COMPUTABILITY AND UNCOUNTABLE LINEAR ORDERS I: 
COMPUTABLE CATEGORICITY

NOAM GREENBERG, ASHER M. KACH, STEFFEN LEMPP, AND DANIEL D. TURETSKY

Abstract. We study the computable structure theory of linear orders of size $\aleph_1$ within the framework of admissible computability theory. In particular, we characterize which of these linear orders are computably categorical.

1. Introduction

Effective properties of countable linear orderings have been studied extensively since the 1960’s. This line of research, surveyed in Downey [1], is part of a broader program of understanding the information content of mathematical structures. Among the notions central to this theory are the notions of computable categoricity and the degree spectrum of a structure. A computable structure is said to be computably categorical if it is effectively isomorphic to any of its computable copies. The degree spectrum of a structure is the collection of Turing degrees which contain a copy of the structure. Examples of major results concerning the effective properties of linear orderings are the Dzgoev [3] and Remmel [15] characterization of the computably categorical linear orderings as those with finitely many successivities and the Richter [16] theorem that the computable order-types are the only ones whose degree spectrum contains a least element.

Traditionally, the domain of computability theory consists of hereditarily finite objects (for example the natural numbers, finite sequences and sets of natural numbers, and so on). For this reason, effectiveness considerations have mostly been applied only to countable mathematical structures. Early on, though, generalizations of the theory of computable functions on domains of larger cardinality were considered. Takeuti [19] and [20] generalized recursion theory to the class of all ordinals. Kreisel and Sacks [9] and [10], following work of Kreisel [8], developed
metarecursion theory, which is the study of computability on the computable ordinals, or equivalently, on their notations. These two approaches were unified by Kripke [11] and Platek [14] in the study of recursion theory on admissible ordinals.

Greenberg and Knight [7] initiated the application of admissible computability theory to the study of effectiveness properties of uncountable structures. Under the assumption that all reals are constructible, they investigate the analogues of classical results about fields and vector spaces, results from pure computable model theory such as the relationship between Scott families and computable categoricity, and results about linear orderings.

A main interest in these investigations is the contrast between the countable and the uncountable case. Some results from classical computability theory, about countable structures, generalize to the uncountable case, albeit with sometimes different proofs. Other classical results fail in the uncountable setting. For example, Greenberg and Knight show that Richter’s result mentioned above fails for uncountable cardinals; in fact, in the uncountable setting, every degree is the least degree of the spectrum of some linear ordering. In either case, the examination of classical results in new surroundings sheds light on the classical theory, often by highlighting essential assumptions that go without notice if generalizations are not considered, and by separating notions that happen to coincide in the countable setting.

A major theme arising from this work is the importance of the notion of true finiteness. In \( \omega_1 \)-computability, the correct analogue for “finite” is “countable”. For example, \( \omega_1 \)-computations take countably many steps and manipulate hereditarily countable objects. Yet, true finiteness has some inherent properties which do not generalize. Lerman and Simpson [12] and Lerman [13] exhibited the effects of the difference between true finiteness and its generalization on the lattice of c.e. sets under inclusion; Greenberg [5] has exhibited the effects on the c.e. degrees. For linear orderings, the important observation is that while a finite set can determine only finitely many cuts in a linear ordering, a countable set may determine uncountably many cuts in a linear ordering. This difference underlies the failure of Richter’s theorem for \( \omega_1 \), as well as many of the other differences we shall see.

In this paper, we continue the investigation started by Greenberg and Knight [7], concentrating on linear orderings of size \( \aleph_1 \). We again assume that \( \mathbb{R} \subset L \), the pertinent effect of which is that \( L_{\omega_1} \) is amenable in \( V \); that is, \( L_{\omega_1} \) coincides with \( H_{\omega_1} \), the collection of hereditarily countable sets. This assumption implies the continuum hypothesis in a strong sense: It gives a \( \Delta_1(H_{\omega_1}) \) bijection between \( 2^{\omega} \) and \( \omega_1 \). Also, amenability and the regularity of \( \omega_1 \) imply that the results of this paper hold when relativized to any subset of \( \omega_1 \).

Again, true finiteness plays a central role. However, we observe new aspects of working with linear orderings of size \( \aleph_1 \). We uncover hidden effectiveness conditions which become vacuous when working with countable linear orders. We also rely heavily on the Hausdorff analysis of countable linear orders. Unlike all other results so far, the results in this paper do not easily generalize to cardinalities beyond \( \aleph_1 \).

In this paper we investigate the Dzgoev-Remmel characterization of the computably categorical linear orderings mentioned above. In Theorem 3.1, we find the correct analogue of this characterization for linear orderings of size \( \aleph_1 \). We begin though (Theorem 2.4) with the easier case of uniform effective categoricity. In the sequel to this paper [6] we study degree spectra, both of linear orders and of the successor relation on computable linear orders.
1.1. Notation, Terminology, Background. We refer the reader to Sacks [18] for additional background on admissible computability, and to Greenberg and Knight [7] for specific background on $\omega_1$-computability theory, for definitions and basic facts on $\omega_1$-computable model theory, and for effectiveness properties of linear orderings of size $\aleph_1$ in particular. In order to distinguish computability in the countable case from computability on the admissible ordinal $\omega_1$, we will usually denote computability in the former case as $\omega$-computability and the latter case as $\omega_1$-computability; in this paper, though, when we omit the prefix, we mean $\omega_1$-computability. For the entire paper, we assume that every real is constructible.

We also refer the reader to Rosenstein [17] for additional background on order-types and linear orders. We use the following notation and terminology for linear orders.

Definition 1.1. Let $L = (L, \prec_L)$ be a linear order.

1. A subset $X$ of $L$ is convex, or an $L$-interval, if for all $x, y \in X$ and $z \in L$, if $x \prec_L z \prec_L y$ then $z \in X$.
2. If $A, B \subseteq L$, then we write $A \prec_L B$ if $a \prec_L b$ for all $a \in A$ and $b \in B$; in this case we let $(A, B)_L$ be the $L$-interval determined by $A$ and $B$, be the convex set $\{x \in L : A \prec_L x \prec_L B\}$. If $A = \{a\}$, we also write $a \prec_L B$ and $(a, B)_L$; if $A = \emptyset$ we write $(-\infty, B)_L$, and so on.
3. If $Q \subseteq L$, then a cut of $Q$ is a partition of $Q$ into subsets $Q_1$ and $Q_2$ such that $Q_1 \prec_L Q_2$.
4. If $Q \subseteq L$, then a $Q$-interval of $L$ is an $L$-interval determined by some cut of $Q$.
5. A block of $L$ is a nonempty convex subset $X$ of $L$ such that for all $a, b \in X$, the interval $(a, b)_L$ is finite. Note that every block is at most countable in size.
6. Let $L$ be a linear ordering. A pair of elements $a \prec_L b$ in $L$ are adjacent if $(a, b)_L$ is empty. We say that $a$ is the predecessor of $b$ (in $L$) and $b$ is the successor of $a$ (in $L$).

An order-type is an isomorphism class of linear orderings, although we often identify the order-type of a well-ordering with the unique ordinal it contains. We write $\text{otp}(L)$ for the order-type of a linear ordering $L$. An element of an order-type $\lambda$ is also called a presentation of $\lambda$.

If $P$ is a property of linear orderings, then we say that an order-type $\lambda$ has property $P$ if some presentation of $\lambda$ has property $P$. Hence, we say that $\lambda$ is computable if it has a computable presentation, and that the size of $\lambda$ is some cardinal $\kappa$ if the presentations of $\lambda$ have cardinality $\kappa$.

Remark 1.2. In this paper, we assume that the universe of any linear ordering is a subset of $H_{\omega_1}$. So each order-type $\lambda$ is a set (rather than a proper class). If its presentations are of size $\aleph_1$, then the set $\lambda$ has size $2^{\aleph_1}$. Nonetheless we say that the size of $\lambda$ is $\aleph_1$, as we only care about the size of the presentations of $\lambda$.

We fix notation for some order-types which will appear often in this paper.

Definition 1.3. We denote the order-type of the natural numbers by $\omega$ and of the least uncountable ordinal by $\omega_1$.

We denote the order-type of the rational numbers by $\eta$ (also by $\eta_0$). This is the saturated countable order-type. We denote the saturated order-type of size $\aleph_1$ by $\eta_1$. 

We fix notation for some order-types which will appear often in this paper.
We denote the order-type of the integers \( \mathbb{Z} \) by \( \zeta \) and the order-type of the real numbers \( \mathbb{R} \) by \( \rho \).

We note the existence of \( \eta_1 \) follows from the continuum hypothesis. By Cantor’s argument, a linear order \( \mathcal{L} \) of size \( \aleph_1 \) is saturated if and only if for any at most countable sets \( A, B \subseteq L \) such that \( A <_\mathcal{L} B \), the interval \( (A, B)_\mathcal{L} \) is nonempty. Note that as \( A \) or \( B \) may be empty, this implies that \( \mathcal{L} \) has uncountable cofinality and cofinality.

It will often be important whether a linear order has a subset of order-type \( \eta_0 \).

**Definition 1.4.** A linear order is *nonscattered* if it has a subset of order type \( \eta_0 \) and *scattered* otherwise.

We use standard sum and product notation: \( A + B \) for appending \( B \) to the right of \( A \) and \( A \cdot B \) for replacing every point of \( B \) by a copy of \( A \). As these operations are invariant under isomorphisms, we extend the notation to order-types as well.

We also use restrictions of linear orderings.

**Definition 1.5.** Let \( A = (A, <_A) \) be a linear ordering. If \( B \subseteq A \), then we let \( A | B \) be the linear ordering \( (B, <_A|B^2) \). If \( A \subseteq \omega_1 \) and \( \alpha < \omega_1 \), then we let \( A | \alpha \) be \( A | (A \cap \alpha) \), recalling that a von Neumann ordinal is the collection of its predecessors. We also denote \( A | \alpha \) by \( A_\alpha \).

We recall the basic definitions of \( \omega_1 \)-computability. We work with the structure \( (H_{\omega_1}; \in) \) enriched by constants naming all the elements of the structure. Note that under the assumption \( \mathbb{R} \subset L, H_{\omega_1} = L_{\omega_1} \). A formula (with parameters from \( H_{\omega_1} \)) is \( \Delta_0(H_{\omega_1}) \) if all of its quantifiers are bounded. A formula is \( \Sigma_1(H_{\omega_1}) \) if it is of the form \( \exists \vec{x} \varphi \) where \( \varphi \) is \( \Delta_0(H_{\omega_1}) \).

**Definition 1.6.** A relation \( R \subseteq (H_{\omega_1})^n \) is \( \omega_1 \)-computably enumerable if it is definable by a \( \Sigma_1(H_{\omega_1}) \) formula. A relation \( R \subseteq (H_{\omega_1})^n \) is \( \omega_1 \)-computable if both it and its complement \( H_{\omega_1} \setminus R \) are \( \omega_1 \)-c.e. A partial function \( f : (H_{\omega_1})^n \rightarrow H_{\omega_1} \) is partial \( \omega_1 \)-computable if its graph \( \{(\bar{a}, f(\bar{a})) : \bar{a} \in \text{dom } f\} \) is an \( \omega_1 \)-c.e. relation. An \( \omega_1 \)-computable function \( f : (H_{\omega_1})^n \rightarrow H_{\omega_1} \) is a partial \( \omega_1 \)-computable function whose domain is \( \omega_1 \)-computable.

The main tool of computability is recursion.

**Proposition 1.7.** Let \( I : H_{\omega_1} \rightarrow H_{\omega_1} \) be a computable function. Then there is a unique computable function \( f : \omega_1 \rightarrow H_{\omega_1} \) such that for all \( \alpha < \omega_1 \), \( f(\alpha) = I(f|\alpha) \).

Defining computable functions by recursion allows us to view them dynamically, as is common in countable computability. Processes of computation, described by \( \Delta_0(L_{\omega_1}) \) formulas, for instance, and taking only countably many steps, can be used to define computable functions.

Familiar facts about computability in the countable setting lift to \( \omega_1 \)-computability with identical reasoning. For example, a set \( A \subseteq H_{\omega_1} \) is \( \omega_1 \)-computable if and only if its characteristic function is \( \omega_1 \)-computable. There is an \( \omega_1 \)-computable bijection between \( \omega_1 \) and \( H_{\omega_1} \). The standard well-ordering of \( L_{\omega_1} = H_{\omega_1} \) of order-type \( \omega_1 \) is \( \omega_1 \)-computable. A nonempty subset \( A \) of \( H_{\omega_1} \) is \( \omega_1 \)-c.e. if and only if it is the domain of a partial \( \omega_1 \)-computable function if and only if it is the range of an \( \omega_1 \)-computable function. We can effectively list (in order-type \( \omega_1 \)) all \( \Sigma_1(H_{\omega_1}) \).
formulas and so can effectively list all partial $\omega_1$-computable functions (as $(\Phi_e)_{e<\omega_1}$) and $\omega_1$-c.e. sets (as $(W_e)_{e<\omega_1}$). This means that the set $\{(e,x): x \in W_e\}$ is $\omega_1$-c.e.; we often denote it by $\emptyset'$. There is an effective, uniform enumeration of all $\omega_1$-c.e. sets: a uniformly $\omega_1$-computable double sequence $(W_{e,s})_{s,e<\omega_1}$ such that for all $e$, $W_e = \bigcup_s W_{e,s}$. With a standard proof, the Fixed Point Theorem (Recursion Theorem) holds: If $f: \omega_1 \to \omega_1$ is $\omega_1$-computable then there is some $e < \omega_1$ such that $\Phi_e = \Phi f(e)$.

An intuition for informal definitions of such computable objects develops with experience. As an example we observe the following

**Fact 1.8.** The collection of countable scattered linear orderings is $\omega_1$-computable.
We mean the subset of $H_{\omega_1}$ which consists of binary relations which define a linear ordering of some set (of course, also an element of $H_{\omega_1}$). To see this, first note that the collection of linear orderings is definable by a formula only using bounded quantifiers.

However we can give a “decision procedure” for the set of scattered linear orders. We observe that the Hausdorff analysis of scattered linear orderings can be defined by effective recursion. Given a countable linear order $L$, we let $L'$ be the linear order obtained by identifying points which are finitely far apart. The graph of the function $L \mapsto L'$ is definable by a bounded formula. By effective recursion, we can now iterate the Hausdorff derivative (transfinitely if necessary, taking direct limits at limit steps) until we get either a dense or empty linear ordering. Which is the case can be observed effectively.

Since we focus on linear orderings, in this paper we do not need the general definition of an $\omega_1$-computable structure. A linear ordering $L$ of $\omega_1$ is $\omega_1$-computable if it is $\omega_1$-computable as a relation (a set of pairs). As in the countable context, we can effectively list $\omega_1$-computable order-types:

**Fact 1.9.** There is a uniformly $\omega_1$-computable list $(L_\beta)_{\beta<\omega_1}$ of $\omega_1$-computable linear orderings such that for any $\omega_1$-computable linear ordering $A$ there is some $\beta < \omega_1$ such that $A \cong L_\beta$.

2. Uniform $\omega_1$-Computable Categoricity

In this paper, we characterize the $\omega_1$-computably categorical and uniformly $\omega_1$-computably categorical linear orders. We recall the appropriate definitions.

**Definition 2.1.** Fix a cardinal $\kappa \in \{\omega, \omega_1\}$.

A $\kappa$-computable order-type $\lambda$ is $\kappa$-computably categorical if for all $\kappa$-computable $A, B \in \lambda$ there is a $\kappa$-computable isomorphism $f: A \cong B$.

A $\kappa$-computable order-type $\lambda$ is uniformly $\kappa$-computably categorical if there is a $\kappa$-computable function mapping a pair of indices of two $\kappa$-computable presentations $A, B \in \lambda$ to an index of a $\kappa$-computable isomorphism between them.

If $L$ is a $\kappa$-computable linear order, then we say that $L$ is (uniformly) $\kappa$-computably categorical if its order-type is (uniformly) $\kappa$-computably categorical.

As mentioned in the introduction, in the countable framework, Dzgoev [3] (see also [4]) and Remmel [15] independently showed that a computable linear ordering...
is computably categorical if and only if it has finitely many adjacencies (see Definition 1.1). Equivalently, a computable linear ordering \( L \) is computably categorical if and only if there is a finite set \( C \subseteq L \) such that every \( L \)-interval determined by \( C \) is either finite or has order-type \( \eta_0 \).

While these two characterizations are equivalent for \( \omega \)-computable linear orderings, their generalizations to uncountable linear orderings are not. The first does not generalize to a characterization of \( \omega_1 \)-computably categorical linear orderings. Na"ively, one would guess that an \( \omega_1 \)-computable linear ordering is \( \omega_1 \)-computably categorical if and only if it has only countably many adjacencies (or perhaps countably many countable intervals). The next two examples show that these conditions are neither necessary nor sufficient for \( \omega_1 \)-computable categoricity.

Example 2.2. The order-type \( 2 \cdot \rho \) is \( \omega_1 \)-computably categorical. To see this, fix computable presentations \( A \) and \( B \) of \( 2 \cdot \rho \). We may fix the “dense” countable subsets \( 2 \cdot \eta \) of \( 2 \cdot \rho \) in both \( A \) and \( B \) as a parameter. Then for any point in \( A \) or \( B \), we can determine whether it is the “left” or “right” point of its pair simply by waiting until both have shown up in the same interval determined by the copy of \( 2 \cdot \eta \).

Example 2.3. The order-type \( \eta \cdot \omega_1 \) is not \( \omega_1 \)-computably categorical. To see this, we construct computable presentations \( A \) and \( B \) of \( \eta \cdot \omega_1 \) meeting the requirement

\[ R_e : \text{The function } \Phi_e \text{ is not an isomorphism from } A \text{ to } B. \]

for all \( e \in \omega_1 \).

In order to satisfy \( R_e \), we wait for \( \Phi_e \) to be completely defined on some copy of \( \eta \) in \( A \), where the image of this copy in \( B \) is greater than the restraint. We then add an extra point to \( B \) within the image and move the restraint (for \( R_j \) with \( j > e \)) to a point in \( B \) greater than the image of this copy.

As a step towards characterizing the \( \omega_1 \)-computably categorical linear orderings, we treat the uniform case.

**Theorem 2.4.** An order-type \( \lambda \) is uniformly \( \omega_1 \)-computably categorical if and only if \( \lambda \) is finite or \( \lambda = \eta_1 \).

**Remark 2.5.** We note that not only is the order-type \( \eta_1 \) uniformly \( \omega_1 \)-computably categorical, the effective back-and-forth argument demonstrating uniform \( \omega_1 \)-computable categoricity shows that if \( A \) and \( B \) are computable presentations of \( \eta_1 \), we can effectively extend any countable partial embedding \( \psi : A \rightarrow B \) to an isomorphism between \( A \) and \( B \). This is uniform given \( \psi \) and \( \omega_1 \)-computable indices for \( A \) and \( B \).

**Proof of Theorem 2.4.** Every finite order-type is clearly uniformly \( \omega_1 \)-computably categorical. An effective back-and-forth argument of length \( \omega_1 \) shows that \( \eta_1 \) is uniformly \( \omega_1 \)-computably categorical. This establishes one direction of the theorem.

In order to prove the other direction, let \( \lambda \) be an infinite, uniformly \( \omega_1 \)-computably categorical order-type, and let \( L \) be a computable presentation of \( \lambda \). We show that \( L \) is \( \kappa_1 \)-saturated. To do this, given countable subsets \( A \) and \( B \) of \( L \) such that \( A <_L B \), we “force” \( L \) to enumerate a point between \( A \) and \( B \).

This is done by building an auxiliary \( \omega_1 \)-computable linear ordering \( K \). We ensure that \( K \) is isomorphic to \( L \). By the Fixed Point Theorem, we know an \( \omega_1 \)-computable
We ensure (even in the case that \( \Phi \) is \( \partial \), \( \omega \)) shape to our wishes.

In greater detail, the Fixed Point Theorem is applied as follows. For each \( e < \omega_1 \), we perform a separate construction. The \( e \)th construction observes the \( e \)th partial \( \omega_1 \)-computable function \( \Phi^e \) and builds an \( \omega_1 \)-computable linear order \( K^e \). We ensure (even in the case that \( \Phi^e \) is not total) that \( K^e \) is isomorphic to \( L \). Since the construction of \( K^e \) is effective, uniformly in \( e \), we obtain an \( \omega_1 \)-computable function \( f: \omega_1 \rightarrow \omega_1 \) such that for all \( e \), \( \Phi f(e) \) is an isomorphism from \( K^e \) to \( L \). By the Fixed Point Theorem, there is some \( e^* < \omega_1 \) such that \( \Phi^e^* = \Phi f(e^*) \). The construction we eventually use is given by this \( e^* \). Letting \( K = K^{e^*} \) and \( \Phi = \Phi^e^* \), during this “real” construction, we build \( K \) while knowing the \( \omega_1 \)-computable function \( \Phi \), which is an isomorphism from \( K \) to \( L \).

However, we need to ensure that each \( K^e \) is isomorphic to \( L \). To do this, we define, for each \( e \) and each \( s < \omega_1 \), an isomorphism from \( K^e_s \) (our stage \( s \) approximation to \( K^e \)) to \( L_s = L \upharpoonright s \). We will ensure that if \( \Phi^e \) is not an isomorphism from \( K^e \) to \( L \), then the sequence \( \langle F^e_s \rangle_{s < \omega_1} \) reaches a limit \( F^e \) which we will ensure is an isomorphism from \( K^e \) to \( L \). From the point of view of the correct construction \( e^* \), at each stage \( s \) we need to define an isomorphism \( F_s = F^e_s \) from \( K_s = K^e_s \) to \( L_s \), even though we do not need the maps \( F_s \) to converge to an isomorphism from \( K \) to \( L \).

We restrict ourselves now to the correct construction \( e^* \) and go back to explaining how the auxiliary order \( K \) and the isomorphism \( \Phi \) are used to control the structure of \( L \). At some stage \( s \), we observe countable subsets \( C \) and \( D \) of \( L_s \) with \( C <_L D \) and \( (C, D)_{L_s} = \emptyset \). The plan is to add a point \( z \) to \( K_{s+1} \) between \( \Phi^{-1}(C) \) and \( \Phi^{-1}(D) \). If we do this, since \( \Phi \) is indeed an isomorphism from \( K \) to \( L \), \( \Phi(z) \) must be a point in \( L \) between \( C \) and \( D \). If we keep track correctly, we can thus treat any pair \( C <_L D \) of countable sets and so show that \( L \) is \( \aleph_1 \)-saturated.

Recall, however, that we need to define an isomorphism \( F_{s+1}: K_{s+1} \rightarrow L_{s+1} \). The eventual point \( \Phi(z) \) may be enumerated into \( L \) much later. As a result, we need to find an embedding \( g: K_s \cup \{ z \} \rightarrow L_{s+1} \) and then add more points to \( K_s \cup \{ z \} \) to define \( K_{s+1} \) such that \( g \) can be extended to the desired \( F_{s+1} \). The map \( g \) (and so \( F_{s+1} \)) may disagree with \( \Phi \). See Figure 1.

![Figure 1. Saturating \( L \). The point \( z \) is added to \( K_{s+1} \), and the result is embedded by \( F_{s+1} \) into \( L_{s+1} \). \( F_{s+1} \) does not agree with \( \Phi \).](image)

Unfortunately, we cannot always guarantee the existence of an embedding of \( K_s \cup \{ z \} \) into \( L_{s+1} \). If \( L_s \) is nonscattered, then such an embedding is ensured as every countable linear order is embeddable into any nonscattered linear order. If \( L_{s+1} \) is
scattered, then there may not be an embedding $g$ as desired. Thus, before executing the above strategy, we work towards guaranteeing that $L$ is nonscattered.

Again, we utilize the auxiliary order $K$. If $L_s$ is infinite and scattered, there is necessarily an infinite block $B$ in $L_s$. We then add a point between any adjacent points in $\Phi^{-1}(B)$. Again, since $\Phi: K \to L$ is an isomorphism, this means that $B$ is not really a block of $L$. If we keep books wisely, we will be able to arrange that $L$ does not have infinite blocks and so will be nonscattered. In turn, this would mean that for some $s$, $L_s$ is nonscattered, and so eventually we could return to the strategy described earlier for making $L_s$ saturated. Again, we need to define $F_{s+1}$; here we let $F_{s+1}$ agree with $\Phi$ outside $\Phi^{-1}[B]$, but can “correct” $\Phi$ on $\Phi^{-1}[B]$ together with the new points in $K_{s+1}$ to an isomorphism with $B$; this depends on the shape of the block. See Figure 2 for the case that $B \cong \mathbb{Z}$.

![Figure 2. Descattering $L$.](image)

Points are added between adjacent points in $\Phi^{-1}[B]$. $F_{s+1}$ (dashed) is an isomorphism between the new $\zeta$ in $K_{s+1}$ and $B$.

The construction is thus split into two phases: a descattering phase and a saturating phase. We employ the descattering strategy while $L_s$ is scattered; once $L_s$ becomes nonscattered, we follow the saturating strategy. Note that both strategies above rely on the fact that at stage $s$ we have access to $\Phi \restriction K_s$ and that $\Phi \restriction K_s$ is an isomorphism from $K_s$ to $L_s$. The regularity of $\omega_1$ implies the existence of a closed and unbounded set of stages $s$ at which this is the case, and so we restrict our action to these stages. While we are waiting for the next such stage (which can be forever in the $e^{th}$ construction for some $e \neq e^*$), we need to ensure that $K_s$ is isomorphic to $L_s$. This can be done without changing the values of $F$, and so on intervals of stages $t$ on which we don’t act we will obtain an increasing sequence of isomorphisms $F_t$. This allows us to define $F_s$ for all limit stages $s$ (and in particular ensure that $K_s \equiv L_s$ for all limit stages). Either $F_t$ stabilizes below $s$ and the union map $F_s$ is an isomorphism; or we act cofinally in $s$, in which case $s$, too, is a stage at which we see $\Phi \restriction K_s$ to be an isomorphism from $K_s$ to $L_s$, in which case we can simply let $F_s = \Phi \restriction K_s$. If $e \neq e^*$ is a “failed” construction then we eventually cease changing $F^e$, which will ensure that $F^e$ is an isomorphism from $K^e$ to $L$ as required.

**Construction e:** Since $L$ is infinite, we may assume that $L_\omega$ is infinite. We define an increasing and continuous sequence $\langle K^e_\omega \rangle_{\omega \leq s < \omega_1}$ of countable linear orderings; and for each $s$ with $\omega \leq s \leq \omega_1$, an isomorphism $F_s^e: K_s \to L_s$. We start with $K^e_\omega := L_\omega$ and $F_\omega^e := id_{K^e_\omega}$.

Let $s < \omega_1$ be infinite, and suppose that $K_s^e$ and $F_s^e$ are already defined. We first define an embedding $\tilde{F}_s^e$ of $K_s^e$ into $L_s$. After $\tilde{F}_s^e$ is defined, we let $K_{s+1}^e$ be an
extension of $K^e$ to a countable linear ordering such that we can extend $F^e$ to an isomorphism $F^e_{s+1}: K^e_{s+1} \rightarrow L_{s+1}$; this will conclude stage $s$.

We define $F^e_s$. Let $\Phi^e_s$ be the function $\Phi^e$, restricted to the inputs $x$ such that $\Phi^e(x)$ converges before stage $s$. At stage $s$, we check if $\Phi^e_s$ is an isomorphism from $K^e_s$ to $L_s$. If not, then we let $F^e_s := F^e_s$. This means that in this case, $F^e_{s+1}$ will extend $F^e_s$.

Suppose that $\Phi^e_s: K^e_s \rightarrow L_s$ is an isomorphism. There are two cases, depending on whether $L_s$ is scattered or not.

- **Descattering**: If $L_s$ is scattered, we let $B_s = \prec_{\omega_1}$-least infinite block of $L_s$. Since the order-type of $B_s$ is either $\omega$, $\omega^*$, or $\zeta$, there is a self-embedding $f_s$ of $L_s$ (which we can take to be the identity outside $B_s$, though this is unimportant) such that for all adjacent $a <_L b$ in $B_s$, $f_s(a)$ and $f_s(b)$ are not adjacent in $L_s$. We pick such an embedding $f_s$. We let $F^e_s := f_s \circ \Phi^e_s$.

- **Saturating**: If $L_s$ is nonscattered, we let $(C_s,D_s)$ be the $\prec_{\omega_1}$-least pair of countable sets $C,D \subseteq L_s$ such that $C <_L D$ and $(C,D)_{L_s}$ is empty. Since $L_s$ is nonscattered, there is a self-embedding $f_s$ of $L_s$ such that $(f_s[C_s],f_s[D_s])_{L_s}$ is nonempty (add a point to $L_s$ between $C_s$ and $D_s$ and embed the result into $L_s$). We let $F^e_s := f_s \circ \Phi^e_s$.

To complete the construction, we need to define $F^e_s$ for limit stages $s$, since we already stipulated that $K^e_s := \bigcup_{s < t} K^e_t$ for limit $t$. Let $J^e$ be the set of stages $t$ such that $\Phi^e_t$ is an isomorphism from $K^e_t$ to $L_t$. Let $s$ be a limit stage. If $J^e \cap s$ is bounded below $s$, then (by induction) for all $r < s$ in the interval $(\sup(J^e \cap s), s)$, we have $F^e_r \subseteq F^e_s$. It then follows that $F^e_s := \bigcup_{t \in (\sup(J^e \cap s), s)} F^e_t$ is an isomorphism between $K^e_s$ and $L_s$. If $J^e \cap s$ is unbounded below $s$, then $s \in J^e$ and so we let $F^e_s := \Phi^e_s$.

**Verification**: Let $K^e := \bigcup_{s < \omega_1} K^e_s$. We first show that $K^e$ and $L$ are isomorphic (for all $e$). One point is that $J^e$ is unbounded in $\omega_1$ if and only if $\Phi^e$ is total and is an isomorphism from $K^e$ to $L$; in the right-to-left direction we use the fact that $\omega_1$ is regular and that the sequences $\langle K^e_s \rangle$ and $\langle L_s \rangle$ are continuous. So if $J^e$ is unbounded in $\omega_1$ then $\Phi^e$ witnesses that $K^e$ and $L$ are isomorphic. On the other hand, if $J^e$ is bounded below $\omega_1$, then after stage $\sup(J^e)$, no action is taken to change $F^e_s$, and so $F^e := \bigcup_{s > \sup(J^e)} F^e_s$ is an isomorphism between $K^e$ and $L$. Hence in either case $K^e$ and $L$ are isomorphic.

Now that we know that $K^e$ and $L$ are isomorphic for all $e$, we can carry out the plan using the Fixed Point Theorem. We obtain $e^*$ such that $\Phi^{e^*}: K^{e^*} \rightarrow L$ is an isomorphism. From now we only consider the $e^*$-construction. Let $K = K^{e^*}$, $J = J^{e^*}$, and so on. We know that $J$ is unbounded in $\omega_1$.

We show that $L$ is nonscattered. Suppose, for a contradiction, that $L$ is scattered. Hence for all $s$, the order $L_s$ is scattered. Now we observe that if $s < t$ are both in $J$, then $B_s \neq B_t$ (where recall that $B_s$ is the $\prec_{\omega_1}$-least infinite block of $L_s$). For let $a,b \in B_s$ be adjacent in $B_s$. The definition of $F_s$ and the fact that $f_s(a)$ and $f_s(b)$ are not adjacent in $L$ means that $K_{s+1}$ contains a point $z$ between $\Phi^{-1}(a)$ and $\Phi^{-1}(b)$. Then $z \in K_s$. Since $\Phi_s$ is an isomorphism between $K_s$ and $L_s$ (and, of course, $\Phi_t$ extends $\Phi_s$), we see that $a$ and $b$ cannot be adjacent in $L_t$. In particular, $B_s$ is not a block of $L_t$, so $B_s \neq B_t$. 
Now the fact that \( J \) is unbounded in \( \omega_1 \) shows that \( \mathcal{L} \) is nonscattered. For if \( \mathcal{L} \) is scattered, then it contains an infinite block. Let \( B \) be the \( \prec_{\omega_1} \)-least infinite block of \( \mathcal{L} \). Being an infinite block of \( \mathcal{L} \) is a \( \Pi^0_1 \) property; this and the regularity of \( \omega_1 \) implies that for all but countably many \( s \in J \), \( B_s = B \). This contradicts the fact that \( J \) is unbounded and the fact that \( s < t \) in \( J \) implies \( B_s \neq B_t \).

Let \( s_0 \) be the least stage such that \( \mathcal{L}_{s_0} \) is nonscattered. We now show that \( \mathcal{L} \) is \( \aleph_1 \)-saturated. The proof is similar. First we observe that if \( s_0 < s < t \) and \( s, t \in J \), then \( (C_s, D_s) \neq (C_t, D_t) \). For the definition \( \Phi_s = f_s \circ \Phi \), and the property of \( f_s \) imply that the interval \((\Phi^{-1}(C_s, \Phi^{-1}(D_s))_{K^{s+1}} \) is nonempty, and so as \( \Phi_t \) is an isomorphism from \( K_t \) to \( L_t \), the interval \((C_s, D_s)_{K^{s+1}} \) is nonempty. We can then show that no pair \((C, D)\) can be the \( \prec_{\omega_1} \)-least pair of countable subsets \( C <_{\mathcal{L}} D \) such that \((C, D)_{C} \) is empty, as this would contradict that \( J \) is unbounded in \( \omega_1 \); again, the property defining the pair \((C, D)\) is \( \Pi^0_1 \). Hence \( \mathcal{L} \) is \( \aleph_1 \)-saturated, which completes the proof.

3. \( \omega_1 \)-Computable Categoricity

We turn to the main result of this paper, the characterization of \( \omega_1 \)-computably categorical linear orderings. During earlier work on this subject, trying to generalize the Remmel-Dzgoev criterion, Knight conjectured that a linear ordering \( \mathcal{L} \) is \( \omega_1 \)-computably categorical if and only if there is a countable subset \( Q \) of \( \mathcal{L} \) and a number \( n \) such that every \( Q \)-interval of \( \mathcal{L} \) is either empty, contains exactly \( n \) points, or is \( \aleph_1 \)-saturated. While not quite correct, this conjecture does contain an important ingredient which is correct: If \( \mathcal{L} \) is \( \omega_1 \)-computably categorical, then there is some countable subset \( Q \) of \( \mathcal{L} \) such that every \( Q \)-interval is either finite or has order-type \( \eta_1 \).

The added ingredient is effectiveness. An ordering \( \mathcal{L} \) with a countable subset \( Q \) can be \( \omega_1 \)-computably categorical, witnessed by \( Q \), even if \( \mathcal{L} \) contains finite \( Q \)-intervals of different sizes. However, for each \( n \), we need to effectively enumerate those cuts of \( Q \) that define intervals that may have size \( n \). This added ingredient sheds light on the countable case as well. The characterization below of \( \omega_1 \)-computable categoricity is a correct characterization of \( \omega \)-computable categoricity if we replace “countable” by “finite”. The special properties of the cardinal \( \omega \) make the effectiveness condition redundant in the countable case. The uncountable case allows us to recover this important aspect of the criterion, which is invisible if one only sees the countable context.

The effectiveness condition of Theorem 3.1 implies another difference between countable and uncountable linear orderings. Given the theorem (and relativizing it), it is easy to construct an order-type \( \omega \) of size \( \aleph_1 \) with \( \omega_1 \)-computable presentations which is not \( \omega_1 \)-computably categorical but is relatively \( \omega_1 \)-computably categorical above \( \mathbf{d} \): There is a degree \( \mathbf{d} \) such that any two presentations \( \mathcal{L}_1, \mathcal{L}_2 \geq \mathbf{d} \) of \( \lambda \) are \( (\mathcal{L}_1 \oplus \mathcal{L}_2) \)-\( \omega_1 \)-computably isomorphic. There are no such countable order-types: If \( \lambda \) is a countable order-type with \( \omega \)-computable elements that is not \( \omega \)-computably categorical, then for every \( \omega \)-Turing degree \( \mathbf{d} \) there are \( \mathbf{d} \)-computable presentations \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) of \( \lambda \) which are not isomorphic by any \( \mathbf{d} \)-computable isomorphism.

**Theorem 3.1.** An \( \omega_1 \)-computable linear order \( \mathcal{L} \) is \( \omega_1 \)-computably categorical if and only if there are a countable set \( Q \subset \mathcal{L} \) and a collection \( \{V_n : 0 < n < \omega\} \) of pairwise disjoint \( \omega_1 \)-c.e. sets of cuts of \( Q \) with the following properties:
Every $Q$-interval of $\mathcal{L}$ is either finite or has order-type $\eta_1$.

(2) For any cut $(Q_1, Q_2)$ of $Q$, if the $Q$-interval $(Q_1, Q_2)_\mathcal{L}$ has size $n > 0$, then $(Q_1, Q_2) \in V_n$.

Note that since the c.e. sets $V_n$ are pairwise disjoint, it follows that if $(Q_1, Q_2) \in V_n$ then the interval $(Q_1, Q_2)_\mathcal{L}$ is either empty, has size $n$, or is $\aleph_1$-saturated.

Proof. $(\iff)$ Let $\mathcal{L}$ be an $\omega_1$-computable linear order, equipped with sets $Q$ and $\{V_n\}$ as described in the theorem. To show that $\mathcal{L}$ is $\omega_1$-computably categorical, let $K$ be an $\omega_1$-computable linear order which is isomorphic to $\mathcal{L}$, and let $g: \mathcal{L} \to K$ be an arbitrary (not necessarily effective) isomorphism. We define an $\omega_1$-computable isomorphism $f: \mathcal{L} \to K$ by starting with $g \upharpoonright Q$. We extend $g \upharpoonright Q$ to a map $f$ on $\mathcal{L}$ by defining $f$ on every $Q$-interval. Let $A := (Q_1, Q_2)_\mathcal{L}$ be a $Q$-interval; let $B := g[A] = (g[Q_1], g[Q_2])_K$. If $A$ is empty, we do not need to define $f$ on $A$. If $A$ is nonempty, we wait for a stage $s$ at which either $A_s := (Q_1, Q_2)_{\mathcal{L}\upharpoonright s}$ is infinite; or $(Q_1, Q_2) \in V_n$ at stage $s$, $|A_s| = n$, and $B_s := (g(Q_1), g(Q_2))_{\mathcal{K}\upharpoonright s}$ also has size $n$ for some positive $n < \omega$. At least one of the two has to happen. Here we use the fact that since the collection of sets $\{V_n\}$ is countable, the sequence $\langle V_n \rangle$ is uniformly c.e. Now in the latter case, we define $f$ to be the order-preserving bijection between $A_s$ and $B_s$. In the former case, we know that both $A$ and $B$ are $\aleph_1$-saturated, so an $\omega_1$-computable isomorphism between $A$ and $B$ can be built uniformly from our indices $(Q_1, Q_2)_\mathcal{L}$ and $(g(Q_1), g(Q_2))_K$ for $A$ and $B$. If we first see that $|A_s| = n = |B_s|$ and $(Q_1, Q_2) \in V_n$ and define $f$ on $A_s$, and then more points are added to $A$, it must be that $A$ and $B$ have order-type $\eta_1$. The map $f \upharpoonright A_s$ can be uniformly extended to an $\omega_1$-computable isomorphism between $A$ and $B$.

$(\implies)$ Let $\mathcal{L}$ be an $\omega_1$-computable, $\omega_1$-computably categorical linear order. We want to find sets $Q$ and $V_n$ as in the theorem. We attempt to emulate the proof of Theorem 2.4. To show that $\mathcal{L}$ has the desired form, we construct an auxiliary $\omega_1$-computable linear ordering $\mathcal{K}$ isomorphic to $\mathcal{L}$ and use an $\omega_1$-computable isomorphism between $\mathcal{K}$ and $\mathcal{L}$ in order to force $\mathcal{L}$ to add points in locations we choose. Since the $\omega_1$-computable categoricity of $\mathcal{L}$ may fail to be uniform, this time we only have one construction (we construct one $\mathcal{K}$ rather than $\omega_1$ many $\mathcal{K}^*$); but we need to guess which $\omega_1$-computable function is the $\omega_1$-computable isomorphism between $\mathcal{K}$ and $\mathcal{L}$. Let $(\Phi_j)_{j<\omega_1}$ list all partial $\omega_1$-computable functions. The guess $R_j$ guesses that $\Phi_j$ is an isomorphism from $\mathcal{K}$ to $\mathcal{L}$.

As in the previous proof, we build $\mathcal{K}$ as the union of an increasing, continuous, $\omega_1$-computable sequence $\langle K_s \rangle$ of countable linear orders. When $\Phi_j, s$ is an isomorphism between $K_s$ and $L_s$, we guess that $R_j$ is correct. If we succeed in making $\mathcal{K}$ isomorphic to $\mathcal{L}$ then some $R_j$ will be correct. On the stages at which we guess this $R_j$ is correct we would like to implement the strategy employed for proving Theorem 2.4.

It is more difficult to ensure that $\mathcal{K}$ is indeed isomorphic to $\mathcal{L}$. As before we construct maps $F_s: K_s \to L_s$. The aim is that in the contradictory event that no $R_j$ is correct, $\lim_{s \to \omega_1} F_s$ exists and is an isomorphism from $\mathcal{K}$ to $\mathcal{L}$. We need to consider the possibility that eventually, each $R_j$ does not appear correct any more, but the stages at which some $R_j$ appears correct are unbounded in $\omega_1$. For notational convenience, we will define (Definition 3.4) for each $j < \omega_1$ a set of stages $J_j$, a subset of the set of stages $s$ such that $\Phi_{j,s}$ is an isomorphism from $K_s$
to $L$. Let $s_j = \sup J_j$ and for simplicity suppose that $s_0 < s_1 < s_2 < \cdots$ are all countable but $\sup_{j<\omega_1} s_j = \omega_1$. If at stage $s_j$ we naively define $F_{s_j} = \Phi_{j,s}$ then there is no guarantee that the sequence $(F_s)$ converges pointwise. For this reason we view the construction as a priority construction, with $R_j$ assigned a stronger priority than $R_i$ if $j < i$. If a guess $R_j$ receives attention at some stage $t$ and defines $F_{t+1}$, and a weaker guess $R_i$ wishes to define $F_{s+1}$ at a later stage $s$, then unless $R_j$ acted between stages $t$ and $s$, $R_i$ is required to let $F_{s+1}$ extend $F_{t+1}$ (see Claim 3.3).

The restraint imposed by stronger guesses complicates the individual strategy of each guess. Suppose that $R_j$ is the strongest guess which is correct. For all $i < j$, we eventually stop believing that $R_i$ is correct. If $s^* - 1$ is the last stage at which any $R_i$ for $i < j$ receives attention (or $s^*$ is the limit of the stages at which any $R_i$ for $i < j$ receives attention) then at all stages $s \geq s^*$ we are required to let $F_s$ extend $F_{s^*}$. This means that during stages $s \in J_j$ beyond $s^*$, the guess $R_j$ must play its strategy on each $K_{s^*}$-interval separately (as $K_{s^*} = \text{dom} F_{s^*}$). The set range $F_{s^*}$ is a first approximation of the desired set of parameters $Q$.

This approximation to the definition of $Q$ is not quite correct because of an annoying fact: $\Phi_j$ can be an isomorphism from $K$ to $L$ which does not extend $F_{s^*}$. Fix a cut $(S_1, S_2)$ of $K_{s^*}$. For $s \geq s^*$ (including $s = \omega_1$) we let $A_s = (S_1, S_2)|_{\text{dom} \Phi_j}$; $B_s = (F_s[S_1], F_s[S_2])|_{\text{dom} \Phi_j}$; and $C_s = (\Phi_j[S_1], \Phi_j[S_2])|_{\text{dom} \Phi_j}$. Disagreement between $\Phi_j$ and $F_{s^*}$ could cause $B_s$ and $C_s$ to be distinct. If $s \in J_j$ then $\Phi_j, s \upharpoonright A_s$ is an isomorphism from $A_s$ to $C_s$. Since $F_s$ extends $F_{s^*}$, the map $F_s \upharpoonright A_s$ is an embedding of $A_s$ into $B_s$, but we will not always be able to ensure that it is onto $B_s$. See Figure 3.

![Figure 3](image_url)

**Figure 3.** The intervals $A_s$, $B_s$ and $C_s$.

Let $s \geq s^*$ be in $J_j$, and suppose, for example, that we want to force our opponent to enumerate a point in $L$ between some points $x$ and $y$ in $C_s$ (in order to make $C_{s^*}$ non-scattered, for example, or saturated). The only thing we can do is to enumerate a point in $K_{s+1}$ between $\Phi_j^{-1}(x)$ and $\Phi_j^{-1}(y)$. However, we are required to let $F_{s+1}$ map $A_{s+1}$ into $B_{s+1}$, not $C_{s+1}$, and there may be no way to do that. In the two cases which occur in the construction for Theorem 2.4, we arrange the following.

- If $C_s$ is non-scattered (and we are trying to saturate it), and $B_s$ is non-scattered as well, then we can always embed $A_{s+1}$ into $B_{s+1}$. So we just need to ensure that when we try to saturate $C_s$, $B_s$ is non-scattered as well.
- If $C_s$ is scattered, then we will ensure (see Claim 3.9 and the discussion following it) that $C_s$ and $B_s$ are isomorphic, indeed that $F_s \circ \Phi_j^{-1}$ gives an isomorphism from $C_s$ to $B_s$ (equivalently that $F_s \upharpoonright A_s$ is onto $B_s$). In
this case, we could imagine that $C_s$ and $B_s$ are identical and carry out the scattering strategy of the previous construction.

Here we have two related tasks. The first is defining $F_s$ at stages $s > s^*$ which are limit points of $J_j$.\(^1\) The second is indeed ensuring that if $A_s$ is scattered then $F_s | A_s$ maps $A_s$ onto $B_s$. Consider the difficulty of making $F_s$ onto $B_s$. In the previous construction this issue was skirted by defining $F_s = \Phi_s$ at such stages $s$. In the current construction we cannot do this because of the restraint imposed on $R_j$, that $F_s$ must extend $F_s^\star$. Suppose for example that $s^* < s_0 < s_1 < \ldots$ are stages in $J_j$, that $C_{s_0}$ is nonscattered, and that at each stage $s_n$ we enumerate points into $A_{s_n + 1}$ in order to make $C_{\omega_1}$ saturated. Let $s = \sup_n s_n$. At each stage $s_n$ we use a self-embedding of $B_{s_n}$ to redefine $F_{s_n + 1} | A_{s_n + 1}$. There is no reason to believe that $F_{s_n} | A_{s_n}$ reaches a limit. Indeed, again the only thing we can do at stage $s$ is to notice that $B_s$ is nonscattered and so let $F_s | A_s$ be an arbitrarily chosen embedding of $A_s$ into $B_s$; $A_s$ and $B_s$ may fail to be isomorphic, in which case $F_s$ will not be onto $B_s$.

In the case that $B_s$ is scattered, we need to ensure that $F_{s_n} | A_{s_n}$ reaches a limit. If each $F_{s_n}$ is onto $B_{s_n}$, then the limit $F_s | A_s$ will be onto $B_s$. In other words, both tasks – defining $F_s$ and ensuring it is onto $B_s$ – will be successfully performed if we ensure that the maps $F_{s_n} | A_{s_n}$ reach a limit. Indeed, we arrange that for $x \in A_{s_m}$, $F_{s_n}(x)$ changes only two or three times at stages $s_n > s_m$.

To do this we need to consider not only the intervals $A_s$, $B_s$ and $C_s$, but also their $j$-conjugates. The idea is to ensure that there is a cut $(S_1', S_2')$ such that $C_s(S_1', S_2') = B_s(S_1, S_2)$ and also a cut $(S_1'', S_2'')$ such that $B_s(S_1'', S_2'') = C_s(S_1, S_2)$ and this is repeated. This is not automatically so, as $\Phi_j$ may not map $K_{s^*}$ isomorphically onto $L_{s^*}$. For this reason we need to increase the sets $Q$ and $S = \Phi_j^{-1}[Q] = F_s^{-1}[Q]$ of parameters. Thus, the guess $R_j$ needs to wait after stage $s^*$ for a stage $t \in J_j$ at which $K_{s^*}$ is contained in the range of $\Phi_j^{-1} \circ F_t$, $(\Phi_j^{-1} \circ F_t)^2$, and so on. To ensure that $F_s$ eventually stabilizes to give this containment, in the meantime $R_j$ may need to impose further restraint on weaker guesses.

Once we have a set $S \subseteq K_j$ which contains $K_{s^*}$ and is invariant under $\Phi_j^{-1} \circ F_t$, we can fix both $S$ and $F | S$, let $Q = F_t[S] = \Phi_j[S]$, and define, for any cut $(S_1, S_2)$ of $S$ and $n \in \mathbb{Z}$, the conjugate cuts $(S_1, S_2)^n = ((\Phi_j^{-1} \circ F_t)^n[S_1], (\Phi_j^{-1} \circ F_t)^n[S_2])$. For $s \geq t$, letting $A_{n,s} = (S_1, S_2)^n_{K_j}$, and similarly defining $B_{n,s}$ and $C_{n,s}$, we see that $B_{n,s} = C_{n,s+1}$ for all $n$. The intervals $A_{n,s}$ are called the $j$-conjugates of $A_s$. Note that if $\Phi_j[S_1] = F_t[S_1]$ then all the conjugates coincide, we have $C_s = B_s$, and so the situation for this interval is similar to the one in the proof of Theorem 2.4. Otherwise, all conjugates are disjoint (without loss of generality, $\Phi_j^{-1}(F_t(x)) \in S_1$ for some $x \in S_2$, in which case $x$ separates $A_{0,s}$ and $A_{1,s}$). See Figure 4.

Once we have these conjugate intervals, we act on them (for descattering, saturating, etc.) at the same time. For descattering, the action is identical on all conjugates. If we wish to destroy adjacencies in some block $D_0$ of $C_{0,s}$, then at the same time we destroy the corresponding adjacencies of the corresponding blocks $D_n = (F_s \circ \Phi_j^{-1})^n[D_0]$ of $C_{n,s}$. This gives a multiplying effect, ensuring that by the next stage in $J_j$, not only has each adjacency in each $D_n$ been broken, but in

\(^1\)Defining $F_t$ at other limit stages $s$ is made simpler by various guesses $R_t$ imposing further restraint on weaker guesses. This allows us then to pick a sequence of stages $t$ cofinal in $s$ on which $F_t$ is increasing. See cases (A), (B) and (C) in the formal construction below, page 20.
fact infinitely many points must be inserted in between (see the proof of Claim 3.10 on page 23). In turn, this means that in further instances of descattering we can fix the points which have been affected before, and so not have to change $F_s$ on them again. This gives the desired convergence of $F_{s_n}$ above in the case that the intervals under discussion are scattered and infinite.

A new case not present in the proof of Theorem 2.4 is that of finite intervals. Let $A_s$ be an interval as above. Consider the simple case that $\Phi_1[S_1] = F_1[S_1]$, so $A_s$ coincides with all of its conjugates. At some stage $u$ we may see that $|A_u| = |C_u| = m$ for some $m < \omega$, and then wish to enumerate $(\Phi_j[S_1], \Phi_j[S_2])$ into $V_m$, the set of cuts of $Q$ which contains all the cuts that define intervals of $L$ of size $m$. We only do this if there is evidence that this situation is stable; we want $u \in J_j$. We then need to ensure that if some points are going to be later added to $C_{s_u}$, then $C_{s_u}$ is infinite. The idea is similar to the strategy for descattering and saturating. Restricted to $A_u$, $\Phi_1$ and $F_s$ agree and are the unique isomorphism between $A_u$ and $C_u$. If at some stage $v > u$ some point is added to $C_{s_v}$, then to maintain isomorphism we must add a new point to $A_{s_v}$, but we can add this point in a place which doesn’t match the new point in $C_{s_v}$. This precludes $\Phi_j$ from being an isomorphism between $A_{s_v}$ and $C_{s_v}$ (we need to change $F_{s_v}$ on $A_v$, however). Since $R_j$ is correct, this means that yet more points must be added later to $C_u$. Once this symmetry has been broken we can repeat this strategy until $C_s$ is infinite. In fact, once the symmetry is broken we can keep matching the opponent without changing the values of $F_s$ again. This allows the maps $F_{s_n} \upharpoonright A_{s_n}$ above to reach a limit in the case that each $A_{s_n}$ is finite (but $A_s$ is not).

![Figure 4. The conjugates of $A_s$.](image4)

![Figure 5. To diagonalize, a point should be added to $A_{s+1}$ anywhere except between $x$ and $y$. $F_{s+1} \upharpoonright A_{s+1}$ will be the unique order-preserving bijection between $A_{s+1}$ and $B_{s+1}$.](image5)
Note, however, that implementing this strategy requires immediate action. When we observe the new point in \( C_{v+1} \) we must quickly respond, even if \( v \notin J_j \). We cannot allow a weaker guess to act first, since that guess might reply to the new point in \( C_{v+1} \) by adding a matching point in \( A_{v+1} \), restoring symmetry and thus allowing \( C_{\omega_1} \) to be finite, but of size different from the one we guessed at first.

In the more complicated case that the conjugates of \( A_s \) are distinct, we again need to use the strategy of working with all conjugates simultaneously. An extra difficulty is, however, that at stages \( v \notin J_j \), the various conjugate intervals \( B_{n,s} \) need not be isomorphic. We did not have this problem when the \( B_{n,s} \) are infinite since once they are infinite, we may restrict all action to stages in \( J_j \). When we act, we need \( |B_{s+1}| = |C_{v+1}| > |A_n| \) [see Figure 5]. We need to balance the need to limit action, so that \( F \) does not change on \( A_s \) too often (in fact, more than once) while \( A_s \) is finite; and the need to act quickly enough so that symmetry can be broken and never repaired. The correct mix is described in Definition 3.7.

These are the ideas behind the construction; we are now ready for the formalities.

**Construction:** Given an \( \omega_1 \)-computably categorical linear ordering \( L \), we define an increasing, continuous, and \( \omega_1 \)-computable sequence \( \langle K_s \rangle_{s < \omega_1} \) of countable linear orderings. For each \( s < \omega_1 \), we also define an embedding \( F_s : K_s \to L_s \). If \( s \) is a successor ordinal, then \( F_s \) will actually be an isomorphism between \( K_s \) and \( L_s \).

Before we describe what we do at each stage, we define some auxiliary notions. At each stage \( s \), we will decide (Definition 3.8) which guess \( R_j \) requires attention at stage \( s \). The guess \( R_j \) will usually require attention at a stage \( s \) if \( \Phi_{j,s} \) is an isomorphism between \( K_s \) and \( L_s \).

**Definition 3.2.** We let \( I_j \) be the collection of stages at which \( R_j \) requires attention.

We will require the sets \( I_j \) to be closed. This means that if \( s \) is a limit of stages at which \( R_j \) requires attention, then \( R_j \) requires attention at stage \( s \) as well. For simplicity, no guess \( R_j \) requires attention at a stage \( s \leq j \), so \( I_j \subseteq (j, \omega_1) \). A guess \( R_j \) receives attention at stage \( s \) if \( j \) is least such that \( R_j \) requires attention at stage \( s \). If a guess \( R_j \) requires attention at stage \( t \), then all guesses \( R_i \) for \( i > j \) are initialized at that stage. Auxiliary notions defined for \( R_0 \) before that stage are abandoned and may be redefined at a later stage. One of the effects of this initialization is that until \( R_j \) itself is initialized, \( F_s \) will extend \( F_t+1 \). Formally, at a stage \( s < \omega_1 \) we let, for each \( j \leq s \),

\[
r_{j,s} := \sup \left\{ t + 1 : t \in \bigcup_{i < j} (I_i \cap s) \right\} ;
\]

so \( r_{j,s} \leq s \). For \( j > s \) we let \( r_{j,s} = s \). The map \( F_{r_{j,s}} \) is the restraint imposed on \( R_j \) at stage \( s \). We will show (see page 21):

**Claim 3.3.** For all \( s, j < \omega_1 \), \( F_s \) extends \( F_{r_{j,s}} \).

If \( R_j \) receives attention at stage \( s \), then it is \( R_j \)'s task to define \( F_{s+1} \); the guess \( R_j \) must let \( F_{s+1} \) extend \( F_{r_{j,s}} \).

Next we will describe the auxiliary object \( S_{j,s} \). This is the stage \( s \) approximation to \( R_j \)'s version of the eventual image of \( Q \) in \( K \). Once \( S_{j,t} \) is defined, it remains fixed until \( R_j \) is initialized; so \( S_{j,s} = S_{j,t} \) for all \( s > t \) such that \( r_{j,s} = r_{j,t} \). When \( R_j \) is initialized, \( S_{j,s} \) becomes undefined. It will possibly be redefined at a later stage (at
which \( R_j \) receives attention); at that stage \( s \), \( \Phi_{j,s} \) will be an isomorphism from \( K_s \) to \( L_s \). If \( s \) is a limit of stages at which \( R_j \) is initialized then \( S_{j,s} \) is not defined at the beginning of stage \( s \) (but may be defined during that stage).

In general, consider a stage \( s \) at which \( \Phi_{j,s} \) is an isomorphism from \( K_s \) to \( L_s \). Then \( \Phi_{j,s}^{-1} \circ F_s \) is a self-embedding of \( K_s \). It may be a proper self-embedding because \( F_s \) may fail to be onto \( L_s \). Let

\[
N_{j,s} := \bigcap_{n<\omega} (\Phi_{j,s}^{-1} \circ F_s)^n [K_s].
\]

So \( N_{j,s} \) is the largest subset of \( K_s \) restricted to which \( \Phi_{j,s}^{-1} \circ F_s \) is an automorphism. For brevity, we let

\[
h_{j,s} := (\Phi_{j,s}^{-1} \circ F_s) \upharpoonright N_{j,s}.
\]

Dually, we let

\[
M_{j,s} := F_s[N_{j,s}] = \Phi_{j,s}[N_{j,s}]
\]

be the largest subset of \( L_s \) restricted to which the self-embedding \( F_s \circ \Phi_{j,s}^{-1} \) of \( L_s \) is an automorphism; we let

\[
g_{j,s} := (F_s \circ \Phi_{j,s}^{-1}) \upharpoonright M_{j,s}.
\]

The set \( S_{j,s} \) has to contain \( K_{r_{j,s}} = \text{dom} \ F_{r_{j,s}} \), but also be a subset of \( N_{j,s} \). Hence we define the following.

**Definition 3.4.** Let \( j < \omega_1 \). We let \( J_j \) be the set of stages \( s > j \) at which:

1. \( \Phi_{j,s} \) is an isomorphism from \( K_s \) to \( L_s \); and
2. \( K_{r_{j,s}} \subseteq N_{j,s} \).

If \( s \in J_j \) and \( R_j \) receives attention at stage \( s \) then unless already defined, \( R_j \) will define \( S_{j,s} \) at that stage. Thus, if \( s > j \) and \( R_j \) is not initialized at stage \( s \), then \( S_{j,s} \) is defined if and only if \( J_j \cap [r_{j,s}, s) \) is nonempty. We will ensure (see page 21):

**Claim 3.5.** Let \( j < t < s < \omega_1 \). Suppose that \( S_{j,t} \) is defined, and that \( R_j \) is not initialized between stages \( t \) and \( s \) (so \( S_{j,s} = S_{j,t} \)). Then \( F_s \) and \( F_t \) agree on \( S_{j,t} \).

**Claim 3.6.** Let \( j < s < \omega_1 \). Suppose that \( S_{j,s} \) is defined. Then:

1. \( K_{r_{j,s}} \subseteq S_{j,s} \);
2. If \( \Phi_{j,s} \) is an isomorphism from \( K_s \) to \( L_s \), then \( s \in J_j \), \( S_{j,s} \subseteq N_{j,s} \) and \( h_{j,s}[S_{j,s}] = S_{j,s} \).

Suppose that \( S_{j,s} \) is defined for some \( j < s < \omega_1 \). We give notation to \( S_{j,s} \)-intervals of \( K_s \) and the corresponding intervals in \( L_s \) as was informally mentioned above. Let \( (S_1, S_2) \) be a cut of \( S_{j,s} \). We let:

\[
A_s(j, S_1, S_2) := (S_1, S_2)_{K_s},
B_s(j, S_1, S_2) := (F_s[S_1], F_s[S_2])_{L_s}, \text{ and }
C_s(j, S_1, S_2) := (\Phi_j[S_1], \Phi_j[S_2])_{L_s}.
\]

When \( j \) is understood from the context, we write \( A_s(S_1, S_2) \). If also \( (S_1, S_2) \) is fixed then we simply write \( A_s \) for \( A_s(j, S_1, S_2) \). We similarly write \( B_s \) and \( C_s \). If \( t < s \), \( S_{j,t} \) is defined and \( R_j \) is not initialized between stages \( t \) and \( s \) (i.e., \( r_{j,s} = r_{j,t} \)), then \( A_t = A_s \cap K_t \) and \( C_t = C_s \cap L_t \); and since \( F_s \) and \( F_t \) agree on \( S_{j,s} = S_{j,t} \), we also have \( B_t = B_s \cap L_t \). If \( s \in J_j \) then \( C_s = \Phi_j[A_s] \). We always have \( B_s \supseteq F_s[A_s] \).
The correspondence between \( A_s, B_s \) and \( C_s \) can be iterated, again as mentioned above. Let \( A_{0,s}(S_1, S_2) = A_s(S_1, S_2) \). There is a unique interval \( A_s(S'_1, S'_2) \) such that \( B_s(S_1, S_2) = C_s(S'_1, S'_2) \); the cut \( (S'_1, S'_2) \) is defined by \( S'_1 = \Phi_j^{-1}[F_j(S_1)] \), in other words, \( S'_1 = h_{j,t}[S_1] \) for any \( t \in J_j \cap [r_{j,s}, s] \). Here we use Claim 3.6. We let \( A_{1,s}(S_1, S_2) = A_s(S'_1, S'_2) \) and iterate, so \( A_{2,s}(S_1, S_2) = A_{1,s}(S'_1, S'_2) \) and so on; and \( A_{-1,s}(S'_1, S'_2) = A_{s}(S_1, S_2) \) and so on. Formally, choosing any \( t \in J_j \cap [r_{j,s}, s] \), for \( n \in \mathbb{Z} \),

\[
A_{n,s}(j, S_1, S_2) := A_s(j, (h_{j,t})^n[S_1], (h_{j,t})^n[S_2]) ;
\]
we again usually omit \( j \) and even \( (S_1, S_2) \). We call the intervals \( A_{n,s} \) the \( j \)-conjugates of \( A_s \). If \( s \in J_j \) and \( A_s \subseteq N_{j,s} \), then \( A_{n,s} = (h_{j,s})^n[A_s] \), so in this case, the \( j \)-conjugates of \( A_s \) are precisely the elements of the orbit of \( A_s \) under the action of \( h_{j,s} \) on the subsets of \( N_{j,s} \). We let \( B_{n,s} \) and \( C_{n,s} \) be the intervals corresponding to \( A_{n,s} \); we have \( B_{n,s} = C_{n+1,s} \) for all \( n \in \mathbb{Z} \).

There are two possibilities: (1) \( (S'_1, S'_2) = (S_1, S_2) \); in this case, for all \( n \), \( A_{n,s}(S_1, S_2) = A_s(S_1, S_2) \) (and for all \( n \), \( B_{n,s} = C_{n,s} = B_s = C_s \)); or (2) either \( S_2 \cap S'_1 \) or \( S_1 \cap S'_2 \) is nonempty; in this case the intervals \( \{A_{n,s} : n \in \mathbb{Z}\} \) are pairwise disjoint (and the intervals \( \{B_{n,s} : n \in \mathbb{Z}\} \) are also pairwise disjoint).

Having defined the conjugates of an interval \( A_s \), we discuss the instances at which a guess \( R_j \) would like to diagonalize on a finite \( S_{j,s} \)-interval \( A_s \).

**Definition 3.7.** Let \( j < s < \omega_1 \), and suppose that \( S_{j,s} \) is defined. Let \( A_s \) be a nonempty \( S_{j,s} \)-interval of \( K_s \). We say that \( R_j \) diagonalizes on \( A_s \), with \( m \) points (at stage \( s \)), if \( R_j \) receives attention at stage \( s \), \( m = |B_{s+1}| = |C_{s+1}| > |A_s| \), and \( F_{s+1} \mid A_{s+1} \) does not extend \( F_s \mid A_s \). This happens because \( R_j \) adds points to \( A_s \) so that \( \Phi_{j,s} \) cannot be extended to an isomorphism between \( A_{s+1} \) and \( C_{s+1} \).

We say that \( R_j \) has an opportunity to diagonalize on \( A_s \) (with \( m \) points) if \( m = |B_{s+1}| = |C_{s+1}| > |A_s| \), there is a stage \( t \in J_j \cap [r_{j,s}, s] \) at which \( A_t = A_s \cap K_t \) is nonempty, and:

- \( R_j \) did not diagonalize on \( A_r = A_s \cap K_r \) (with any number of points) at any stage \( r \in [r_{j,s}, s] \); and
- for any \( j \)-conjugate \( A'_r \) of \( A_s \), \( R_j \) did not diagonalize on \( A'_r = A'_s \cap K_r \) with \( m \) points at any stage \( r \in [r_{j,s}, s] \).

Finally, we can now describe when a guess \( R_j \) requires attention.

**Definition 3.8.** Let \( j < s < \omega_1 \). A guess \( R_j \) requires attention at stage \( s \) if one of the following holds:

- \( \Phi_{j,s} \) is an isomorphism from \( K_s \) to \( L_s \); or
- \( s \) is a limit ordinal and \( I_s \cap s \) is unbounded in \( s \); or
- \( S_{j,s} \) is defined, and \( R_j \) has an opportunity to diagonalize on some finite \( S_{j,s} \)-interval at stage \( s \).

Having described most of the auxiliary notions, we can now describe the construction. We start with \( K_0 := L_0 \) being the empty ordering, and \( F_0 \) being the empty function. At stage \( s \) of the construction we define \( K_{s+1} \) and \( F_{s+1} \). If \( s \) is a successor ordinal then \( K_s \) and \( F_s \) will have been defined at the previous stage. If \( s \) is a limit ordinal then before defining \( K_{s+1} \) and \( F_{s+1} \) we first need to define \( F_s \), letting \( K_s = \bigcup_{t < s} K_t \).
In the description of what we do at stage $s$ we, of course, use the auxiliary notions described above, which in turn requires the claims we have stated (3.3, 3.5 and 3.6; more will be stated below). This means that both the construction and the claims are defined and verified by simultaneous induction on the stage. At stage $s$, we assume that the construction has been defined up to that stage, and that the claims hold up to that stage, and then define what we do at that stage; after we specify these instructions, we will verify that the claims continue to hold at the end of the stage.

We first describe how to define $K_{s+1}$ and $F_{s+1}$ assuming that both $K_s$ and $F_s$ have already been defined.

If no guess requires attention at stage $s$, then we let $K_{s+1}$ be an extension of $K_s$ such that there is some isomorphism $F_{s+1} : K_{s+1} \rightarrow \mathcal{L}_{s+1}$ extending $F_s$. Otherwise, let $R_j$ be the guess which receives attention at stage $s$.

Now there are a couple of cases. If $S_{j,s}$ is not yet defined, and $s \notin J_j$, then we act as if $R_j$ did not receive attention: we again let $K_{s+1}$ be an extension of $K_s$ such that there is some isomorphism $F_{s+1} : K_{s+1} \rightarrow \mathcal{L}_{s+1}$ extending $F_s$. The reason for $R_j$ officially receiving attention at this stage is merely to impose restraint on weaker guesses.

Next, suppose that $S_{j,s}$ is not yet defined, but that $s \in J_j$. In this case we define $S_{j,s}$ to conform with Claim 3.6:

$$S_{j,s} := \bigcup_{n \in \mathbb{Z}} (h_{j,s})^n[K_{r_{j,s}}].$$

This is the smallest subset of $N_{j,s}$ containing $K_{r_{j,s}}$ which is closed under the action of $h_{j,s}$. In this case, too, we end the stage and let $F_{s+1}$ extend $F_s$.

Now suppose that $S_{j,s}$ is already defined. We will let $F_{s+1}$ agree with $F_s$ on $S_{j,s}$. To define $K_{s+1}$ and $F_{s+1}$, we will define, for every nonempty $S_{j,s}$-interval $A_s$ of $K_s$, an isomorphism $F_{s+1} \upharpoonright A_{s+1}$ between $A_{s+1}$ (which we define) and the corresponding $B_{s+1}$ (which the opponent plays). [Note that by definition of $B_s$, $F_s \upharpoonright A_s$ is an embedding of $A_s$ into $B_s$] Exactly how to define $A_{s+1}$ and $F_{s+1} \upharpoonright A_{s+1}$ depends on the order-type of $A_s$. We will consider all $j$-conjugates of an interval simultaneously. That we can do so relies on the following claim.

**Claim 3.9.** Let $j < s < \omega_1$. Suppose that $S_{j,s}$ is defined (in particular, $R_j$ is not initialized at stage $s$); let $A_s$ be an $S_{j,s}$-interval. If $A_s$ is scattered (either finite or infinite) then $F_s \upharpoonright A_s$ is onto $B_s$.

Since $F_s \upharpoonright A_s$ is always an embedding of $A_s$ into $B_s$, this means that if $A_s$ is scattered then $F_s \upharpoonright A_s$ is an isomorphism of $A_s$ with $B_s$. Suppose in addition that $s \in J_j$. Consider the $j$-conjugates of $A_s$. Suppose that $A_{n,s}$ is scattered for some $n \in \mathbb{Z}$. Then $F_s \upharpoonright A_{n,s}$ is an isomorphism of $A_{n,s}$ to $B_{n,s}$, and so $(\Phi_j^{-1} \circ F_s) \upharpoonright A_{n,s}$ is an isomorphism from $A_{n,s}$ to $A_{n+1,s}$. It follows, of course, that $A_{n+1,s}$ is scattered, too. Similarly, $C_{n,s} = B_{n-1,s}$ is isomorphic to $A_{n,s}$ by $\Phi_j$, and so is scattered. Since $F_s \upharpoonright A_{n-1,s}$ is an embedding of $A_{n-1,s}$ into $B_{n-1,s}$, it follows that $A_{n-1,s}$ is scattered. Thus, for all $n \in \mathbb{Z}$, $A_{n,s}$ is scattered; $A_{n,s} \subseteq N_{j,s}$ for all $n \in \mathbb{Z}$, and similarly, $B_{n,s} \subseteq M_{j,s}$ for all $n \in \mathbb{Z}$; all the intervals $A_{n,s}$ are isomorphic by repeatedly applying $h_{j,s}$, and all the $B_{n,s}$ are isomorphic by repeatedly applying $g_{j,s}$.
Fix an $S_{j,s}$-interval $A_s$ and its $j$-conjugates $A_{n,s}$.

0. Unless one of the cases below holds, we will extend simply, that is, let $A_{s+1}$ be any extension of $A_s$ for which there is an isomorphism from $A_{s+1}$ to $B_{s+1}$ extending $F_s$, and let $F_{s+1} \upharpoonright A_{s+1}$ be any such isomorphism. In particular, we will extend simply if $s \notin J_j$ and $A_s$ is infinite.

1. Suppose that $s \in J_j$ and that $A_s$ is nonscattered. By Claim 3.9 and the discussion which follows it, each $A_{n,s}$ and each $B_{n,s}$ is nonscattered. For each $n \in \mathbb{Z}$, let $(X_{n,s}, Y_{n,s})$ be the $<_{\omega_1}$-least pair of countable subsets $(X, Y)$ of $C_{n,s}$ such that $X <_{L_s} Y$ and $(X, Y)_{L_s}$ is empty. For each $n$, enumerate a new point into $A_{n,s}$ between $\Phi_{j,s}^{-1}[X_{n,s}]$ and $\Phi_{j,s}^{-1}[Y_{n,s}]$. Since $B_{n,s}$ is nonscattered, we can find an embedding of $A_{n,s}$, with the extra point added, into $B_{n,s}$. We can thus add points to define $A_{n,s+1}$ to be isomorphic to $B_{n,s+1}$ and let $F_{s+1} \upharpoonright A_{n,s+1}$ be any isomorphism between $A_{n,s+1}$ and $B_{n,s+1}$.

2. Suppose that $R_j$ has an opportunity to diagonalize on $A_s$ with $m$ points at stage $s$ (so $B_s$ is finite), and that $A_s$ is the $<_{\omega_1}$-least such interval among its $j$-conjugates.

As $|A_s| < m = |C_{s+1}|$, the map $\Phi_{j,s} \upharpoonright A_s$ is not onto $C_s$. We can extend $A_s$ to an ordering $A_{s+1}$ of size $m$ such that $\Phi_{j,s}$ cannot be extended to an isomorphism between $A_{s+1}$ and $C_{s+1}$. To see this, by the definition of having an opportunity to diagonalize (Definition 3.7), we take a stage $t \in J_j \cap [r_{j,s}, s]$ such that $A_t$ is nonempty. As $t \in J_j$, we know that $\Phi_{j,t}$ is an isomorphism of $K_t$ and $L_t$, and so $A_t \subseteq \text{dom } \Phi_{j,t}$; as $\Phi_{j,s}$ extends $\Phi_{j,t}$, we see that $A_s \cap \text{dom } \Phi_{j,s}$ is nonempty.

Since $|C_{s+1}| > |A_s|$, there is some cut $(D, E)$ of $A_t$ such that $(D, E)_A$ is smaller than $\left(\Phi_{j,D}, \Phi_{j,E}\right)_{C_{s+1}}$. Since $A_t$ is nonempty, $(D, E)_A$ is not the only interval of $A_s$, so we can add points to $A_{s+1}$ elsewhere, so that $A_{s+1}$ contains $m$ points, but $(D, E)_A = (D, E)_{A_t}$. Then $\Phi_{j,t} \upharpoonright A_t$ cannot be extended to an isomorphism between $A_{s+1}$ and $C_{s+1}$. See Figure 5.

This defines $A_{s+1}$; since $|A_{s+1}| = |B_{s+1}| = m$, we let $F_{s+1} \upharpoonright A_{s+1}$ be the unique isomorphism between $A_{s+1}$ and $B_{s+1}$.

It is important that at stage $s$, we do not let $R_j$ diagonalize on any $j$-conjugate of $A_s$ with $m$ points other than $A_s$ itself; this is why we demanded that $A_s$ be the $<_{\omega_1}$-least such interval among its $j$-conjugates. So if $A'_s$ is a $j$-conjugate of $A_s$, distinct from $A_s$, and at stage $s$, $R_j$ has the opportunity to diagonalize on $A'_s$ with $m$ points, then we do not let $R_j$ diagonalize on $A'_s$, simply as in (0) above. During stage $s$, we may diagonalize with a different number of points on other $j$-conjugates of $A_s$.

3. If $s \in J_j$ and $A_s$ is infinite and scattered, we again treat all of the conjugates $A_{n,s}$ of $A_s$ in one step. Again, these are all isomorphic by powers of $h_{j,s}$.

We let $t_{j,s}^\text{inf}(A_s)$ be the least $t \in J_j \cap [r_{j,s}, s]$ such that $A_t$ is infinite. Note that the argument following Claim 3.9 shows that this does not depend on the choice of $A_n$ among its $j$-conjugates: for all $n$, $t_{j,s}^\text{inf}(A_n) = t_{j,s}^\text{inf}(A_s)$.

Let $D_{0,s}$ be the $<_{\omega_1}$-least maximal infinite block of $B_{0,s}$. For $n \in \mathbb{Z}$, let $D_{n,s} = (g_{j,s})^n[D_{0,s}]$; so $D_{n,s}$ is a maximal infinite block of $B_{n,s}$.

Claim 3.10. There are self-embeddings $f_{n,s}$ of $D_{n,s}$ with the following four properties.

- Coherence: The functions $f_{n,s}$ are coherent with respect to $g_{j,s}$: For all $n$ and $m$, $f_{n+m,s} = (g_{j,s})^m \circ f_{n,s} \circ (g_{j,s})^{-m}$. 

• Fixed Points: For all $n$, the set $E_{n,s} := \{ a \in D_{n,s} : f_{n,s}(a) = a \}$ is a finite (possibly empty) convex subset of $D_{n,s}$. Note that the coherence of the functions $f_{n,s}$ shows that for all $n$ and $m$, $E_{n+m,s} = (g_{j,s})^m E_{n,s}$.

• Historical Responsibility: For all $n$, if $a \in D_{n,s}$ and there is some stage $u \in I_j \cap \lceil \text{inf}(A_s) \rceil, s$ such that $a \in B_{n,u}$ and $F_{u+1}^{-1}(a) \neq F_u^{-1}(a)$, then $a \in E_{n,s}$.

Dually, if $F_u(x) \in D_{n,s}$ and there is some stage $u \in I_j \cap \lceil \text{inf}(A_s), s \rceil$ such that $x \in A_{n,u}$ and $F_{u+1}(x) \neq F_u(x)$, then $F_u(x) \in E_{n,s}$.

Interpolation: For all $n$, $a \in D_{n,s} \setminus E_{n,s}$ and $b \in D_{n,s}$ distinct from $a$, $f_{n,s}(a)$ and $f_{n,s}(b)$ are not adjacent in $D_{n,s}$.

For the third property, note that if $u \in I_j \cap \lceil \text{inf}(A_s), s \rceil$ and $a \in B_{n,u}$ then by Claim 3.9 (as $A_{n,u}$ is scattered), $u$ is scattered as well. By Claim 3.9, for any nonempty $F$-element of $A$ such that $a \in B_{n,u}$ and there is some stage $u \in I_j \cap \lceil \text{inf}(A_s), s \rceil$ such that $x \in A_{n,u}$ and $F_{u+1}(x) \neq F_u(x)$, then $F_u(x) \in E_{n,s}$.

This completes the instructions for stage $s$, given $K_s$ and $F_s$. At limit stages $s$, we need to define $F_s$, we already stipulated that $K_s = \bigcup_{t < s} K_t$.

There are four cases.

A. Suppose that $r_{j,s} < s$. So between stages $r_{j,s}$ and $s$, no guess requires attention. Then our instructions show that for all $t < t'$ in $(r_{j,s}, s)$, $F_{t'}$ extends $F_t$. In this case we let $F_s = \bigcup_{t \in (r_{j,s}, s)} F_t$.

B. If (A) fails, we let $j$ be the least ordinal $j < s$ such that $r_{j,s} = s$. Suppose that $j$ is a limit ordinal. Let $T = \{ r_{i,s} : i < j \}$. The set $T$ is unbounded in $s$. Let $t < t'$ be elements of $T$, and let $i$ be such that $t = r_{i,s}$. Since $t' \in (r_{i,s}, s)$, $t = r_{i,t'}$, and so Claim 3.3 says that $F_{t'}$ extends $F_t$. Hence we can let $F_s = \bigcup_{t \in T} F_t$.

C. If both (A) and (B) fail, then there is some (unique) $j < s$ such that $r_{j,s} < s$ but $I_j \cap s$ is unbounded in $s$. Suppose that $J_j \cap \lceil r_{j,s}, s \rceil$ is empty: $S_{j,t}$ is not defined for any $t \in [r_{j,s}, s)$. Let $t \in I_j \cap \lceil r_{j,s}, s \rceil$. Since $R_j$ receives attention at stage $t$, the instructions show that $F_{t+1}$ extends $F_t$. By Claim 3.3, if $t'$ is the next element of $I_j$ beyond $t$, then $F_{t'}$ extends $F_{t+1}$, as $r_{j+1,t'} = t+1$. Hence, for all $t < t'$ in $I_j \cap \lceil r_{j,s}, s \rceil$, $F_t \subseteq F_{t+1} \subseteq F_{t'} \subseteq F_{t+1}$. We thus let $F_s = \bigcup_{t \in I_j \cap \lceil r_{j,s}, s \rceil} F_t = \bigcup_{t \in I_j \cap \lceil r_{j,s}, s \rceil} F_{t+1}$.

D. Otherwise, we again take $j < s$ such that $r_{j,s} < s$ but $I_j \cap s$ is unbounded in $s$; and now we suppose that $J_j \cap \lceil r_{j,s}, s \rceil$ is nonempty.

Thus, $S_{j,t}$ is defined at stage $w = \min J_j \cap \lceil r_{j,s}, s \rceil$ and we have $S_{j,t} = S_{j,w}$ for all $t \in [w, s)$. Recall that Claim 3.5 says that $F_t \upharpoonright S_{j,w}$ is constant for $t \in [w, s)$. We then let $F_s$ extend this map. To define the rest of $F_s$, we need to define $F_s$ on any nonempty $S_{j,w}$-interval $A_s$ of $K_s$.

Let $A_s$ be an $S_{j,w}$-interval of $K_s$. If $B_s$ is nonscattered then we let $F_s \upharpoonright A_s$ be any embedding of $A_s$ into $B_s$. Suppose that $B_s$ is scattered. Then for all $u \in [w, s)$, $B_u = B_s \cap L_u$ is scattered. Since $F_u \upharpoonright A_u$ is an embedding of $A_u$ into $B_u$, $A_u$ is scattered as well. By Claim 3.9, for $u \in J_j \cap \lceil w, s \rceil$, $A_u$ is isomorphic to $B_u$ by $F_u \upharpoonright A_u$. We require the following facts.

Claim 3.11. Let $j < t < s < \omega_1$, with $t \in I_j$. Suppose that $S_{j,t}$ is defined and that $R_j$ is not initialized between stages $t$ and $s$. Let $A_t$ be a scattered $S_{j,w}$-interval.
(1) For each \( x \in A_t \) there are at most two stages \( u \in [t, s) \) at which \( F_{u+1}(x) \neq F_u(x) \).

(2) For each \( a \in B_t \) there are at most two stages \( u \in [t, s) \) at which \( F^{-1}_{u+1}(a) \neq F^{-1}_u(a) \).

(Again note that in (2), for all \( u \in [t, s) \), \( a \in \text{range } F_u \) by Claim 3.9.)

Because for all \( x \in A_t \), we can find a stage \( t \in J_j \cap [w, s) \) such that \( x \in A_t \), we see that for all \( x \in A_s \), we can let \( F_s(x) \) be the limit \( \lim_{u \to s} F_u(x) \). It is easy to see that \( F_s | A_s \) is order-preserving (and in fact, onto \( B_s \); this will help us prove Claim 3.9).

This completes the construction of \( K \) and of the sequence \( \langle F_s \rangle_{s \in \omega_1} \).

**Promises Were Made:** As discussed above, to carry out the construction, we relied on various facts about the construction itself. We now establish these facts, by a global induction on the stages. We begin with an observation.

**Lemma 3.12.** Let \( j < i < s \) and suppose that both \( S_{j,s} \) and \( S_{i,s} \) are defined. Then \( S_{j,s} \subseteq S_{i,s} \).

**Proof.** Let \( t = \min J_j \cap [r_{j,s}, s] \) be the stage at which \( S_{j,s} \) is defined. Since \( R_j \) receives attention at stage \( t \), \( r_{i,s} > t \). So \( S_{j,s} = S_{j,t} = \bigcup_{i} K_{i} \subseteq K_{r_{j,s}} \). By Claim 3.6, \( K_{r_{j,s}} \subseteq S_{i,s} \).

**Proof of Claim 3.5.** Suppose first that \( s \) is a successor stage. By induction, \( F_{s-1} \) and \( F_t \) agree on \( S_{j,t} \), so in this case we just need to show that \( F_{s-1} \) and \( F_t \) agree on \( S_{j,t} \). If \( F_s \) extends \( F_{s-1} \) we are, of course, done. Suppose, then, that \( F_s \) is not an extension of \( F_{s-1} \). This means that at stage \( s - 1 \), some requirement \( R_i \) received attention, and \( s - 1 \in J_i \). Since \( R_j \) was not initialized at stage \( s - 1 \), we must have \( i \geq j \). By Lemma 3.12, \( S_{j,t} = S_{j,s-1} \subseteq S_{i,s-1} \). The instructions for \( R_i \) at stage \( s - 1 \) ensure that \( F_s \) and \( F_{s-1} \) agree on \( S_{i,s-1} \), and so agree on \( S_{j,t} \).

Suppose that \( s \) is a limit stage. In cases (A), (B) and (C) of the definition of \( F_s \), we let \( F_t \) be the union of \( F_u \) for some \( u \) in a set cofinal in \( s \). In these cases, as for all \( u \in (t, s) \), \( F_u \) and \( F_t \) agree on \( S_{j,t} \), we have \( F_s \) and \( F_t \) agree on \( S_{j,t} \). In case (D), suppose that \( R_i \) defined \( F_s \), that is, \( r_{i,s} < s \) but \( I_i \cap s \) is unbounded in \( s \). Since \( r_{j,s} < s \), we must have \( j \leq i \), so \( S_{j,t} \subseteq S_{i,s} \), and \( F_s \mid S_{i,s} = F_u \mid S_{i,s} \) for a set of \( u \) cofinal in \( s \). For such \( u \), by induction, \( F_u \) agrees with \( F_t \) on \( S_{j,t} \) and so \( F_s \) also agrees with \( F_t \) on \( S_{j,t} \).

**Proof of Claim 3.3.** Of course, if \( r_{j,s} = s \) then we are done. Hence, we assume that \( r_{j,s} < s \).

First, suppose that \( s \) is a successor stage. Then \( r_{j,s} = r_{j,s} \neq 1 \). By induction, \( F_{r_{j,s}} \) extends \( F_{r_{j,s}} \). If \( F_s \) extends \( F_{r_{j,s}} \), then the claim holds at \( s \). Suppose that \( F_s \) does not extend \( F_{r_{j,s}} \). Let \( R_t \) be the guess which receives attention at stage \( s - 1 \); then \( S_{i,s-1} \) is defined. Since \( r_{j,s} < s \), \( i \geq j \). Hence \( r_{i,s} - 1 \geq r_{j,s} = r_{j,s} \). As argued in the proof of Lemma 3.12, this means that \( K_{r_{j,s}} \) is contained in \( S_{i,s-1} \), so \( F_{s-1} \mid S_{i,s-1} \) extends \( F_{r_{j,s}} \). At stage \( s - 1 \), \( R_i \) is instructed to let \( F_s \) agree with \( F_{s-1} \) on \( S_{i,s-1} \); so \( F_s \) extends \( F_{r_{j,s}} \).

Next, suppose that \( s \) is a limit ordinal. Again we consider the cases defining \( F_t \).

If \( F_s = \bigcup_{t \in T} F_t \) where \( T \) is cofinal in \( T \), then \( F_s \) extends \( F_{r_{j,s}} \), as by induction, \( F_t \) extends \( F_{r_{j,s}} \) for \( t \in [r_{j,s}, s) \). In case (D), let \( R_t \) be the guess which is responsible for defining \( F_{r_{j,s}} \). Since \( r_{j,s} < s \), we have \( i \geq j \). Let \( t \in J_i \cap [r_{i,s}, s) \). \( F_s \) agrees with \( F_t \).
on $S_{r,s} = S_{i,t}$. Since $r_{i,s} \geq r_{j,s}$, the set $S_{r,s}$ contains $K_{r_{i,s}}$. By induction, $F_t$ extends $F_{r_{j,s}}$, so $F_t \upharpoonright S_{i,t}$ extends $F_{r_{j,s}}$, and so $F_s$ extends $F_{r_{j,s}}$.

Proof of Claim 3.6. Let $t = \min(J_j \cap [r_{j,s}, s))$. This is the stage at which $S_{r,s}$ was defined. We have $r_{j,s} = r_{j,t}$. At stage $t$, we define $S_{j,t}$ to be a superset of $K_{r_{j,t}}$. This establishes (1).

By definition, $S_{j,t} \subseteq N_j(t)$, and $(\Phi_j^{-1} \circ F_t)[S_{j,t}] = S_{j,t}$. By Claim 3.5, $F_s \upharpoonright S_{j,t} = F_t \upharpoonright S_{j,t}$. Of course, $\Phi_j \circ F_s[S_{j,t}] = S_{j,t}$. This shows that $S_{j,t} \subseteq N_j$. Thus, $\Phi_j^{-1} \circ F_s|S_{j,t} = F_s|S_{j,t}$. This shows that $S_{j,t} \subseteq S_{j,s}$, this shows that $s \in J_j$. □

Proof of Claim 3.11. Let $t \in I_j$ with $t < s$. We first note that if $u \in I_j \cap [t, s)$ and $u' := \min(I_j \cap (u, s])$ is the successor of $u$ in $I_j$, then $r_{j+1,u'} = u + 1$ and so (Claim 3.3) $F_{u'}$ extends $F_{u+1}$. Also, if $u \in (t, s]$ is a limit ordinal, then the claim holds at $u$ by induction. It suffices, then, to show:

1. For all $x \in A_t$ there are at most two stages $u \in I_j \cap [t, s)$ such that $F_{u+1}(x) \neq F_u(x)$.
2. For all $a \in B_t$ there are at most two stages $u \in I_j \cap [t, s)$ such that $F_{u+1}^{-1}(a) \neq F_u^{-1}(a)$.

Both follow directly from our instructions. If $u \in I_j \cap [t, s)$ then (as $u \geq r_{j,s}$) $R_j$ receives attention at stage $u$. Let $x \in A_t$ and $a \in B_t$.

We first note that there is at most one stage $u \in I_j \cap [t, s)$ at which $B_u$ is finite and at which $F_{u+1} \upharpoonright A_u$ does not extend $F_u \upharpoonright A_u$. This is by the definition of having an opportunity to diagonalize on a finite interval (Definition 3.7).

The “historical responsibility” property of the functions $f_{n,u}$ shows that there is at most one stage $u \in I_j \cap [t, s)$ such that $B_u$ is infinite and such that $F_{u+1}^{-1}(a) \neq F_u^{-1}(a)$. To see this, let $u$ be such a stage. At every stage $v \in I_j \cap (u, s)$, as $u \geq \inf(A_v)$, this property ensures that $f_{n,v}(a) = a$, where $F_{v+1} \upharpoonright A_v$ extends $F_v \upharpoonright A_v$, so $F^{-1}_{v+1}(a) = F^{-1}_u(a)$. Similarly, there is at most one stage $u \in I_j \cap [t, s)$ such that $B_u$ is infinite and $F_{u+1}(x) \neq F_u(x)$. This completes the proof of the claim. □

We note that it is not necessarily the case that for $x \in A_t$, there are at most two stages $u \in [r_{u,s}, s)$ such that $F_{u+1}(x) \neq F_u(x)$. This is because $F_u(x)$ could change often on the interval $[v, r]$, where $v$ is the least such that $x \in A_v$ and $r := \min(I_j \cap (v, s))$ is $v$’s successor in $I_j$. It is true that $F_u(x)$ changes at most finitely many times on this interval, but we do not need this fact.

Proof of Claim 3.9. Let $A_s$ be an $S_{r,s}$-interval and let $B_s$ be the corresponding interval in $L_s$. We need to show that $F_s \upharpoonright A_s$ is onto $B_s$, equivalently $B_s \subseteq R_s$. If $s$ is a successor stage, then $F_s$ is onto $L_s$ by construction. Since $F_s$ is order-preserving, it follows that $F_s \upharpoonright A_s$ is onto $B_s$. Suppose then that $s$ is a limit stage.

Consider the cases defining $F_s$. We claim that in cases (A), (B) and (C), $F_s$ is the union of maps $F_u$ where $u$ ranges over a set $T$ of successor stages cofinal in $s$: this would imply that in these cases, too, $F_s$ is onto $L_s$. In case (A) we can let $T$ be the collection of all successor ordinals in $(r_{s,s}, s)$. In case (C) we let $T$ be the set of successor ordinals in $I_j \cap [r_{s,s}, s)$. In case (B) let $i$ be the least such that $r_{i,s} = s$. In the construction, we let $T' = \{r_{k,s} : k < i\}$ and we let $F_s$ be the union of $F_u$ for $u \in T'$. While $T'$ may contain limit stages, we
show that $T'$ contains a cofinal subset $T$ consisting of successor stages. For $k < i$, let $\alpha_k, s = \sup_{t < k} \alpha_{k', s} (v + 1) = \max(I_k \cap s) + 1$ (recall that $I_k$ is closed). For all $k \leq i$, $r_k, s = \sup_{k < k} \alpha_{k', s}$. Let $T$ be the set of stages $\alpha_k, s$ such that $\alpha_k, s > r_k, s$. Certainly $T$ consists of successor ordinals. The fact that $s = \sup_{k < s} \alpha_k, s$ shows that $T$ is unbounded in $s$. And if $\alpha_k, s \in T$ then $r_{k+1}, s = \alpha_k, s$ and $r_{k+1}, s \in T'$. Thus $T \subseteq T'$.

We discuss case (D). Let $i < s$ such that $r_i, s < s$ but $I_i \cap s$ is unbounded in $s$. Hence $s \in I_i$. By the assumption that $R_j$ is not initialized at stage $s$, we must have $j \leq i$. Let $A'_s$ be a scattered $S_{i, s}$-interval. Let $w := \min(J_i \cap r_{i, s})$. We define $F_s \upharpoonright A'_s$ to be the limit of $F_u \upharpoonright A'_s$ for $u \in [w, s)$. Let $a \in B'_s$. There is a stage $v \in [w, s)$ such that $a \in B'_v$. Since for all $u \in [w, s)$, $A'_u$ is scattered, Claim 3.11 implies that $F_{u-1}(a)$ (which exists by induction) is constant on a final segment of $s$, and so $a \in \text{range } F_s$. Thus $F_s \upharpoonright A'_s$ is onto $B'_s$. Hence if $i = j$ then we are done.

Suppose that $j < i$. In this case the point is that every scattered $S_{i, s}$-interval is the union of scattered $S_{j, s}$-intervals and some points from $S_{i, s}$. By Lemma 3.12, $S_{j, s} \subseteq S_{i, s}$. Let $a \in B_s$. If $a \in F_s[S_{j, s}]$ then, of course, $a \in \text{range } F_s$. Otherwise, there is an $S_{j, s}$-interval $A'_s \subseteq A_s$ such that $a \in B'_s$. Since $A'_s \subseteq A_s$, $A'_s$ is scattered. In the previous paragraph we observed that $F_s[A'_s] = B'_s$ so in this case, too, $a \in \text{range } F_s$.

Proof of Claim 3.10. Let $U_s$ be the set of $x \in \bigcup_n A_{n, s}$ such that there is some $t \in I_j \cap \{^{\text{inf}}_{j, s}(A_s), s\}$ such that $F_{t+1}(x) \neq F_t(x)$; and dually, let $V_s$ be the set of $a \in \bigcup_n B_{n, s}$ such that there is some $t \in I_j \cap \{^{\text{inf}}_{j, s}(A_s), s\}$ such that $F_{t+1}(a) \neq F_t(a)$; noting again, of course, that for all $n$ and $m$, $^{\text{inf}}_{j, s}(A_{m, s}) = ^{\text{inf}}_{j, s}(A_{m, s})$ as that stage is in $J_j$. We claim that $V_s$ is invariant under $g_{j, s}$, that $V_s = \Phi_{j, s}[U_s] = F_s[U_s]$, and that for all $n$, $V_s \cap D_{n, s}$ is at most a singleton.

For this, consider a stage $t \in J_j \cap \{^{\text{inf}}_{j, s}(A_s), s\}$ (assuming that $^{\text{inf}}_{j, s}(A_s) < s$) at which $F_{t+1}(x) \neq F_t(x)$ for some $x \in \bigcup_n A_{n, t}$ (equivalently at which $F_{t+1}^{-1}(a) \neq F_t^{-1}(a)$ for some $x \in \bigcup_n B_{n, t}$). Let $u := \min(J_j \cap (t, s])$ be $t$'s successor in $J_j$. If $v \in I_j \cap (t, u]$ then at stage $v$, $R_j$ is instructed to let $F_{v+1} \upharpoonright A_{v+1}$ extend $F_v \upharpoonright A_v$. It follows that $F_u \upharpoonright A_u$ extends $F_{t+1} \upharpoonright A_{t+1}$. Since $t \geq ^{\text{inf}}_{j, s}(A_s)$, at stage $t$ we act for $A_t$ and its $j$-conjugates as in option (3), so we make use of maximal blocks $D_{n, t}$ and self-embeddings $f_{n, t}$ as given by this claim at stage $t$. As above, for all $n \in \mathbb{N}$ and all $a \in E_{n, t}$, $F_{t+1}^{-1}(a) = F_t^{-1}(a)$.

We show that for all $n \in \mathbb{N}$ and all distinct $a < b \in D_{n, t}$ which are not both in $E_{n, t}$, the interval $(a, b)_{B_{n, t}}$ is infinite. We prove, by induction on $m \geq 0$, that each such interval contains at least $m$ points; the base case is vacuous. Assume we showed this for $m \geq 0$. Let $n \in \mathbb{N}$ and let $a < b \in D_{n, t}$, not both in $E_{n, t}$. Let $x := \Phi_{j, t}^{-1}(a)$ and $y := \Phi_{j, t}^{-1}(b)$; so $x < y$ are elements of $A_{n+1, t}$, and $g_{j, t}(a) = F_t(x)$, and $g_{j, t}(b) = F_t(y)$ are elements of $D_{n+1, t}$. Let $a' := F_{t+1}(x)$ and $b' := F_{t+1}(y)$. Since $\Phi_{j, u}$ extends $\Phi_{j, t}$, and $a' = F_{u}(x)$, $b' = F_{u}(y)$, we see that $a' = g_{j, u}(a)$ and $b' = g_{j, u}(b)$.

The coherence property of the functions $f_{n, t}$ shows that not both of $g_{j, t}(a)$ and $g_{j, t}(b)$ are in $E_{n+1, t}$. The definition of $F_{t+1}$ shows that $a' = f_{n+1, t+1}(g_{j, t}(a))$ and $b' = f_{n+1, t+1}(g_{j, t}(b))$. The interpolation property of the functions $f_{n, t}$ shows that there is some $c' \in (a', b')_{D_{n+1, t}}$. Now since the function $f_{n, t}$ is injective, the definition of the set $E_{n+1, t}$ implies that either $a'$ or $b'$ are not elements of $E_{n+1, t}$. By
induction, either the interval \((a',c')_{B_{n+1,u}}\) or the interval \((c',b')_{B_{n+1,u}}\) contains at least three points, and so \((a,b)_{\mathcal{L}_u}\) must contain at least three points. The dotted arrows denote \(F_t\). The dashed arrows denote \(F_{t+1} = F_u\).

For simplicity \(g_{j,t}(a) = g_{j,u}(a)\) and \(g_{j,t}^2(a) = g_{j,u}^2(a)\).

We note that this proof works if the conjugates \(B_{n,s}\) are all identical, and also if they are pairwise disjoint.

We return to the sets \(U_s\) and \(V_s\). Let \(a \in V_s\); let \(t\) witness this fact. So \(b = f_{n,t}(a) \neq a\) (where \(b \in B_{n,t}\)). Let \(a' := g_{j,t}(a)\). The coherence of \(f_{m,t}\) shows that \(b' := f_{n+1,t}(a') = g_{j,t}(b)\), so \(b' \neq a'\). Let \(x := \Phi_{j,t}^{-1}(a)\); we define \(F_{t+1}(x) := b'\), so \(b' = g_{j,t+1}(a)\). The fact that \(F_t(x) = b'\) and \(b' \neq a'\) means that for all \(u \in J_j \cap (t, s)\) we have \(b' \in V_s\), so inductively \(F_u(x) = F_{u+1}(x) = b'\); so \(b' = g_{j,s}(a)\). This shows that \(g_{j,s}(a) \in V_s\) as well. An identical argument shows that \((g_{j,s})^{-1}(a) \in V_s\); so \(V_s\) is invariant under \(g_{j,s}\).

This argument also shows that if \(a \in V_s\), witnessed by \(t\), then \(\Phi_{j,t}^{-1}(a) \in U_s\). If \(a \in B_{n,s}\) and \(a \notin V_s\), then for all \(t \in J_j \cap \{\text{inf}(A_s), s\}\) such that \(a \in B_{n,t}\), the coherence property shows that \(F_{t+1}(x) = F_t(x)\) for \(x = \Phi_{j,t}^{-1}(a)\); so \(x \notin U_s\). Hence \(U_s = \Phi_{j,t}^{-1}V_s\). Since \(V_s\) is invariant under \(g_{j,s}\), we also have \(V_s = F_j[U_s]\).

Suppose, for a contradiction, that \(n \in \mathbb{Z}\), and that \(a, b \in V_s \cap D_{n,s}\) and \(a < b\). Let \(t_a\) witness that \(a \in V_s\) and \(t_b\) witness that \(b \in V_s\). Without loss of generality, \(t_b \geq t_a\). Then \(b \in D_{n,t_b}\). Since \((a,b)_{B_{n,s}}\) is finite, so is \((a,b)_{B_{n,t_b}}\). Since \(D_{n,t_b}\) is a maximal block, \(a \in D_{n,t_b}\) as well. Since \(b \notin E_{n,t_b}\), the argument above shows that the interval \((a,b)_{B_{n,u}}\), where \(u = \min(J_j \cap (t_b, s))\), is infinite, contradicting that \((a,b)_{B_{n,u}}\) is finite.

This tells us how to define the functions \(f_{n,s}\). The order-type of \(D_{0,s}\) is either \(\zeta\), \(\omega\) or \(\omega^*\). If \(\text{otp}(D_{0,s}) = \zeta\), then we let \(f_{0,s}\) be a self-embedding of \(D_{0,s}\) which fixes the unique element of \(V_s \cap D_{0,s}\), if that element exists, moves every other element, and satisfies the interpolation property; so \(E_{0,s} = V_s \cap D_{0,s}\). If \(\text{otp}(D_{0,s}) = \omega\), then we let \(f_{0,s}\) be a self-embedding of \(D_{0,s}\) which fixes the initial segment of \(D_{0,s}\) determined by the unique element of \(V_s \cap D_{0,s}\), and moves every other element; this initial segment is, of course, finite; we can again define \(f_{0,s}\) to satisfy interpolation. The case \(\text{otp}(D_{0,s}) = \omega^*\) is symmetrical. We then define \(f_{n,s}\) for \(n \neq 0\) so that coherence holds. The fact that \(V_s\) is invariant under \(g_{j,s}\), and that \(V_s = F_j[U_s]\), shows that this definition of \(f_{n,s}\) satisfies the historical responsibility property. \(\square\)
The Correct Guess: We show that some guess is correct, and is eventually able to act as it wishes. We first note that the arguments in cases (A) or (B) for defining $F_s$ for limit stages $s$ show that if for all $j < \omega_1$, $r_{j,\omega_1} < \omega_1$, that is, if for all $j < \omega_1$, $I_j$ is bounded below $\omega_1$, then we can define an isomorphism $F_{w_1}$ from $K = K_{\omega_1} \to L = L_{\omega_1}$ by taking the union of maps $F_t$ where $t$ ranges over some set cofinal in $\omega_1$. This isomorphism is in fact $\Delta^1_2$. The assumption that $L$ is $\omega_1$-computably categorical then implies that there is an $\omega_1$-computable isomorphism from $K$ to $L$.

On the other hand, let $j < \omega_1$, and suppose that $I_j$ is unbounded in $\omega_1$. Then $J_j$ is also unbounded in $\omega_1$. For suppose otherwise; let $t := \max J_j$. The guess $J_j$ requires attention at stage $s > t$ only if $r_{j,s} \leq t$ and $J_j$ has the opportunity, at stage $s$, to diagonalize on some finite $S_{j,t}$-interval $A_s$ such that $A_s$ is nonempty. Since $K_t$ is countable, there are only countably many nonempty $S_{j,t}$-intervals of $K_t$. For each cut $(S_1, S_2)$ of $S_{j,t}$ such that $A_t(j, S_1, S_2)$ is nonempty, there are at most countably many stages $s > t$ at which $R_j$ diagonalizes on $A_s(j, S_1, S_2)$. This is by construction – we never diagonalize at the same cut twice. So $R_j$ receives attention at most countably many times after stage $t$. If $r_{j,\omega_1} > t$, then after stage $r_{j,\omega_1}$, $R_j$ never requires attention, so $I_j$ is bounded below $\omega_1$. Otherwise, $R_j$ receives attention at every stage $s \in I_j \cap \{t, \omega_1\}$, so again $I_j$ is bounded below $\omega_1$.

Certainly, if $J_j$ is unbounded in $\omega_1$, then $\Phi_j$ is an isomorphism from $K$ to $L$. We have established, therefore, that in either case, $K$ and $L$ are $\omega_1$-computably isomorphic. Let $j$ be the least index such that $\Phi_j$ is an $\omega_1$-computable isomorphism from $K$ to $L$. The minimality of $j$ shows that for all $i < j$, $I_i$ is bounded below $\omega_1$; we just argued that this implies that for all $i < j$, $I_i$ is bounded below $\omega_1$. Hence $r_{j,\omega_1} < \omega_1$.

We show that $J_j$ is unbounded in $\omega_1$. Let $r = r_{j,\omega_1}$. We know that the set $H$ of stages $s \geq r$ such that $\Phi_j,s$ is an isomorphism from $K_s$ to $L_s$ is closed and unbounded in $\omega_1$. Claim 3.6 implies that to show that $J_j$ contains a final segment of $H$, it is sufficient to show that $J_j \cap [r, \omega_1)$ is nonempty. Suppose, for a contradiction, that $J_j \subseteq r$. Let $s$ be the least limit point of $H$. As $H \subseteq I_j$, case (C) shows that $F_s$ is the union of maps $F_t$ where $t \in H \cap [r, s)$, and that $F_s$ is onto $L_s$. Hence $N_{j,s} = K_s$, so $K_r$ is contained in $N_{j,s}$; so $s \in J_j$ after all, for the desired contradiction.

We have thus established the existence of $j < \omega_1$ such that $r_{j,\omega_1} < \omega_1$ but $J_j$ is unbounded in $\omega_1$. The guess $R_j$ is the “correct guess” with which we work to establish the structure theorem for $L$.

Enumerating Finite Intervals: From now, we fix $j$ such that $r_{j,\omega_1} < \omega_1$ but $J_j$ is unbounded in $\omega_1$. Let $r := r_{j,\omega_1}$. Let $S := S_{j,\omega_1} = S_{j,s}$ for all $s \in J_j \setminus r$, and let $Q := \Phi_j[S] = F_s[S]$ for such $s$.

The arguments of the proof of Theorem 2.4 show that every infinite $Q$-interval of $L$ is $\aleph_1$-saturated. In slightly more detail, let $B_{\omega_1}$ be an infinite $Q$-interval of $L$. To show that $B_{\omega_1}$ is nonscattered, assume otherwise. For $n \in \mathbb{Z}$, let $D_n$ be the $<_{\omega_1}$-least maximal infinite block of the $j$-conjugate $B_{n,\omega_1}$ of $B_{\omega_1}$. For sufficiently late $s \in J_j$, for all $n$, $D_n$ is the $<_{\omega_1}$-least maximal infinite block of $B_{n,s}$. Let $s \in J_j$ be sufficiently late. Then if $A_{n,s}$ is $<_{\omega_1}$-least among its $j$-conjugates, then $D_{n,s} = D_n$, and at stage $s$, we add points to $A_{n+1,s}$ to ensure that $D_{n,s}$ is in fact not a convex subset of $B_{n,\omega_1}$; this follows from the fact that $E_{n,s} \neq D_{n,s}$, as $E_{n,s}$ is finite. This is a contradiction, and so $B_{\omega_1}$ is nonscattered. Then, an argument identical to the one in Theorem 2.4 shows that $B_{\omega_1}$ is $\aleph_1$-saturated.
It remains to deal with finite intervals. For \( n > 0 \), at stage \( s \in J_j \setminus r \) we enumerate a cut \((Q_1, Q_2)\) of \( Q \) into \( V_n \) if the interval \( B_s = (Q_1, Q_2)_{\leq s} \) contains exactly \( n \) points. Certainly, if \((Q_1, Q_2)_{\leq} \) has size \( n > 0 \) then \((Q_1, Q_2)_{\leq} \in V_n \). We need to show that the sets \( V_n \) are pairwise disjoint. That is, we show that if \( u \in J_j \), \( u \geq r \), \( B_u \) is finite and nonempty, and \( s > u \) is also in \( J_j \), then either \( B_u \) is infinite, or \( B_u = B_v \). Fix such \( u, s \) and \( B_u \), and suppose, for contradiction, that \( B_u \) is finite but that \( B_v \neq B_u \). Let \( t \) be the least stage in \( J_j \setminus r \) such that \( B_t \) is nonempty.

The proof bifurcates into two cases. Either the \( j \)-conjugates \( B_{n,v} \) of \( B_u \) are all identical, or they are pairwise disjoint. First, suppose they are identical. In this case, we first show that if \( v \geq t \) and at stage \( v \), \( R_j \) diagonalizes on \( A_v = \Phi_j^{-1}(B_v) \), then \( B_v \) is infinite, where \( u = \min J_j \cup (v, \omega_1) \). For at stage \( v \), we ensure that \( \Phi_j \mid A_v \) cannot be extended to the unique isomorphism \( F_{v+1} \mid A_{v+1} \) from \( A_{v+1} \) to \( C_{v+1} = B_{v+1} \); so \( \Phi_j \mid A_v \) and \( F_{v+1} \) disagree on some element of \( A_v \). If \( B_v \) is finite, then \( F \mid A_v = \Phi_j \mid A_u \) is an isomorphism between \( A_v \) and \( B_v \). However, \( \Phi_j \mid A_v \) extends \( F_{v+1} \mid A_v \) extends \( F_{v+1} \mid A_{v+1} \), because the instructions don’t allow \( R_j \) to change \( F \) on \( A \) between stages \( v \) and \( w \); we also use Claim 3.3. This is impossible.

This implies that \( R_j \) did not diagonalize on \( B_v \) at any stage \( v \in [t, s) \). However, let \( v \) be the least stage in \([u, s) \) such that \( B_{v+1} = B_s \); so \( B_v \neq B_v \). Then \( R_j \) has the opportunity to diagonalize on \( A_v \) at stage \( v \), because it did not do so at an earlier stage, and \( |A_v| = |B_v| < |B_{v+1}| = |C_v| \). This is a contradiction.

Suppose now that the intervals \( B_{n,s} \) are pairwise disjoint. Let \( m = |B_s| \). Since \( s \in J_j \), \( m = |B_{n,s}| \) for every \( j \)-conjugate \( B_{n,s} \) of \( B_u \). We first show that there is some stage \( v \in [u, s) \) at which \( R_j \) diagonalizes on some conjugate \( A_{n,u} \) with \( m \) points. Suppose otherwise. As for all \( n \in \mathbb{Z} \), \( |B_{n,u}| = |B_u| < m \), certainly \( R_j \) does not diagonalize on any \( A_{n,v} \) with \( m \) points at any stage \( v \in [t, u) \). For each \( m' < m \), there is at most one \( n \) such that \( R_j \) diagonalizes with \( m' \) points on \( A_{n,v} \) (at any stage \( v \in [t, s) \)). Certainly \( R_j \) does not diagonalize on any conjugate \( A_{n,v} \) with more than \( m \) points at any stage before \( s \). Let \( k \) be the maximal integer such that \( R_j \) diagonalized on \( A_{n,v} \) at some \( v \in [t, s) \). This is well-defined as the conjugates \( A_{n,s} \) are pairwise disjoint. Let \( v_0 \) be the least stage \( v < s \) at which \( |B_{l,v+1}| = m \) for some \( l \geq k \). Let \( l \) be the least integer \( l \geq k \) such that \( |B_{l,v_0+1}| = m \). Let \( v_1 \) be the least stage \( v < s \) at which \( |B_{l,v+1}| = m \); so \( v_1 \geq v_0 \). Then \( |B_{l,v_1+1}| = |B_{l+1,v+1}| = m \) and \( |A_{l+1,v_1}| = |B_{l+1,v_1}| < m \). So at stage \( v_1 \), \( R_j \) has the opportunity to diagonalize on \( A_{l+1,v_1} \) with \( m \) points, which is impossible.

Let \( v \in [u, s) \) be a stage at which \( R_j \) diagonalizes on some \( A_{n,v} \) with \( m \) points. At stage \( v \), we ensure that \( \Phi_j \mid A_v \) cannot be extended to an isomorphism of \( A_{n,v+1} \) and \( C_{n,v+1} \). However, \( A_{n,v+1} = A_{n,v} \) and \( C_{n,v+1} = C_{n,v} \), and \( \Phi_j \mid A_{n,v} \) is an extension of \( \Phi_j \mid A_v \) to precisely such an isomorphism. This yields the desired contradiction, with which we conclude the proof of Theorem 3.1.

4. Open Questions

If every infinite linear order had proper self-embeddings, it would seem possible to generalize the characterization of computable categoricity to higher cardinals. As this is not the case for linear orders of size \( \aleph_1 \) (see [2]), we ask for a characterization of computable categoricity at the next cardinal.
Question 4.1. Which $\omega_2$-computable linear orders are $\omega_2$-computably categorical?

The precise analogue of Theorem 3.1 does not hold for $\omega_2$-computably categorical linear orders. One of the obstacles is the existence of linear orders of size $\aleph_1$ which have no proper self-embeddings. A linear ordering obtained by taking a set of parameters of size $\aleph_1$ with $\aleph_2$ many cuts, and inserting a fixed linear order with no self-embedding in each cut, will be $\omega_2$-computably categorical. The anonymous referees gave an even more compelling example. They take the $\omega_2$-sum of linear orders $R_\alpha$ ($\alpha < \omega_2$), no interval of which can be embedded into another (these can be taken to be subsets of $\mathbb{R}$). This counter-example (which is in fact uniformly $\omega_2$-computably categorical) has cofinality $\omega_2$ but every proper initial segment has size $\aleph_1$.

We also pose a methodological question:

**Question 4.2.** What effects do combinatorial principles such as ♦ have on the effectiveness properties of uncountable linear orders (either of size $\aleph_1$ or even $\aleph_2$)?

We note that Jensen’s original proof of ♦ shows the existence of an $\omega_1$-computable ♦-sequence.

**References**


Department of Mathematics, Victoria University of Wellington, Wellington, New Zealand

E-mail address: Noam.Greenberg@msor.vuw.ac.nz
URL: http://homepages.mcs.vuw.ac.nz/~greenberg/

Department of Mathematics, University of Connecticut-Storrs, Storrs, CT 06269-3009, USA

E-mail address: asher.kach@uconn.edu
URL: http://www.math.uconn.edu/~kach/

Department of Mathematics, University of Wisconsin, Madison, WI 53706-1388, USA

E-mail address: lempp@math.wisc.edu
URL: http://www.math.wisc.edu/~lempp/

Kurt Gödel Research Center, University of Vienna, Vienna, Austria

E-mail address: turetsd4@univie.ac.at
URL: http://tinyurl.com/dturetsky