$ and $a(\alpha, s', i'', 1)$ have been enumerated into both $\mu(m)$ and $\nu(g(m))$ but have been inactive so far),

(v) call the old α-markers $a(\alpha, s', i, 0)$, $a(\alpha, s', i, 1)$ *inactive*, and

(vi) proceed to Step 7.

(7) Check whether u is still defined correctly, i.e., whether clause 6a holds for all $p \in P$. If so, then return to Step 3. Otherwise,
- *cancel the u-links* to α from all $\mathcal{S}$-strategies β' as described above in Step 6i–6v,
- make the reduction u completely undefined,
- set $q = 0$, and
- proceed to Step 8.

(8) Fix the least $p \in P$ such that clause 6b applies and
- for all $\widehat{m} < m$ for which $v(\widehat{m})$ is not yet defined and $\widehat{m}$ is free of any link except possibly for links to some α' such that $\alpha'^\frown\langle u\rangle <_L \alpha$, in increasing order of $\widehat{m}$,
  (a) wait until some pair of active α-markers $a(\alpha, s', \hat{\imath}, 0)$ and $a(\alpha, s', \hat{\imath}, 1)$ for the set $\mu(\widehat{m})$ appear in a set $\pi(\hat{p})$ for some $\hat{p}$ (the strategy α will always end the substage with current outcome *fin* while waiting for this, each time restarting at Substep 8a at the next stage at which it is eligible to act, until the wait is over), and
  (b) set $v(\widehat{m}) = \hat{p}$,
- once $v(\widehat{m})$ has been defined for all $\widehat{m} < m$, set $v(m) = p$,
- enumerate m into the stream $S_{\alpha^\frown\langle v\rangle}$ if $m > |\alpha|$,
- call the α-marker $a(\alpha, s', i, 1)$ (which has left $\pi(p)$ and was extracted from $\mu(m)$) the *locking α-marker for $\mu(m)$*, and
- *end the substage* with current outcome v.

In addition to the action described above, unless the $\mathcal{R}_\pi$-strategy has already stopped, it will also perform the following *background actions* at the end of substage $|\alpha|$ of stage s:

Let l be the least μ-index such that no α'-marker has ever been active for $\mu(l)$ and for any α'. Set $g(l) = l$. For each μ-index $m \leq l$ such that $\mu(m)$ does not contain any active α-marker $a(\alpha, s', j, k)$ for s' as above and any $j \in \omega$ and $k \leq 1$, in increasing order:

(1) fix the least number e such that the α-markers $a(\alpha, s', e, 0)$ and $a(\alpha, s', e, 1)$ have not been used before;
(2) enumerate $a(\alpha, s', e, 0)$ and $a(\alpha, s', e, 1)$ into $\mu(m)$ and $\nu(g(m))$; and
(3) call $a(\alpha, s', e, 0)$ and $a(\alpha, s', e, 1)$ the *active α-markers* for both $\mu(m)$ and $\nu(g(m))$.

This ends the description of the action of an $\mathcal{R}$-strategy.

The full $\mathcal{S}_f$-strategy: An $\mathcal{S}_f$-strategy β tries to diagonalize μ against $\nu \circ f$ at one of the μ-indices chosen from an auxiliary set D_β of witnesses. If at some stage we have $\mu \neq \nu \circ f$, then there is nothing to diagonalize, in this case we set $D_\beta = \emptyset$.

By Convention 2 and Step 5 of the $\mathcal{R}$-strategy, we are guaranteed that there exists at least one $\mathcal{R}$-strategy α with $\alpha\widehat{\ }\langle u\rangle \subseteq \beta$.

We first let $\alpha_0 \subset \ldots \subset \alpha_l \subset \beta$ be all the $\mathcal{R}_{\pi_k}$-strategies α_k with $\alpha_k \subset \beta$. For every $k \leq l$, let s_k be the least stage at which α_k has been eligible to act since α_k's most recent initialization. Let u_k and v_k be the functions built by the $\mathcal{R}_{\pi_k}$-strategy α_k since stage s_k.

The $\mathcal{S}_f$-strategy β now proceeds as follows (where s' is the least stage $\leq s$ at which β has been eligible to act since β's most recent initialization):

(1) If for some μ-index m, $f(m)$ and $g(m)$ are defined and $f(m) \neq g(m)$ then set $D_\beta = \emptyset$, *stop*, and *end the stage*. (At all future substages, β will immediately *end the substage* with current outcome *fin* unless initialized.) Otherwise, proceed to Step 2.
(2) If $f(m)$ is undefined at stage s for some μ-index $m \in D_\beta$, then *end the substage* with current outcome *fin*. Otherwise, proceed to Step 3.
(3) If there exists $m \in D_\beta$ such that
   (a) $f(m) > s', |\beta|$,
   (b) for every $k \leq l$ and for all $\mathcal{S}$-strategies β', $\text{vlink}(\beta', \alpha_k) \neq m$, and
   (c) for every $k \leq l$ and for all $\mathcal{S}$-strategies β', $\text{ulink}(\beta', \alpha_k)_1 \neq m$,

(d) for every $k \leq l$, if $v_k(m)$ is defined then some α-marker $a(\alpha_k, s_k, \widehat{\imath_k}, 1)$, say, is a locking α_k-marker for $\mu(m)$,

then proceed to Step 5, otherwise, proceed to Step 4. (If $D_\beta = \emptyset$ then also proceed to Step 4.)

(4) Pick the least μ-index m such that
   (a) $m \notin \bigcup_{\beta' \subseteq \beta} D_{\beta'}$,
   (b) $m \in S_\beta$ (and, therefore, by Property 9, m is in the stream $S_{\alpha^\frown\langle v\rangle}$ for all $\mathcal{R}$-strategies α with $\alpha^\frown\langle v\rangle \subseteq \beta$, if any).

   If no such m is available now then *stop and end the stage*. Otherwise, enumerate m into D_β and *end the substage* with current outcome *fin*.

(5) Pick the least $m \in D_\beta$ for which all of 3a, 3b, 3c, and 3d hold, and wait for a pair of active α_k-markers to appear in $\mu(m)$ for each $k \leq l$. (β will always *end the substage* with current outcome *fin* while waiting for this.)

(6) Set $n = f(m)$ (which equals $g(m)$ by hypothesis). For each $k \leq l$, denote by $a(\alpha_k, s_k, i_k, 0)$ and $a(\alpha_k, s_k, i_k, 1)$ the pair of active α_k-markers for $\mu(m)$, and choose new α_k-markers $a(\alpha_k, s_k, i'_k, 0)$, $a(\alpha_k, s_k, i'_k, 1)$, $a(\alpha_k, s_k, i''_k, 0)$, and $a(\alpha_k, s_k, i''_k, 1)$ for new $i''_k > i'_k > i_k$ never used before.

   Now proceed as follows.
   (a) For all $k \leq l$:
      (i) enumerate $a(\alpha_k, s_k, i'_k, 0)$ and $a(\alpha_k, s_k, i'_k, 1)$ into $\nu(n)$ (these new α_k-markers are, however, not yet called "active"),
      (ii) enumerate $a(\alpha_k, s_k, i''_k, 0)$ and $a(\alpha_k, s_k, i''_k, 1)$ into $\mu(m)$ (these new α_k-markers are not yet called "active"),
      (iii) extract the active α_k-marker $a(\alpha_k, s_k, i_k, 1)$ from $\mu(m)$ (note, however, that the α-markers $a(\alpha_k, s_k, i_k, 0)$ and $a(\alpha_k, s_k, i_k, 1)$ remain the active α_k-markers for $\mu(m)$ and $\nu(n)$).

   In addition do the following:
   (b) For each $k \leq l$, if $v_k(m)$ is defined (and hence, some α-marker, $a(\alpha_k, s_k, \widehat{\imath_k}, 1)$, say, is a locking α_k-marker for $\mu(m)$), then:
      (i) enumerate the locking α_k-marker $a(\alpha_k, s_k, \widehat{\imath_k}, 1)$ into $\nu(n)$,
      (ii) *create a v-link* from β to α_k by setting $\text{vlink}(\beta, \alpha_k) = m$.
   (c) For each $k \leq l$, if $m \notin \text{dom}(v_k)$ and $n \notin \text{range}(u_k)$, then:

(i) cancel $a(\alpha_k, s_k, i_k, 0)$ and $a(\alpha_k, s_k, i_k, 1)$ as active α_k-markers for both $\mu(m)$ and $\nu(n)$,
(ii) call the α-markers $a(\alpha_k, s_k, i'_k, 0)$ and $a(\alpha_k, s_k, i'_k, 1)$ the *active α_k-markers* for $\nu(n)$, and
(iii) call the α-markers $a(\alpha_k, s_k, i''_k, 0)$ and $a(\alpha_k, s_k, i''_k, 1)$ the *active α_k-markers* for $\mu(m)$.

(d) For each $k \leq l$, if $m \notin \mathrm{dom}(v_k)$ and $n \in \mathrm{range}(u_k)$ then:
(i) let P_k be the set of all π_k-indices p with $u_k(p) = n$, and
(ii) *create a u-link* from the $\mathcal{S}_f$-strategy β to the $\mathcal{R}_{\pi_k}$-strategy α_k by setting $\mathrm{ulink}(\beta, \alpha_k) = \langle m, P_k \rangle$.

(e) Fix the least unused μ-index m' and:
(i) Redefine $g(m) = m'$ and $g(m') = n$.
(ii) Copy the set $\nu(n)$ into $\mu(m')$, and the set $\mu(m)$ into $\nu(m')$. (Here, *copying a set A into a set B* means enumerating all elements of $A \setminus B$ into B and extracting all elements of $B \setminus A$ from B as well as preserving the attributes of α-markers such as "active" and "locking". Note that some α-markers may be active for both $\mu(m)$ and $\mu(m')$ although these sets are different!)

(f) *Stop* and *end the stage*. (The current outcome of the strategy β is not defined since we end stage s. At all future stages, β will immediately end the substage with current outcome *fin* unless initialized.)

This ends the description of the action of an $\mathcal{S}$-strategy.

End of a stage: Let ξ_s be the longest strategy eligible to act at stage s. In particular, ξ_s may have stopped at substage $|\xi_s|$ of stage s. *Initialize* all strategies $\xi \geq \xi_s{}^\frown\langle \mathit{fin} \rangle$ by performing the following actions for each strategy $\xi \geq \xi_s{}^\frown\langle \mathit{fin} \rangle$:

(1) If ξ is an $\mathcal{R}_\pi$-strategy α then
- declare all active α-markers to be no longer active,
- declare all locking α-markers to be no longer locking,
- set both functions u_α and v_α completely undefined, and set $q_\alpha = 0$,
- cancel all links to α, i.e., set both $\mathrm{ulink}(\beta, \alpha)$ and $\mathrm{vlink}(\beta, \alpha)$ to be undefined for all $\mathcal{S}$-strategies β, and
- let the stream S_α be empty.

(2) If ξ is an $\mathcal{S}_f$-strategy β then set D_β equal to the empty set. (Note that we do not cancel any link from β.)

It is easy to check that all the Properties 1–10 will hold at the beginning of stage $s+1$.

This ends the description of the construction.

4. The Verification for the Main Theorem

We now verify that the above construction satisfies the requirements for our Main Theorem in a sequence of lemmas. First, note that, by induction, the Properties 1–10 of the construction are true at all stages s. Now we prove some technical lemmas on more important properties of our construction.

Lemma 3. *(i) For every μ-index m, there exists a stage s after which a value $g(m)$ is defined but never redefined.*

(ii) For every ν-index n, there exist a stage s and a μ-index m such that $g(m) = n$ at all stages $\geq s$.

(iii) For every μ-index m, there exists a stage t_m such that, for all stages $s \geq t_m$, either m is not the parameter of any link at stage s, i.e., $m \neq \mathrm{vlink}(\beta, \alpha)$ and $m \neq \mathrm{ulink}(\beta, \alpha)_1$ for all β and all α, or m is the permanent parameter of such a link at stage s.

Proof. By Property 2, $g(m)$ is defined at stage m and remains defined from now on. The value $g(m)$ can be changed only at stages at which some $\mathcal{S}_f$-strategy β uses the μ-index m from the stream S_β to proceed to Step 6e.

Statement (i) now is a consequence of the following observation. The definition of a *stream* (in particular, the condition in Step 8 of the full $\mathcal{R}_\pi$-strategy) implies that if a μ-index m is enumerated into a stream S_ξ, then $m > |\xi|$. Hence, m may enter the streams of a finite number of strategies only, and, therefore, m can be picked by $\mathcal{S}$-strategies only finitely often. (The point here is that an $\mathcal{R}$-strategy α may produce in Step 8 infinitely many μ-indices (at different stages) and enumerate them into the stream $S_{\alpha^\frown\langle v\rangle}$. These indices may later reach lower and lower streams.)

To prove statement (ii), note that if at stage s, some $\mathcal{S}_f$-strategy β picks a μ-index m in Step 5 and diagonalizes in Step 6, then $f(m) > |\beta|$. It follows that only finitely many $\mathcal{S}$-strategies can change the value of $g^{-1}(n)$ for fixed n, and each at most finitely often by the clause $f(m) > s'$ where s' is the first stage at which β was eligible to act since its most recent diagonalization. Now statement (ii) follows easily from Property 2.

By statement (i) above, there exists a stage t'_m after which $g(m)$ is not redefined any more. By our construction, any u- or v-link from some β to α with parameter m may be created only at a stage s at which $g(m)$ is redefined. By Property 8, m can be the parameter for only one link at any stage s. So, if a u- or v-link from some β to α with the parameter m created at stage t'_m is never canceled, then set $t_m = t'_m$; otherwise let t_m be the least stage at which such a link is canceled. □

We define the *true path* of the construction $TP \in [T]$ as usual for tree arguments as the leftmost path through T of nodes eligible to act infinitely often. By Convention 2, $\langle u\rangle \subset TP$.

Lemma 4. *For every strategy $\xi \subset TP$ and its immediate successor $\xi^\frown\langle o\rangle$ in TP, the following hold:*

(i) The strategy ξ stops at most finitely often.
(ii) The strategy $\xi^\frown\langle o\rangle$ is initialized at most finitely often.
(iii) The stream $S_{\xi^\frown\langle o\rangle}$ is infinite.
(iv) If ξ is an $\mathcal{S}_f$-strategy then, for some stage t_ξ, no new elements enter D_ξ at any stage $t \geq t_\xi$.

Proof. We prove the statements of the lemma by induction on the length of ξ. Assume that the statements (i)–(iv) hold for all the strategies $\xi' \subset \xi$; and we will prove them for ξ. Denote by s_ξ the least stage at which ξ is eligible to act and after which it is not initialized any more. Now we consider two cases.

Case 1: ξ is an $\mathcal{R}_\pi$-strategy α: Statement (iv) is vacuous. Statements (i) and (ii) are obvious since, by the construction, the $\mathcal{R}$-strategy α can stop at most once after initialization, and, therefore, $\xi^\frown\langle o\rangle$ can be initialized at most twice after stage s_ξ.

If $\alpha^\frown\langle o\rangle \subset TP$ and $o \in \{u, \mathit{fin}\}$, then $S_{\alpha^\frown\langle o\rangle} = S_\alpha \cap (s_\alpha, s]$ for every s. Therefore, in this case, (iii) is true by the induction hypothesis.

Consider now the case $o = v$. Since $\alpha^\frown\langle v\rangle \subset TP$, there exist infinitely many stages at which the $\mathcal{R}$-strategy α ends substage $|\alpha|$ in Step 8 with current outcome v. At each of these stages, some μ-index $m > |\alpha|$ is enumerated into the stream $S_{\alpha^\frown\langle v\rangle}$, and, simultaneously, some v-link with m as the parameter is canceled. Lemma 3(iii) implies that the number m can be enumerated into $S_{\alpha^\frown\langle v\rangle}$ only finitely many times. Furthermore, $\alpha^\frown\langle v\rangle$ is not initialized after some stage $s_{\alpha^\frown\langle v\rangle}$, and hence, $S_{\alpha^\frown\langle v\rangle}$ cannot become empty at any stage $s > s_{\alpha^\frown\langle v\rangle}$. Therefore, the stream $S_{\alpha^\frown\langle v\rangle}$ is infinite.

Case 2: ξ is an $\mathcal{S}_f$-strategy β: In this case, we must have $o = \mathit{fin}$. Since for every $s \geq s_\beta$, $S_{\beta^\frown\langle \mathit{fin}\rangle} = S_\beta \cap (s_\beta, s]$, it follows that (iii) is true.

Note that the strategy $\beta^\frown\langle fin\rangle$ can be initialized at stage $s > s_\beta$ only if the strategy β stops (and immediately ends) stage s. By the full $\mathcal{S}_f$-strategy, β can stop the stage only in Steps 1 and 6 (each at most once) and in Step 4. Thus, to prove statements (i) and (ii), it is sufficient to show that β cannot end the stage in Step 4 infinitely often.

The induction hypothesis easily implies that the set $\bigcup_{\beta'\subset\beta} D_{\beta'}$ does not grow after some stage t'. In Step 4, the strategy β tries to enumerate new elements from the infinite set

$$S_\beta \setminus \bigcup_{\beta'\subset\beta} D_{\beta'}$$

into the set D_β provided that, for every $m \in D_\beta$, at least one of the following conditions holds:

(a) $f(m) \leq s_\beta$, or
(b) m is the parameter of some v-link to an $\mathcal{R}$-strategy $\alpha \subset \beta$, or
(c) m is the parameter of some u-link to an $\mathcal{R}$-strategy $\alpha \subset \beta$, or
(d) for some $\mathcal{R}$-strategy $\alpha \subset \beta$, $v_\alpha(m)$ is defined but there is no locking α-marker for $\mu(m)$.

If clause (a) holds for more than $\max\{s_\beta, |\beta|\} + 1$ many μ-indices, then $f(m_1) = f(m_2)$ for two different indices m_1 and m_2, and, therefore, $g(m_1) \neq f(m_1)$ or $g(m_2) \neq f(m_2)$. As a result, the strategy β ends the stage in Step 1 when this happens.

By Lemma 3(iii), some of the μ-indices $m \in D_\beta$ can be parameters of permanent u- or v-links. Since the strategy β deals with only finitely many $\mathcal{R}$-strategies (namely, with the strategies $\alpha \subset \beta$), it follows that only finitely many μ-indices can be parameters of such permanent links. Lemma 3(iii) implies also that all other μ-indices will be eventually released from the u- or v-links to all strategies $\alpha \subset \beta$.

Clause (d) may also be checked easily. Indeed, $\mathrm{dom}(v_\alpha)$ is finite for all $\mathcal{R}$-strategies α with $\alpha^\frown\langle u\rangle \subseteq \beta$ or $\alpha^\frown\langle fin\rangle \subseteq \beta$. On the other hand, if $\alpha^\frown\langle v\rangle \subseteq \beta$, then Properties 9 and 10 imply that $v_\alpha(m)$ is defined and that there is a locking α-marker for $\mu(m)$.

Thus, at some stage $s_1 > s_\beta$, a new μ-index m from $S_\beta \setminus \bigcup_{\beta'\subset\beta} D_{\beta'}$ is enumerated into D_β, and the conditions (a), (b), (c), and (d) fail for this m.

If $f(m)$ is not defined, then at all stages $s \geq s_1$ at which β is eligible to act, β ends the substage of stage s with current outcome *fin*. If $f(m)$ is defined, then at some stage $s_2 \geq s_1$, when β is eligible to act, the strategy β picks m at Step 5 as a witness to diagonalize $\mu(m)$ against $\nu(f(m))$. At stage s_2 or a bit later, the strategy β ends the stage in

Step 6. Thus, in both these cases, the strategy β eventually no longer ends the stage.

It is clear now that the stage t_ξ in statement (iv) may be taken to be the least stage after which the strategy β does not end the stage. □

Lemma 5. *Each $\mathcal{R}$- and $\mathcal{S}$-requirement is represented by a unique strategy $\xi \subset TP$.*

Proof. Immediate by the definition of the tree. □

We can thus, for each strategy $\xi \subset TP$, define the least stage s_ξ at which ξ is eligible to act after the last time it is initialized. Before verifying the satisfaction of all requirements, we note some facts about how the active α-markers distinguish the sets in our family $\mathcal{F}$.

Lemma 6. *For each $\mathcal{R}_\pi$-strategy α with $\alpha\widehat{\ }\langle u\rangle \subset TP$ or $\alpha\widehat{\ }\langle v\rangle \subset TP$,*

(i) any link to α is eventually canceled; and
(ii) each μ-set eventually contains exactly two active α-markers, namely, $a(\alpha, s_\alpha{}', i_\alpha, 0)$ and $a(\alpha, s_\alpha{}', i_\alpha, 1)$, for some fixed i_α.

Proof. (i) First, note that a new value for the function u_α can be defined only by the strategy α in Step 5. At each stage $s > s_\alpha$ at which the strategy α defines a new value for u_α in Step 5, any u- or v-links to α (from any $\mathcal{S}$-strategy) must have been canceled by the beginning of Step 5.

If $\alpha\widehat{\ }\langle u\rangle \subset TP$, then the function u_α is total. If $\alpha\widehat{\ }\langle v\rangle \subset TP$, then the function v_α is total, and, by the construction, this is possible only if a new version of u_α is defined at infinitely many stages. Therefore, in both these cases, each link to α is eventually canceled.

(ii) For each m, there exist a stage s_0 and a number i_0 such that the pair of α-markers $a(\alpha, s_0, i_0, 0)$ and $a(\alpha, s_0, i_0, 1)$ is enumerated into $\mu(m)$ at stage s_0 (by the background actions at the end of substage $|\alpha|$ of each stage at which α is eligible to act and has not yet stopped, or as a result of the copying in Step 6e of some $\mathcal{S}_f$-strategy), and they are called *active α-markers* for $\mu(m)$. From now on, the set $\mu(m)$ has two active α-markers (but $\mu(m)$ may contain one or both of them) by Property 4 and background actions.

Let t_m satisfy statement (iii) of Lemma 3. Obviously, $t_m \geq s_\alpha$. Let $a(\alpha, s_\alpha{}', i_\alpha, 0)$ and $a(\alpha, s_\alpha{}', i_\alpha, 1)$ be the pair of α-markers which are active for $\mu(m)$ at stage t_m. Statement (i) implies now that $m \neq \mathrm{vlink}(\beta, \alpha)$ and $m \neq \mathrm{ulink}(\beta, \alpha)_1$ for all β and all $s \geq t_m$. Therefore, by Properties 5–7, $a(\alpha, s_\alpha{}', i_\alpha, 0)$ and $a(\alpha, s_\alpha{}', i_\alpha, 1)$ form a pair of permanent active α-markers for $\mu(m)$ from stage s_α on. □

We are now ready to establish that our construction satisfies our requirements.

Lemma 7. *The global requirement $\mathcal{G}$ is satisfied, i.e., the families $\mathcal{F}_\nu$ and $\mathcal{F}_\mu$ of d.c.e. sets enumerated by ν and μ, respectively, coincide. Furthermore, ν is in fact a c.e. numbering $\mathcal{F}_\nu$. (We call this family $\mathcal{F}$.)*

Proof. Property 2 and Lemma 3 imply that g is, in the limit, a permutation of the natural numbers. The statement $\mu = \nu \circ g$ now follows from Property 3 and Lemma 3. Therefore, $\mathcal{F}_\mu = \mathcal{F}_\nu$.

For every m, the set $\mu(m)$ consists of the permanent active α-markers for $\mu(m)$, and inactive α-markers which were previously active for $\mu(m)$ (including the locking α-markers). Markers which are not locking never leave sets once they have been enumerated. And the locking α-markers are enumerated only into the sets which never contained them before. The locking α-markers are never extracted from the sets after they have been called locking. All these considerations show that $\mu(m)$ is a difference of c.e. sets.

As for the numbering ν, no number is ever extracted from a ν-set. $\square$

Lemma 8. *Let $\alpha \subset TP$ be an $\mathcal{R}_\pi$-strategy α with $\mathcal{F}_\pi = \mathcal{F}$. Then*

(i) every link to the strategy α is eventually canceled,
(ii) exactly one of $\alpha^\frown\langle u\rangle \subset TP$ or $\alpha^\frown\langle v\rangle \subset TP$ holds.

Proof. (i) *Case 1: u-links.* Suppose, at a stage s_0, we set $\mathrm{ulink}(\beta, \alpha) = \langle m, P\rangle$ for some $\mathcal{S}_f$-strategy β, some μ-index m, and some set of π-indices P. We can suppose that s_0 is the stage at which this u-link was created. Then the strategy β proceeds to Step 6 at stage s_0. The set P consists of all π-indices $p \in \mathrm{dom}(u)$ for which $u(p) = g(m)$ at stage s_0. Denote the value $g(m)$ at stage s_0 by n.

Note that, even if β is initialized at some stage, all the links from β are preserved in the future until canceled. On the other hand, if α is initialized, then all links to α are canceled, and we need to start our considerations from the beginning. So, for simplicity, we can assume that $s_0 \geq s_\alpha$.

At the beginning of Step 6 at stage s_0, m is not a parameter of any u- or v-link to α and, hence, by Properties 6 and 7, $\mu(m)$ (as well as $\nu(n)$, which is still identical to $\mu(m)$ at that moment) contains a pair of active α-markers $a(\alpha, s_\alpha, i, 0)$ and $a(\alpha, s_\alpha, i, 1)$, say. On the other hand, at the beginning of stage s_0, the α-markers $a(\alpha, s_\alpha, i, 0)$ and $a(\alpha, s_\alpha, i, 1)$ are also contained in $\pi(p)$ for every $p \in P$ (otherwise, the α-marker which has left is enumerated into all the sets of $\mathcal{F}$, the numbering π is killed, and $\mathcal{F}_\pi \neq \mathcal{F}$).

In Substep 6a, the active α-marker $a(\alpha, s_\alpha, i, 1)$ is extracted from $\mu(m)$, while $a(\alpha, s_\alpha, i, 0)$ remains in $\mu(m)$. Furthermore, the fresh α-markers $a(\alpha, s_\alpha, i', 0)$ and $a(\alpha, s_\alpha, i', 1)$ are enumerated into $\nu(n)$, and two more unused α-markers $a(\alpha, s_\alpha, i'', 0)$ and $a(\alpha, s_\alpha, i'', 1)$ are enumerated into $\mu(m)$.

In Substep 6e, the modified set $\nu(n)$ is copied into a fresh μ-set $\mu(m')$.

By Property 8, m can be a parameter for at most one link at every stage. And the strategy α tries to cancel a link with the least parameter. So, we can assume additionally that m is the least μ-index among the parameters of all links to α at stage s_0. Then, at every stage $s > s_0$, when the strategy α is eligible to act, it tries to cancel the u-link to α with this parameter m.

By Properties 6 and 7, the active α-marker $a(\alpha, s_\alpha, i, 0)$ is contained in $\mu(m)$ and $\mu(m')$ only, and it never leaves them in the future. $a(\alpha, s_\alpha, i, 0)$ is also contained in $\pi(p)$ at stage s_0. It cannot leave $\pi(p)$ later until the u-link is canceled, otherwise the strategy α enumerates it in Step 1 into all the sets of $\mathcal{F}$, and we obtain $\pi(p) \notin \mathcal{F}$, a contradiction.

Therefore, for each $p \in P$, $\pi(p)$ can equal either $\mu(m)$ or $\mu(m')$. If $\pi(p) = \mu(m)$ then eventually $a(\alpha, s_\alpha, i'', 0)$ and $a(\alpha, s_\alpha, i'', 1)$ are enumerated in $\pi(p)$, while $a(\alpha, s_\alpha, i, 1)$ leaves $\pi(p)$. If $\pi(p) = \mu(m')$, then $a(\alpha, s_\alpha, i', 0)$ and $a(\alpha, s_\alpha, i', 1)$ will be enumerated in $\pi(p)$. This implies that the u-link from β to α is eventually canceled by the strategy α in Step 6.

(i) *Case 2: v-links.* We can assume, as we did in Case 1, that a link $\mathrm{vlink}(\beta, \alpha) = m$ was created by some $\mathcal{S}_f$-strategy β to α at some stage $s_0 \geq s_\alpha$. At some stage $s_1 < s_0$ the strategy α defined $v(m)$ and called some α-marker $a(\alpha, s_\alpha, \hat{\imath}, 1)$, say, the locking α-marker for $\mu(m)$. This locking α-marker was extracted from $\mu(m)$ before stage s_1 and has left $\pi(v(m))$ not later than stage s_1. The point is that $a(\alpha, s_\alpha, \hat{\imath}, 1)$ is contained in $\nu(n)$ at the beginning of Step 6 at stage s_0.

In Step 6b at stage s_0, the locking α-marker $a(\alpha, s_\alpha, \hat{\imath}, 1)$ is enumerated into $\nu(n)$. And from now on, the active α-marker $a(\alpha, s_\alpha, i, 0)$ remains in $\mu(m)$, $\nu(n)$, and $\pi(p)$ until the v-link is canceled (the active α-marker $a(\alpha, s_\alpha, i, 0)$ cannot leave $\pi(p)$ for the same reasons as in Case 1). So, $\pi(p)$ can only equal one of the two sets $\mu(m)$ and $\nu(n)$. But $\pi(p) \neq \nu(n)$ since $\nu(n)$ contains the locking α-marker $a(\alpha, s_\alpha, \hat{\imath}, 1)$. Therefore, there exists a stage $s_2 > s_0$ at which α is eligible to act and such that $a(\alpha, s_\alpha, i'', 0)$ and $a(\alpha, s_\alpha, i'', 1)$ have been enumerated into $\pi(p)$ while $a(\alpha, s_\alpha, i, 1)$ has left $\pi(p)$ by the beginning of stage s_2. Then, at stage s_2, the strategy α declares $\mathrm{vlink}(\beta, \alpha)$ undefined in Step 4.

(ii) It is obvious that, for any $\mathcal{R}$-strategy $\alpha \subset TP$, either $\alpha^\frown\langle u\rangle \subset TP$, or $\alpha^\frown\langle v\rangle \subset TP$, or $\alpha^\frown\langle fin\rangle \subset TP$. Statement (i) precludes the case $\alpha^\frown\langle fin\rangle \subset TP$. □

Corollary 9. *μ and ν are Friedberg numberings.*

Proof. Let π be any numbering of $\mathcal{F}$, and let α be the unique $\mathcal{R}_\pi$-strategy α with $\alpha \subset TP$. Then $\alpha^\frown\langle u\rangle \subset TP$ or $\alpha^\frown\langle v\rangle \subset TP$. By Lemma 6(ii), every μ-set eventually contains a pair of active α-markers. Properties 6 and 7 imply that μ is a one-to-one numbering. And ν is a Friedberg numbering as well since $\mu = \nu \circ g$ and g is a permutation of ω. □

Corollary 10. *Let $\alpha \subset TP$ be an $\mathcal{R}_\pi$-strategy α with $\mathcal{F}_\pi = \mathcal{F}$. Then, for every m and p, $\mu(m) = \pi(p)$ if and only if there exists a pair of permanent active α-markers contained in both $\mu(m)$ and $\pi(p)$.*

Proof. This is immediate by Property 6 and 7 and Lemmas 6(ii) and 8(ii). □

Lemma 11. *Each $\mathcal{R}_\pi$-requirement is satisfied by the unique $\mathcal{R}_\pi$-strategy $\alpha \subset TP$ provided $\mathcal{F}_\pi = \mathcal{F}$.*

Proof. In view of Lemma 8(ii), we consider the following two cases.

Case 1: $\alpha^\frown\langle u\rangle \subset TP$: Let $s_0 > s_\alpha$ be the least stage such that the strategy α does not have current outcome v at substage $|\alpha|$ of any stage $s \geq s_0$ at which α is eligible to act. Thus the function $u = u_\alpha$ is total, and its values are not redefined after stage s_0. In contrast to u, $\mathrm{dom}(v)$ is a finite set. Then we can suppose (because of Lemmas 3(iii) and 8(i)) that no v-link exists or is created after stage s_0. We will prove that, for all p such that $u(p)$ has been defined after stage s_0, $\pi(p) = \nu(u(p))$.

Fix any such p and let s_1 be the stage when $u(p)$ was set equal to n, say. From now on, $u(p) = n$ holds permanently. At stage s_1, for some μ-index m_0 with $n = g(m_0)$, we have $\mu(m_0) = \nu(n)$, and the sets $\pi(p)$ and $\mu(m_0)$ contain the same pair of active α-markers $a(\alpha, s_\alpha, i_0, 0)$ and $a(\alpha, s_\alpha, i_0, 1)$, say. If, after stage s_1, m_0 is not the parameter of any link, then at all stages $s > s_1$, the α-markers $a(\alpha, s_\alpha, i_0, 0)$ and $a(\alpha, s_\alpha, i_0, 1)$ are active and are contained in $\mu(m_0)$, and so $\mu(m_0) = \nu(n)$. If one of these α-markers leaves $\pi(p)$, then it is enumerated into each set of $\mathcal{F}$, the numbering π is killed, and we obtain $\mathcal{F}_\pi \neq \mathcal{F}$, a contradiction. Thus, $a(\alpha, s_\alpha, i_0, 0)$ and $a(\alpha, s_\alpha, i_0, 1)$ are also contained in $\pi(p)$ at all stages $s > s_1$, and therefore, $\pi(p) = \mu(m_0) = \nu(n)$ by Corollary 10.

Consider now the possibility that at some stage $s_2 > s_1$, the μ-index m_0 becomes the parameter for some u-link. Then $\mathrm{ulink}(\beta, \alpha) = \langle m_0, P\rangle$ at stage s_2 for some $\mathcal{S}$-strategy β and some set P. At stage s_2, the strategy β proceeds to Step 6a, followed by Step 6d. In Step 6a, a fresh pair of α-markers $a(\alpha, s_\alpha, i_1, 0)$ and $a(\alpha, s_\alpha, i_1, 1)$, say, is enumerated into $\nu(n)$. (A bit later, at the same stage, in Step 6e, this modified set $\nu(n)$ is copied into a fresh μ-set $\mu(m_1)$, say.) One of the active α-markers, namely, $a(\alpha, s_\alpha, i_0, 0)$, remains in both $\mu(m_0)$ and $\nu(n) = \mu(m_1)$. In Step 6d, p is put into the set P since $u(p) = n$. Note that, for all $p' \in P$, $u(p') = n$, and $\pi(p')$ contains both the α-markers $a(\alpha, s_\alpha, i_0, 0)$ and $a(\alpha, s_\alpha, i_0, 1)$ at stage s_2.

Since $\alpha\widehat{\ }\langle u\rangle \subset TP$, it follows by Lemma 6(i) that at some stage $s_3 > s_2$, the strategy α cancels the u-link from β to α and declares $a(\alpha, s_\alpha, i_1, 0)$ and $a(\alpha, s_\alpha, i_1, 1)$ to be the active α-markers for $\mu(m_1)$ in Step 6. After that, the strategy α proceeds to Step 7. Since $\mathrm{dom}(v)$ is not growing any more after stage s_0, it follows that both the α-markers $a(\alpha, s_\alpha, i_1, 0)$ and $a(\alpha, s_\alpha, i_1, 1)$ are eventually enumerated into the sets $\pi(p')$ for all $p' \in P$, and in particular into $\pi(p)$. Thus, by the end of stage s_3, the situation is analogous to the situation at stage s_1.

So, after k many such cycles, we obtain a pair of permanent active α-markers $a(\alpha, s_\alpha, i_k, 0)$ and $a(\alpha, s_\alpha, i_k, 1)$, say, which are eventually contained in both $\pi(p)$ and some set $\mu(m_k)$ with $g(m_k) = n$. Then Corollary 10 implies that $\mu(m_k) = \nu(n) = \pi(p)$.

Case 2: $\alpha\widehat{\ }\langle v\rangle \subset TP$: The values of the function v are defined by the strategy α only at stages s_1 when α is eligible to act and some u-link is canceled. Then the strategy α proceeds to Steps 6–8. This process can be delayed by waiting in Substep 8a. But for simplicity, we will assume that all three Steps 6–8 are performed by α at the same stage s_1.

Let that u-link be created at some stage ${s_1}' < s_1$, i.e., some $\mathcal{S}$-strategy β sets $\mathrm{ulink}(\beta, \alpha) = \langle m, P\rangle$ at a stage ${s_1}'$. We will consider only links with ${s_1}' \geq s_\alpha$. In Step 6 of substage $|\alpha|$ of stage s_1, the strategy α cancels this u-link from $\mathcal{S}$-strategy β, while in Step 8, it defines the values $v(m)$ and $v(\tilde{m})$ for all $\tilde{m} < m$ such that $\tilde{m} \notin \mathrm{dom}(v)$ and $\tilde{m}$ is free of all links except the links to the strategies $\alpha'\widehat{\ }\langle u\rangle <_L \alpha$. Note that we consider the case when $v(m)$ is still undefined at the beginning of stage s_1. We will consider the two subcases $v(m)$ and $v(\tilde{m})$ separately.

Subcase $v(m)$: By the end of substage $|\alpha|$ of stage s_1, we have the following:

- m is enumerated into the stream $S_{\alpha\widehat{\ }\langle v\rangle}$,
- m is not a parameter for any link,
- $v(m)$ is defined,
- a pair of active α-markers $a(\alpha, s_\alpha, i_0, 0)$ and $a(\alpha, s_\alpha, i_0, 1)$, say, is contained in $\mu(m)$,

- the same pair of active α-markers is contained in $\pi(v(m))$, and
- some α-marker $a(\alpha, s_\alpha, \hat{\imath}, 1)$, say, is declared to be the locking α-marker for $\mu(m)$ (and this locking α-marker has been extracted from $\mu(m)$ at stage ${s_1}'$ and later has left $\pi(v(m))$).

Since from now on, $v(m)$ is defined and $m \in S_{\alpha^\frown\langle v\rangle}$, it follows that furthermore, m can be a parameter for v-links only.

If, after stage s_1, m is not a parameter of any v-link, then at all stages $s > s_1$ the α-markers $a(\alpha, s_\alpha, i_0, 0)$ and $a(\alpha, s_\alpha, i_0, 1)$ remain active and do not leave $\mu(m)$. On the other hand, they are both contained in $\pi(v(m))$. (Otherwise, π is killed as in Case 1 and $\mathcal{F}_\pi \neq \mathcal{F}$.) Therefore, $\mu(m) = \pi(v(m))$ by Corollary 10.

Consider now the possibility that at some stage $s_2 > s_1$, the μ-index m becomes a parameter of some v-link, i.e., $\text{vlink}(\beta_1, \alpha) = m$ for some $\mathcal{S}$-strategy β_1. At stage s_2, the strategy β_1 proceeds to Step 6a, followed by Step 6b. In Step 6a, a fresh pair of α-markers $a(\alpha, s_\alpha, i_1, 0)$ and $a(\alpha, s_\alpha, i_1, 1)$, say, is enumerated into $\mu(m)$, the active α-marker $a(\alpha, s_\alpha, i_0, 1)$ is extracted from $\mu(m)$, while another active α-marker $a(\alpha, s_\alpha, i_0, 0)$ remains in $\mu(m)$. In Step 6b, the locking α-marker $a(\alpha, s_\alpha, \hat{\imath}, 1)$ is enumerated into $\nu(n)$.

Since $\alpha^\frown\langle v\rangle \subset TP$, Lemma 6(i) implies that at some stage $s_3 > s_2$, the strategy α cancels the v-link from β_1 to α and declares $a(\alpha, s_\alpha, i_1, 0)$ and $a(\alpha, s_\alpha, i_1, 1)$ the active α-markers for $\mu(m)$ in Step 4. While performing Step 4, the strategy α ensures also that $a(\alpha, s_\alpha, i_1, 0)$ and $a(\alpha, s_\alpha, i_1, 1)$ are enumerated into $\pi(v(m))$. Note that the α-marker $a(\alpha, s_\alpha, \hat{\imath}, 1)$ remains locking for the set $\mu(m)$ and, therefore, can be used for the next cycle of making m a parameter of some new v-link and releasing it of the parameter role later on. Thus, eventually, we obtain a pair of active α-markers $a(\alpha, s_\alpha, i_k, 0)$ and $a(\alpha, s_\alpha, i_k, 1)$, say, which are contained in both $\mu(m)$ and $\pi(v(m))$ permanently. Then $\mu(m) = \pi(v(m))$ by Corollary 10.

Subcase $v(\tilde{m})$: By the end of substage $|\alpha|$ of stage s_1, we have the following:

- $\tilde{m}$ is not enumerated into the stream $\alpha^\frown\langle v\rangle$,
- $v(\tilde{m})$ is defined,
- a pair of active α-markers $a(\alpha, s_\alpha, \tilde{\imath}, 0)$ and $a(\alpha, s_\alpha, \tilde{\imath}, 1)$, say, is contained in $\mu(\tilde{m})$, and
- the same pair of active α-markers is contained in $\pi(v(\tilde{m}))$.

Therefore, by the end of stage s_1, $\tilde{m} \notin S_{\alpha^\frown\langle v\rangle}$. Since $v(\tilde{m})$ is defined by the end of stage s_1, it follows that $\tilde{m}$ cannot be chosen as a witness for diagonalization by any $\mathcal{S}$-strategy at any stage $s > s_1$. Hence, $\tilde{m}$ can never become the parameter for any u- or v-link, and $a(\alpha, s_\alpha, \tilde{\imath}, 0)$

and $a(\alpha, s_\alpha, \tilde{\imath}, 1)$ form a permanent pair of α-markers for $\mu(\tilde{m})$. As in Case 1 and Subcase $v(m)$, $a(\alpha, s_\alpha, \tilde{\imath}, 0)$ and $a(\alpha, s_\alpha, \tilde{\imath}, 1)$ never leave $\pi(v(\tilde{m}))$. Thus, by Corollary 10, $\mu(\tilde{m}) = \pi(v(\tilde{m}))$. □

Lemma 12. *Each $\mathcal{S}_f$-requirement is satisfied by the unique $\mathcal{S}_f$-strategy $\beta \subset TP$.*

Proof. Let f be a computable function. Assume that f is total and let β be the unique $\mathcal{S}_f$-strategy with $\beta \subset TP$. In view of Lemma 4(i) and (iv), we can choose the least stage $s_0 > s_\beta$ at which β is eligible to act and such that

- the strategy β does not stop in Step 4 at any stage $s \geq s_0$,
- the set D_β does not grow any more at any stage $s \geq s_0$, and
- for all $m \in D_\beta$, $f(m)$ is defined at stage s_0.

We can also suppose in addition that $f(m) = g(m)$ for all $m \in D_\beta$. (If, for any $m \in \mathrm{dom}(f)$ at stage s_0, $f(m) \neq g(m)$, then there is nothing to prove since ν is a Friedberg numbering.) Thus, at stage s_0, the strategy β picks some μ-index m_0 in Step 5 and proceeds to Step 6. We conclude $f(m_0) \neq g(m_0)$ at stage $s_0 + 1$. □

These lemmas establish our Main Theorem. □

5. Concluding Remarks

Our Main Theorem leaves open the question of whether the Rogers semilattice of a family of d.c.e. sets can be finite of size ≥ 2. Before stating our conjecture, we recall the definition of distributivity for upper semilattices and then a weakening of this definition from Badaev/Goncharov/Sorbi [BGS03a, Definition 3.2]:

Definition 13. (1) An upper semilattice $\mathcal{U} = \langle U, \leq, \vee \rangle$ is *distributive* if for any $a_0, a_1, b \in U$ with $b \leq a_0 \vee a_1$, there are $b_0 \leq a_0$ and $b_1 \leq a_1$ with $b = b_0 \vee b_1$.
(2) (Badaev/Goncharov/Sorbi [BGS03a, Definition 3.2]) An upper semilattice $\mathcal{U} = \langle U, \leq, \vee \rangle$ is *weakly distributive* if $\mathcal{U}^\perp$ is distributive, where $\mathcal{U}^\perp$ is simply $\mathcal{U}$ with a new least element $\perp$.

(We remark that the notion of "weak distributivity" has been used in other meanings by lattice theorists in earlier work.)

Note that the definition of weak distributivity is quite natural for upper semilattices since it "removes" a "trivial" reason why an upper semilattice is not distributive, namely, in the situation where $b \leq a_0 \vee a_1$ and for some $i \leq 1$, there is no element $b_i \leq a_i, b$ in U. Furthermore, any finite upper semilattice with least element is a lattice, and so a finite upper semilattice $\mathcal{U}$ is weakly distributive iff $\mathcal{U}^{\perp}$ is a finite distributive lattice.

We now state the following sweeping conjecture, which would refute Khutoretskii's theorem for families of d.c.e. sets and also answer a number of open questions on the number of minimal and Friedberg numberings for such families:

Conjecture 14. Every finite weakly distributive upper semilattice is isomorphic to the Rogers semilattice of a family $\mathcal{F}$ of d.c.e. sets. (Indeed, $\mathcal{F}$ is a family of c.e. sets with exactly one c.e. numbering up to equivalence.) In particular,

- for any $n \geq 1$, there is a family of d.c.e. sets whose Rogers semilattice has size n; and
- for any $n \geq 1$, there is a family of d.c.e. sets which has exactly n many minimal numberings (up to equivalence), and all of these are Friedberg numberings.

The two consequences in the above conjecture are immediate from the conjecture's main statement, taking the upper semilattice to be a finite chain of n elements (for (1)) and a finite n-atom Boolean algebra with the least element removed (for (2)), respectively.

Preliminary work of the authors with Kastermans suggests that this conjecture is indeed correct. This would, however, still leave open the exact characterization of the finite upper semilattices which can be isomorphic to Rogers semilattices of d.c.e. sets, since it is conceivable that such a finite Rogers semilattice is not necessarily weakly distributive.

We finally speculate that similar results can also be obtained for Rogers semilattices of families of n-c.e. sets for $n > 2$.

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