Differences of computably enumerable sets

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Abstract
We consider the lower semilattice $\mathcal{D}$ of differences of c.e. sets under inclusion. It is shown that $\mathcal{D}$ is not distributive as a semilattice, and that the c.e. sets form a definable subclass.

1 Introduction

A persistent open problem about the lattice $\mathcal{E}$ of computably enumerable (c.e.) sets under inclusion is to determine the least number $k$ such that the $\Sigma_k$-theory is undecidable. Lachlan [6] proved that the $\Sigma_2$-theory in

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the language of lattices is decidable, while one of the various known proofs of undecidability for $Th(\mathcal{E})$, in that case due to Harrington, shows that in fact the $\Sigma_8$-theory in the language of lattices is undecidable (see [10], p. 381 for a sketch of that proof). Thus a very unsatisfying gap of 6 quantifier alternations remains. The reason why the undecidability proofs are so “bad” is that the coding used is very indirect. For instance, first one codes the class of finite symmetric graphs (which has an hereditarily undecidable $\Sigma_2$-theory) in the class of recursive Boolean pairs ([1]), and in a second step, the latter class is coded in $\mathcal{E}$ (see [8] for details about how to prove undecidability of fragments of theories). Each step involves quite a complex coding, which yields an increase of three quantifier alternations.

Since $\mathcal{E}$ seems to be very well behaved for properties definable with few quantifier alternations, it is not at all clear how to obtain a better coding. Ideally, one would want to have a coding of a sufficiently complex class, like the class of finite symmetric graphs, using only $\Sigma_1$-formulas with parameters. By the methods developed in [8], this would give the undecidability of the $\Sigma_3$-theory. However, such a proof is not possible since it would show that the class of finite distributive lattices with the reduction property has an undecidable theory, contrary to a result of Ershov. The argument is as follows: Suppose that, via some scheme of $\Sigma_1$ formulas we can code each finite symmetric graph $(V, E)$, using appropriate parameters $\mathcal{P}$ (see [8] for definitions). Let $L$ be a finite distributive sublattice of $\mathcal{E}$ which contains $\mathcal{P}$, all the elements of $\mathcal{E}$ representing the vertices in $V$ and also witnesses for all $\Sigma_1$-formulas involved to code $(V, E)$. Then $L$, and in fact any distributive lattice $H$ such that $L \subseteq H \subseteq \mathcal{E}$ codes $(V, E)$ via the same scheme and parameters. Now let $H$ be such a lattice which is also finite and satisfies the reduction property. In this way, we have obtained a uniform coding of a complex class in the class of finite distributive lattices with the reduction property. We conclude that the best we can hope for to obtain by the standard coding methods is undecidability of the $\Sigma_4$-theory, which still would require a far more direct coding than the ones presently known.

Here we propose a new view of $\mathcal{E}$, which may eventually lead to such a more direct coding. We consider the structure $\mathcal{D}$ of differences of c.e. sets under inclusion. This structure coincides with the second level of Ershov’s difference hierarchy. $\mathcal{D}$ is closed under intersections, since

\[(A_1 - B_1) \cap (A_2 - B_2) = (A_1 \cap A_2) - (B_1 \cup B_2). \tag{1.1}\]

$\mathcal{D}^*$ will denote the structure $\mathcal{D}$ modulo finite differences. For notational reasons, we formulate most of our results in the setting of $\mathcal{D}^*$, not $\mathcal{D}$. We
first collect some simple facts about $\mathcal{D}$ and $\mathcal{D}^*$. The first one shows that any undecidability proof for a low-level fragment of Th($\mathcal{D}$) would yield the same result for $\mathcal{E}$, which justifies the program suggested above.

**Proposition 1.1** The lower semilattice $\mathcal{D}$ can be interpreted in the lattice $\mathcal{E}$, using only quantifier free formulas without parameters. The same holds for $\mathcal{D}^*$ and $\mathcal{E}^*$.

**Proof.** We ambiguously represent the element $A - B$ of $\mathcal{D}$ by the pair of c.e. sets $(A, B)$. Then (1.1) gives a formulas to define the infimum (and hence inclusion) in terms of pairs as desired. The same formula works for $\mathcal{D}^*$. An explicit way to define inclusion is

$$A_1 - B_1 \subseteq A_2 - B_2 \iff (A_1 \subseteq A_2 \cup B_1 \land A_1 \cap B_2 \subseteq B_1).$$

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**Conventions.** We use the notational convention that, if a set $X$ in $\mathcal{D}$ is given, the corresponding lower case letter $x$ denotes the element $X^*$ of $\mathcal{D}^*$. Conversely, if $y \in \mathcal{D}^*$ is given, then $Y$ denotes an element of $\mathcal{D}$ such that $Y^* = y$. We denote $\emptyset^*, \omega^*$ by 0, 1 respectively. If $X = A - B$, $A, B$ c.e. then we assume that $A, B$ are equipped with enumerations such that $\forall s[B_s \subseteq A_{s+1}]$. We say that an element $m$ enters $X$ when $m$ is enumerated into $A$, and leaves $X$ when, later, $m$ is enumerated into $B$. The elements of $\mathcal{D}$ are called difference c.e. (d.c.e.) sets. Letters $A, B, C, D$ always denote c.e. sets. A split of a c.e. set $A$ is a c.e. set $B \subseteq A$ such that $A - B$ is c.e.

**Proposition 1.2** $\mathcal{D}^*$ is not a lattice.

**Proof.** We construct $D_0, D_1 \in \mathcal{D}$ such that sup$(d_0, d_1)$ fails to exist. If $x = \sup(d_0, d_1)$ then w.l.o.g. we can assume that $D_0, D_1 \subseteq X$. So it is enough to build for each $X$ a “counterexample” $D_X \supseteq D_0, D_1$ such that the following requirements are met:

$$P_n^X : X \supseteq D_0, D_1 \Rightarrow |X - D_X| \geq n.$$  

For in that case, $d_x \notin x$, so not $x = \sup(d_0, d_1)$. The strategy for $P_n^X$ is as follows: if $P_{n-1}^X$ is met, i.e. if already $|X - D_X| \geq n - 1$, enumerate a new candidate $m$ for $P_n^X$ into $D_0$. If $m$ does not appear in $X$, we win $P$. If $m$ appears in $X$ then we extract $m$ from $D_0$, so if $m$ stays in $X$, we win because we have increased $|X - D_X|$ by one. If, however, $m$ leaves $X$ later, then we enumerate $m$ into $D_1$. So $m \in D_1 - X$, so we win again. Combining the strategies is routine. (A less elementary proof using major subsets also works here.)

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**Proposition 1.3** If $A - B$ is co-finite, then there is a co-finite $c.e. C$ such that $A - B \subseteq C$.

*Proof.* If $A$ is co-finite, let $C = A$. Else $\overline{B}$ is co-finite, so pick an infinite computable $R \subseteq B$ and let $C = R$. ♦

We next consider lattice embeddings. The property that a finite lattice $L$ can be embedded (preserving meets and joins) into $D^*$ is a $\Sigma^0_2$ property of $D^*$ in the language of partial order, and hence, by proposition 1.1, a $\Sigma^0_2$ property of $E^*$ in the language of lattices. So the question whether this property holds can in principle be answered by invoking Lachlan’s decision procedure for the $\Sigma^0_2$-theory of $E^*$ [6]. However, due to the technical complexity of Lachlan’s procedure it is often easier to give a direct proof. In this way we establish the following result, which shows that $D^*$ is not distributive as a lower semilattice (see [9]). This supports our expectation that coding in $D^*$ on a low quantifier level may turn out to be less restricted than coding in $E^*$.

**Proposition 1.4 (Implicit in [6])** The five-element nonmodular lattice, $N_5$, can be embedded into $D^*$, preserving meet, join and least element.

![Figure 1: An embedding of $N_5$](image)

*Sketch of a proof.* We build c.e. sets $A, B, C$ such that $\emptyset \subset C \subset B \subset A$, and $\text{sup}(a - b, c) = a$. Then an embedding of $N_5$ is obtained as shown in
Figure 1. Along with making \( C, A - B \) and \( B - C \) infinite, we meet the supremum requirements

\[
N_V : A - B, C \subseteq V \Rightarrow A \subseteq^* V.
\]

The main problem is to make \( B - C \) infinite, i.e. to meet for each \( k \) the requirement \( P_k : |B - C| \geq k \). We outline a strategy to put a new element \( x \) into \( B - C \) in the context of a higher priority \( N_V \) requirement. First, we enumerate \( x \) into \( A \). Then \( x \) must appear in \( V \), else we win \( N_V \) in a finitary way (and \( P_k \) starts with a new \( x \), but can now disregard \( N_V \)). Now we put \( x \) into \( B \). Then we have succeeded (\( x \) is in \( B - C \)), unless \( x \) leaves \( V \), threatening to make us loose \( N_V \). But in this case, we enumerate \( x \) into \( C \) and, again, we win \( N_V \) in a finitary way.

Also by a direct proof, we obtained the opposite result for \( M_5 \).

\[\Diamond\]

**Proposition 1.5 (Implicit in [6])** The five element modular nondistributive lattice \( M_5 \) cannot be embedded into \( \mathcal{D}^* \).

\[\Diamond\]

2 The definability of \( \mathcal{D}^* \) in \( \mathcal{E}^* \)

We now analyze cupping properties and complementation in \( \mathcal{D}^* \). This leads to our main result that \( \mathcal{E}^* \), viewed as a subclass, is definable in \( \mathcal{D}^* \) without parameters. Analogous definability results have been obtained by Cooper for the c.e. degrees [2], viewed as a subclass of the \( \Delta_2^0 \)-degrees, and by the second author for \( \mathcal{E}^* \), viewed as a subclass of the uppersemilattice of c.e. equivalence relations modulo finite differences [7]. Recall that \( A \subset_m B \) iff

\[
A \subset_\infty B \text{ and } \forall C (B \cup C = \omega \Rightarrow A \cup C = \ast \omega).
\]

We write \( A \subseteq_m^\ast B \) if \( A \subseteq^\ast B \) and

\[
(A \subset_m B \cup A \text{ or } B - A \text{ finite}).
\]

Note that \( A \subseteq_m^\ast B \) means that \( B^* \) has no “cupping partner” above \( A^* \). Thus, \( A \subseteq_m^\ast B \Leftrightarrow \)

\[
\forall D[B \cup D = \ast \omega \land D \subset_\infty \omega \Rightarrow A \not\subseteq^* D],
\]

which is equivalent to \( \forall D[D \subseteq^* B \land |D| = \infty \Rightarrow A \cap \overline{D} \neq \emptyset] \). Therefore, for \( A \subset_\infty B, A \subset_m B \) iff \( B - A \) does not \( \ast \)-contain an infinite co-c.e. set (and therefore no infinite c.e. set – compare this to proposition 1.3).

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A complement of \( x \in \mathcal{D}^* \) is a \( y \) such that \( x \land y = 0, x \lor y = 1 \). If \( A \) is c.e. then clearly \( \overline{a} = \overline{A} \) is a complement of \( a \) in \( \mathcal{D}^* \). We now prove that all complements are of the form \( \overline{B} \) for \( B \) “close to” \( A \).

**Lemma 2.1** Suppose \( A \) is c.e. Then the complements of \( A^* \) in \( \mathcal{D}^* \) are precisely those sets \( \overline{B}^* \) s.t. \( A \subseteq_m B \), \( B \) c.e.

**Proof.** If \( B \) is such, then \( A \cap \overline{B} = \emptyset \). Moreover, \( \sup(A^*, \overline{B}^*) = 1 \) : else there would be an c.e. \( w \) s.t. \( A^*, \overline{B}^* \leq w < 1 \), whence \( B^* \) would have a cupping partner above \( A^* \). Now suppose \( (C - D)^* \) is a complement of \( A^* \). Then \( A \cup C = \omega^* \), so by the reduction principle,

\[
A \cup R = \omega^* \land R \subseteq \omega^* C
\]

for some recursive \( R \). Since \( A \cap (C - D) = \emptyset \), this implies

\[
C - D = \omega^* R - D.
\]

So \( C - D \) is co-c.e. Let \( \overline{B} = C - D \), \( B \) c.e. Then \( A \subseteq_m B \). Otherwise there would be a c.e. \( E \) s.t. \( A \subseteq E \subsetneq \omega \) but \( B \cup E = \omega^* \), i.e. \( a, \overline{b} \leq e < 1 \). \( \diamond \)

**Lemma 2.2** Suppose \( B \subseteq A \) and \( D \subseteq C \). Then the following are equivalent:

(i) \( (A - B)^* \lor (C - D)^* = 1 \)

(ii) \( A \cup C = \omega^* \land B \cap D \) finite \( \land A \subseteq_m A \cup D \land C \subseteq_m C \cup B \)

**Proof.** (See figure 2). (i) \( \rightarrow \) (ii). Clearly \( (C - D), (A - B) \subseteq A \cup C \) and \( C - D, A - B \subseteq \overline{B} \cap \overline{D} \). So \( A \cup C \) and \( B \cap D \) must be cofinite. Now suppose \( A \subseteq E \) and \( A \cup D \) cofinite. We have to show that \( E \) is cofinite. Clearly \( \overline{A} \cap \overline{D} \subseteq E \) and \( A \subseteq E \) implies \( \overline{D} \subseteq \omega^* E \). Thus \( A - B, C - D \subseteq \omega^* E \).

So by (i), \( E \) is cofinite. Thus \( A \subseteq_m A \cup D \). Similarly \( C \subseteq_m C \cup B \).

(ii) \( \rightarrow \) (i). By proposition 1.3, it is enough to show that, if \( A - B, C - D \subseteq \omega^* X \), \( X \) c.e., then \( X \) is cofinite. Clearly \( A \subseteq X \cup B \). Moreover, since \( A \cup C = \omega^* \),

\[
\omega^* (C - D) \cup (D \cup A) \subseteq X \cup (D \cup A).
\]

So \( A \subseteq_m A \cup D \) implies that \( X \cup B = \omega^* \). Symmetrically, \( X \cup D = \omega^* \) and hence because \( B \cap D = \emptyset \),

\[
X = \omega^* (X \cup B) \cap (X \cup D) = \omega^*.
\]

\( \diamond \)

The following corollary gives an approximation to distributivity.
Corollary 2.3 For each \( v \in \mathcal{D}^* \), \( F = \{ w : v \lor w = 1 \} \) is closed under infima.

Proof. Suppose \( v = (A - B)^* \), and \((C_i - D_i)^* \in F\) for \( i = 0, 1\). We show that \(((C_0 \cap C_1) - (D_0 \cup D_1))^* \in F\). Clearly \( A \cup (C_0 \cap C_1) =^* \omega \) and \( B \cap (D_0 \cup D_1) =^* \emptyset \). A routine computation shows the major subset relations: e.g., for \( A \subseteq^*_m (D_0 \cup D_1) \), if \( A \cup D_0 \cup D_1 \cup E =^* \omega \), then \( A \cup D_1 \cup E =^* \omega \), so \( A \cup E =^* \omega \).

We now obtain the definability of \( \mathcal{E}^* \) as a subclass of \( \mathcal{D}^* \).

Theorem 2.4 An element of \( \mathcal{D}^* \) is c.e. iff it is the supremum (in \( \mathcal{D}^* \)) of two elements which have a unique complement.

Proof. Let \( v \in \mathcal{D}^* \). First, if \( v \) is c.e., then choose c.e. disjoint sets \( A_0, A_1 \) of low degree such that \( V = A_0 \cup A_1 \). Then, in \( \mathcal{D}^* \), \( v = a_0 \lor a_1 \). Moreover, since each \( a_i \) is low, but major subsets are high [So 87, XI.1.19], Lemma 2.1 implies that \( a_0, a_1 \) have a unique complement: \( \overline{A}_i^* \) is a complement of \( a_i \), and, if \((C - D)^* \) is a complement, where \( A_i \cup C = \omega \) and \( A_i \cap C \subseteq^* D \), then \( A_i \subseteq^*_m A_i \cup D \). Therefore \( D \subseteq^* A_i \) which implies \( C - D =^* \overline{A}_i^* \).

For the other direction, it is sufficient to show that, if \((A - B)^* \) has a unique complement \((C - D)^* \) then \( A - B \) must be c.e. Then, if an element
of $\mathcal{D}^*$ is the sup of two elements with a unique complement, it is c.e. as the supremum of two c.e. elements of $\mathcal{D}^*$. To show this, we use Lemma 2.2. Since $A \cup C =^* \omega$, we can choose a computable set $R$ s.t. $R \subseteq^* A$ and $\overline{R} \subseteq^* C$.

Case 1. $C$ is recursive. By Lemma 2.2, $C \subseteq^*_m C \cup B$, which implies $C =^* C \cup B$, i.e. $B \subseteq^* C$. Since $A \cap C \subseteq^* B \cup D$ and $B \cap D =^* \emptyset$, it follows that $A - B =^* (A - C) \cup (A \cap D)$. So $A - B$ is c.e. (see Fig. 2).

Case 2. Otherwise, by [6], choose a small major subset $U \subseteq_m C$ (see [10] for a definition). Notice that $U$-contains precisely those splits of $C$ which are recursive ($U \subseteq_m C$ implies that $U$-contains all recursive splits, and $U \subseteq_s C$ implies that all such splits are recursive). Thus, $\overline{R} \subseteq U$. Let $\tilde{C} = U \cup D$. We claim that also $(\tilde{C} - D)^*$ is a complement of $(A - B)^*$. To verify this, we check Lemma 2.2. Clearly, $\tilde{C} - D \subseteq C - D$, so $(A - B) \cap (\tilde{C} - D) = \emptyset$. Moreover, $\overline{R} \subseteq \tilde{C}$, so $A \cup \tilde{C} =^* \omega$. Finally, $\tilde{C} \subseteq_m C \subseteq_m C \cup B$, so $\tilde{C} \subseteq_m \tilde{C} \cup B$. Since we didn’t change $D$, the hypotheses of Lemma 2.2 are satisfied.

Assume that $A - B$ is not c.e. We claim that, in this case,

$$\tilde{C} - D \subset \infty \ C - D$$

so that $(A - B)^*$ has two different complements. Assume that $C - D =^* \tilde{C} - D$. Since $B \cap D =^* \emptyset$, $B \cap C \subseteq^* C - D \subseteq^* \tilde{C} - D$. Thus $B \cap C \subseteq^* U$. But $S = R \cap B \cap C$ is a split of $C$ with $\overline{R} \cup D$ as the other component since $R \cap C \subseteq^* A \cap C \subseteq^* B \cup D$. So $S$ must be recursive. Let $X = \overline{R} \cup S \cup \overline{B}$ (X is co-c.e.). Then $A - B \subseteq^* X$ and $C \subseteq^* S \cup \overline{R} \cup D \subseteq^* X$. We will show that $\overline{X} =^* R \cap (B - S)$ is infinite. This contradicts $\sup((A - B)^*, (C - D)^*) = 1$. Assume $\overline{X}$ is finite. Then, “on $R^*$, $B =^* S$”, so we have $B \subseteq^* C$. We will show that in this case,

$$A - B =^* (R - S) \cup (D \cap A)$$

so that $A - B$ is c.e. because $S$ is recursive. Note that, if $B \subseteq^* C$, then $R - S =^* R - B$. So we get the inclusion “$\geq^*$” immediately from $R \subseteq A$ and $B \cap D = \emptyset$. For the inclusion “$\subseteq^*$”, consider $n \in A - B$. We can assume that $n \notin R$. Then $n \in A \cap C \subseteq^* B \cup D$, so for almost all relevant $n$, $n \in A \cap D$ because $n \notin B$. ◊

Corollary 2.5 \{ $v \in \mathcal{D}^*$ : $v$ c.e. \} is definable in $\mathcal{D}^*$ without parameters.

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Corollary 2.6 $\text{Aut}(\mathcal{E}^*) \cong \text{Aut}(\mathcal{D}^*)$ via the map $H$ defined by

$$H(\Phi)(a - b) = \Phi(a) - \Phi(b).$$
Proof. First we show that, for each $\Phi \in \text{Aut}(\mathcal{E}^*)$, $H(\Phi)$ is well-defined and an automorphism: Notice that for $a, b, c, d \in \mathcal{E}^*$

$$a - b \leq c - d \iff a \land d \leq b \land a \leq c \lor b$$
$$\iff \Phi(a) \land \Phi(d) \leq \Phi(b) \land \Phi(a) \leq \Phi(c) \lor \Phi(b)$$
$$\iff H(\Phi)(a - b) \leq H(\Phi)(c - d)$$

Next, $\mathcal{E}^* = \{v \in \mathcal{D}^* : v \text{ c.e.}\}$ is a definable set which generates $\mathcal{D}^*$ under $\land$ and complementation. Hence $H$ is $1 - 1$ and, for each $\Psi \in \text{Aut}(\mathcal{D}^*)$, $\Psi = H(\Psi \upharpoonright (\mathcal{E}^*))$. \hfill \diamondsuit

For the notions from model theory used in the following corollary, see [4], Ch. 5.

**Corollary 2.7** The structures $\mathcal{E}^*$ and $\mathcal{D}^*$ are bi-interpretable.

Proof. In the introduction, we gave an interpretation $\mathcal{D}^* = \Gamma(\mathcal{E}^*)$, representing $v \in \mathcal{D}^*$ by pairs $a, b \in \mathcal{E}^*$ s.t. $v = a - b$. Theorem 2.4 gives an interpretation $\mathcal{E}^* = \Delta(\mathcal{D}^*)$.

We have to show that the isomorphisms

$$G : \mathcal{E}^* \cong \Delta(\Gamma(\mathcal{E}^*))$$

$$H : \mathcal{D}^* \cong \Gamma(\Delta(\mathcal{D}^*))$$

are definable in $\mathcal{E}^*$ and $\mathcal{D}^*$ respectively. For the first, note that $G$ maps $a \in \mathcal{E}^*$ to a pair $(b, c)$ representing $a$ in $\mathcal{D}^*$. Then it is enough to notice that $\{(a; b, c) : a = b - c\}$ is definable in $\mathcal{E}^*$. For $H$, we have to show that the relation

$$(*) \quad \{x; a, b : a, b \text{ c.e.} \land x = a - b\}$$

is definable in $\mathcal{D}^*$. Since $\mathcal{D}^*(\leq a) \cong \mathcal{D}^*$ for any c.e. $a \neq 0$, we can first assume $a = 1$ to obtain a formula $\tilde{\psi}(b, x)$ defining "$b$ c.e. $\land x = \overline{b}$". Then the formula $\psi(a, b, x) = \overline{\psi}^{1, a}(b, x)$ ( $\psi$ relativized to $[0, a]$) will define $(*)$. Recall from the proof of Theorem 2.4 that

$$\{v \in \mathcal{D}^* : v \text{ c.e. low}\} \subseteq \{v : v \text{ has a unique complement}\} \subseteq \{v : v \text{ c.e.}\}$$

So the desired formula is $\tilde{\psi}(b, x) \equiv \exists b_0, b_1$

"$b_0, b_1$ have unique complements" $\land b = b_0 \lor b_1 \land x = \overline{b_0} \land \overline{b_1}$. 

9
REFERENCES


