COMPUTABILITY AND UNCOUNTABLE LINEAR ORDERS

NOAM GREENBERG, ASHER M. KACH, STEFFEN LEMPP, AND DANIEL D. TURETSKY

Abstract. We study the computable structure theory of linear orders of size $\aleph_1$ within the framework of admissible computability theory. In particular, we study degree spectra, computable categoricity, and the successor relation.

1. Introduction

Effective properties of countable linear orderings have been studied extensively since the 1960's. This line of research, surveyed in Downey [4], is part of a broader program of understanding the information content of mathematical structures. Among the notions central to this theory are the notions of *computable categoricity* and the *degree spectrum* of a structure. A computable structure is said to be computably categorical if it is effectively isomorphic to any of its computable copies. The degree spectrum of a structure is the collection of Turing degrees which contain a copy of the structure. Examples of major results concerning the effective properties of linear orderings are the Dzgoev [9] and Remmel [27] characterization of the computably categorical linear orderings as those with finitely many successivities and the Richter [29] theorem that the computable order-types are the only ones whose degree spectrum contains a least element.

Traditionally, the domain of computability theory consists of hereditarily finite objects (for example the natural numbers, finite sequences and sets of natural numbers, and so on). For this reason, effectiveness considerations have mostly been applied only to countable mathematical structures. Early on, though, generalizations of the theory of computable functions on domains of larger cardinality were considered. Takeuti [32] and [33] generalized recursion theory to the class of all ordinals. Kreisel and Sacks [19] and [20], following work of Kreisel [18], developed metarecursion theory, which is the study of computability on the computable ordinals, or equivalently, on their notations. These two approaches were unified by Kripke [21] and Platek [26] in the study of recursion theory on admissible ordinals.

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Greenberg and Knight [15] initiated the application of admissible computability theory to the study of effectiveness properties of uncountable structures. Under the assumption that all reals are constructible, they investigate the analogues of classical results about fields and vector spaces, results from pure computable model theory such as the relationship between Scott families and computable categoricity, and results about linear orderings.

A main interest in these investigations is the contrast between the countable and the uncountable case. Some results from classical computability theory, about countable structures, generalize to the uncountable case, albeit with sometimes different proofs. Other classical results fail in the uncountable setting. For example, Greenberg and Knight show that Richter's result mentioned above fails for uncountable cardinals; in fact, in the uncountable setting, every degree is the least degree of the spectrum of some linear ordering. In either case, the examination of classical results in new surroundings sheds light on the classical theory, often by highlighting essential assumptions that go without notice if generalizations are not considered, and by separating notions that happen to coincide in the countable setting.

A major theme arising from this work is the importance of the notion of true finiteness. In ω_1-computability, the correct analogue for "finite" is "countable". For example, ω_1-computations take countably many steps and manipulate hereditarily countable objects. Yet, true finiteness has some inherent properties which do not generalize. Lerman and Simpson [22] and Lerman [23] exhibited the effects of the difference between true finiteness and its generalization on the lattice of c.e. sets under inclusion; Greenberg [14] has exhibited the effects on the c.e. degrees. For linear orderings, the important observation is that while a finite set can determine only finitely many cuts in a linear ordering, a countable set may determine uncountably many cuts in a linear ordering. This difference underlies the failure of Richter's theorem for ω_1, as well as many of the other differences we shall see.

In this paper, we continue the investigation started by Greenberg and Knight [15], concentrating on linear orderings of size $\aleph_1$. We again assume that $\mathbb{R} \subset L$, the pertinent effect of which is that L_{ω_1} is amenable in V; that is, L_{ω_1} coincides with H_{ω_1}, the collection of hereditarily countable sets. This assumption implies the continuum hypothesis in a strong sense: It gives an ω_1-computable bijection between 2^ω and ω_1.

Again, true finiteness plays a central role. However, we observe a new aspect of working with linear orderings of size $\aleph_1$. In particular, in Theorem 3.6, we give a characterization of computable categoricity of linear orderings of size $\aleph_1$, which is a generalization of the Dzgoev-Remmel theorem mentioned above.

In Section 2, we study degree spectra of (order-types of) linear orderings of size $\aleph_1$. Jockusch and Soare [16] showed that there is a countable order-type having low presentations but no computable presentation. Various strengthenings of this result included the construction of R. Miller [24] of a countable linear ordering which has a copy in every nonzero Δ^0_2 Turing degree, but no computable copy; Downey later observed that in fact this ordering has a copy in every hyperimmune degree. In Theorem 2.7, we give an uncountable analogue of R. Miller's result.

In the countable context, Goncharov, Harizanov, Knight, McCoy, R. Miller, and Solomon [13] showed that there are structures whose degree spectra consist of exactly the nonlow degrees; it is unknown if there is a countable linear ordering with this degree spectrum. In Theorem 2.18, we show that for any finite n, there is a linear ordering of size $\aleph_1$ whose degree spectrum is the collection of ω_1-nonlow$_n$ degrees. This again is a testament to the stronger (or at least easier) coding power vested in uncountable linear orderings.

In the same section, we also discuss finite jump degrees. As mentioned above, Richter [29] showed that the only degree of a countable order-type is $\mathbf{0}$. Knight [17] showed that the only jump degree of a countable order-type is $\mathbf{0}'$. However, Downey and Knight [5] (building on work of Ash, Jockusch, and Knight [1] and Ash and Knight [2]) showed that for all computable ordinals $\alpha \geq 2$, every degree $\mathbf{d} \geq \mathbf{0}^{(\alpha)}$ is the proper α^{th} jump degree of a countable order-type. As mentioned above, Greenberg and Knight [15] showed that every ω_1-Turing degree is the degree of an order-type. We show in Theorem 2.21 that every ω_1-Turing degree $\mathbf{d} \geq \mathbf{0}^{(n)}$ is the proper n^{th} jump degree of an order-type. In Theorem 2.10, however, we show that the primary tool used by Downey and Knight for the countable case does not carry over to the ω_1-setting.

In Section 3, as mentioned above, we characterize the ω_1-computably categorical linear orderings. We also characterize the uniformly ω_1-computably categorical linear orderings (Theorem 3.4). In Section 4 we study the complexity of the successor relation on a linear ordering. Recently, Downey, Lempp, and Wu [6] complemented work by Frolov [10] to show that for any ω-computable linear ordering $\mathcal{L}$, the collection of degrees of the successor relation in computable copies of $\mathcal{L}$ is upward closed in the c.e. degrees. For orderings of size $\aleph_1$, the situation is radically different. For example, in Example 4.2, we show that the successor relation can be intrinsically computable, that is, there is an ω_1-computable order-type λ such that the successor relation is computable in any computable presentation of λ. We identify a dichotomy between two kinds of linear orderings of size $\aleph_1$: Roughly speaking, between those which contain a copy of the rational numbers which demarcates the successivities of the linear ordering, and those which do not. The latter case behaves similarly to countable linear orderings in that the degrees of the successor relation in computable copies are upward closed in the c.e. degrees (Theorem 4.4). The other case is interesting; we identify an interval in the c.e. degrees which contains all the degrees of the successor relation in computable copies of the given linear ordering. The top and bottom degree in this interval are always realized as the degrees of the successor relation in some copy, but not all degrees in the interval need to be so realized (although they can be). As a corollary, we see that for any ω_1-c.e. degree $\mathbf{d}$, there is an ω_1-computable linear ordering $\mathcal{L}$ such that the degree of the successor relation in every ω_1-computable copy of $\mathcal{L}$ is $\mathbf{d}$.

1.1. **Notation, Terminology, Background.** We refer the reader to Sacks [31] for additional background on admissible computability, and to Greenberg and Knight [15] for specific background on ω_1-computability theory, for definitions and basic facts on ω_1-computable model theory, and for effectiveness properties of linear orderings of size $\aleph_1$ in particular. In order to distinguish computability in the countable case from computability on the admissible ordinal ω_1, we will usually denote computability in the former case as *ω-computability* and the latter case as *ω_1-computability*; in this paper, though, when we omit the prefix, we mean ω_1-computability. For the entire paper, we assume that every real is constructible.

We also refer the reader to Rosenstein [30] for additional background on order-types and linear orders. We use the following notation and terminology for linear orders.

Definition 1.1. Let $\mathcal{L} = (L, <_\mathcal{L})$ be a linear order.

(1) We denote by $\mathcal{L}^* := (L, <_{\mathcal{L}^*})$ the reverse ordering on L, where $x <_{\mathcal{L}^*} y$ if and only if $y <_\mathcal{L} x$.
(2) A subset X of L is *convex*, or an $\mathcal{L}$-*interval*, if for all $x, y \in X$ and $z \in L$, if $x <_\mathcal{L} z <_\mathcal{L} y$ then $z \in X$.
(3) If $A, B \subseteq L$, then we write $A <_\mathcal{L} B$ if $a <_\mathcal{L} b$ for all $a \in A$ and $b \in B$. If $A = \{a\}$, we also write $a <_\mathcal{L} B$, etc.
(4) If $A, B \subseteq L$ and $A <_\mathcal{L} B$, then we let $(A, B)_\mathcal{L}$, the $\mathcal{L}$-*interval* determined by A and B, be the convex set $\{x \in L : A <_\mathcal{L} x <_\mathcal{L} B\}$. Note that this is well-defined even if A or B is empty. If $A = \{a\}$, we also write $(a, B)_\mathcal{L}$; if $A = \emptyset$, we also write $(-\infty, B)_\mathcal{L}$; etc.
(5) If $Q \subseteq L$, then a *cut* of Q is a partition of Q into subsets Q_1 and Q_2 such that $Q_1 <_\mathcal{L} Q_2$.
(6) If $Q \subseteq L$, then a Q-*interval* of $\mathcal{L}$ is an $\mathcal{L}$-interval determined by some cut of Q.
(7) A *block* of $\mathcal{L}$ is a nonempty convex subset X of L such that for all $a, b \in X$, the interval $(a, b)_\mathcal{L}$ is finite. Note that every block is at most countable in size.

Definition 1.2. Let $\mathcal{L}$ be a linear ordering. A pair of elements $a <_\mathcal{L} b$ in $\mathcal{L}$ are *adjacent* if $(a, b)_\mathcal{L}$ is empty. We say that a is the *predecessor* of b (in $\mathcal{L}$) and b is the *successor* of a (in $\mathcal{L}$).

We let $\mathrm{Succ}(\mathcal{L})$ be the collection of all pairs (a, b) of elements of $\mathcal{L}$ such that b is the successor of a.

We note that $\mathrm{Succ}(\mathcal{L})$ has a $\Pi^0_1(\mathcal{L})$-definition, namely $(a, b) \in \mathrm{Succ}(\mathcal{L})$ if and only if $a <_\mathcal{L} b$ and there is no $c \in \mathcal{L}$ lying between a and b.

An *order-type* is an isomorphism class of linear orderings, although we often identify the order-type of a well-ordering with the unique ordinal it contains. We write $\mathrm{otp}(\mathcal{L})$ for the order-type of a linear ordering $\mathcal{L}$. An element of an order-type λ is also called a *presentation* of λ.

If P is a property of linear orderings, then we say that an order-type λ has property P if some presentation of λ has property P. Hence, we say that λ is *computable* if it has a computable presentation, and that the *size* of λ is some cardinal κ if the presentations of λ have cardinality κ.

Remark 1.3. We remark that we never refer to the size of λ as a collection of linear orderings. In this paper, we assume that the universe of any linear ordering is a subset of H_{ω_1}. So each order-type is indeed a set, rather than a proper class, but if its presentations are of size $\aleph_1$, the set has size $2^{\aleph_1}$.

We fix notation for some order-types which will appear often in this paper.

Definition 1.4. We denote the order-type of the least infinite ordinal by ω, which is, of course, the order-type of the natural numbers. We denote the order-type of the least uncountable ordinal by ω_1, which is, of course, the order-type of the collection of countable ordinals.

We denote the order-type of the rational numbers by η (also by η_0). This is the saturated countable order-type. We denote the saturated order-type of size $\aleph_1$ by η_1.

We denote the order-type of the integers $\mathbb{Z}$ by ζ and the order-type of the real numbers $\mathbb{R}$ by ρ.

We note the existence of η_1 follows from the continuum hypothesis. By Cantor's argument, a linear order $\mathcal{L}$ of size $\aleph_1$ is saturated if and only if for any at most countable sets $A, B \subset L$ such that $A <_{\mathcal{L}} B$, the interval $(A, B)_{\mathcal{L}}$ is nonempty. Note that as A or B may be empty, this implies that $\mathcal{L}$ has uncountable coinitiality and cofinality.

It will often be important whether a linear order has a subset of order-type η_0.

Definition 1.5. A linear order is *nonscattered* if it has a subset of order type η_0 and *scattered* otherwise.

We use standard sum and product notation.

Definition 1.6. If $\mathcal{A} = (A, <_{\mathcal{A}})$ and $\mathcal{B} = (B, <_{\mathcal{B}})$ are linear orderings, then $\mathcal{A} + \mathcal{B}$ is the linear ordering obtained by appending $\mathcal{B}$ to the right of $\mathcal{A}$.

The addition operation can be extended to arbitrary collections of linear orderings, indexed by elements of a linear ordering. If $\mathcal{A} = (A, <_{\mathcal{A}})$ is a linear ordering, and $\langle \mathcal{L}_a \rangle_{a \in A}$ is a sequence of linear orderings, then $\sum_{a \in \mathcal{A}} \mathcal{L}_a$ is the linear ordering obtained by replacing every point $a \in A$ by a copy of $\mathcal{L}_a$.

If $\mathcal{A}$ and $\mathcal{B}$ are linear orderings, then $\mathcal{A} \cdot \mathcal{B}$ is $\sum_{b \in \mathcal{B}} \mathcal{A}$, the result of replacing every point in $\mathcal{B}$ by a copy of $\mathcal{A}$. Equivalently, it is the linear ordering on $A \times B$ defined by the right-lexicographic product of $<_{\mathcal{A}}$ and $<_{\mathcal{B}}$.

As these operations are invariant under isomorphisms, we extend the notation to order-types as well.

We also use restrictions of linear orderings.

Definition 1.7. Let $\mathcal{A} = (A, <_{\mathcal{A}})$ be a linear ordering. If $B \subseteq A$, then we let $\mathcal{A} \restriction B$ be the linear ordering $(B, <_{\mathcal{A}} \restriction B^2)$. If $A \subseteq \omega_1$ and $\alpha < \omega_1$, then we let $\mathcal{A} \restriction \alpha$ be $\mathcal{A} \restriction (A \cap \alpha)$, recalling that a von Neumann ordinal is the collection of its predecessors. We also denote $\mathcal{A} \restriction \alpha$ by $\mathcal{A}_\alpha$.

If $B \subseteq A$, we let

$$\mathrm{dcl}(B) := \{b \in A : (\exists c \in B)[b \leq_{\mathcal{A}} c]\}$$

and

$$\mathrm{ucl}(B) := \{b \in A : (\exists c \in B)[b \geq_{\mathcal{A}} c]\}$$

be the downward closure and upward closure of B, respectively. When $\mathcal{A}$ is possibly ambiguous, we write $\mathrm{dcl}_{\mathcal{A}}(B)$ and $\mathrm{ucl}_{\mathcal{A}}(B)$, respectively.

We will make use of the linear orderings $\mathbb{Z}^\alpha$, where $\alpha \leq \omega_1$.

Definition 1.8. By recursion on ordinals α, we define a directed system of linear orderings and embeddings $\langle \mathbb{Z}^\alpha, \iota_{\beta,\alpha} \rangle$. We let $\mathbb{Z}^0 := 1$. Given $\mathbb{Z}^\alpha$, we let $\mathbb{Z}^{\alpha+1} := \mathbb{Z}^\alpha \cdot \mathbb{Z}$, and define $\iota_{\alpha,\alpha+1} \colon \mathbb{Z}^\alpha \to \mathbb{Z}^{\alpha+1}$ by letting $\iota_{\alpha,\alpha+1}(x) := (x, 0)$. In other words, $\mathbb{Z}^{\alpha+1}$ is obtained from $\mathbb{Z}^\alpha$ by adding ω many copies of $\mathbb{Z}^\alpha$ to the right, and ω^* many copies of $\mathbb{Z}^\alpha$ to the left. For $\beta < \alpha$, we let $\iota_{\beta,\alpha+1} := \iota_{\alpha,\alpha+1} \circ \iota_{\beta,\alpha}$. At limit stages δ, we let $\mathbb{Z}^\delta$ be the direct limit of the system $\langle \mathbb{Z}^\alpha, \iota_{\beta,\alpha} \rangle_{\beta < \alpha < \delta}$, and the maps $\iota_{\beta,\delta}$ be the limit of the maps $\langle \iota_{\beta,\alpha} \rangle_{\beta < \alpha < \delta}$.

By induction, it is easy to see that every map $\iota_{\beta,\alpha}$ is a *convex* embedding of $\mathbb{Z}^\beta$ into $\mathbb{Z}^\alpha$ (i.e., its image is convex), that each $\mathbb{Z}^\alpha$ is discrete, and that the maximal blocks in each $\mathbb{Z}^\alpha$ (for $\alpha > 0$) are all infinite.

Lemma 1.9. *Let $\alpha \leq \omega_1$.*

(1) $(\mathbb{Z}^\alpha)^* \cong \mathbb{Z}^\alpha$.
(2) *There is no embedding of* $\mathbb{Z}^\alpha$ *into a proper initial segment of itself; so, a fortiori, if* $\gamma < \beta \leq \omega_1$*, then there is no embedding of* $\mathbb{Z}^\beta$ *into* $\mathbb{Z}^\gamma$*.*

Proof. (1) is proved by induction on α, taking direct limits on both sides at limit stages.

(2) is proved by induction on α. Suppose this is known for α. Suppose that there is an embedding of $\mathbb{Z}^{\alpha+1}$ into a proper initial segment of itself. Then there is an embedding $f\colon \mathbb{Z}^{\alpha+1} \to \mathbb{Z}^\alpha \cdot \omega^*$. By taking a rightmost copy of $\mathbb{Z}^\alpha$ in $\mathbb{Z}^\alpha \cdot \omega^*$ intersecting the range of f, we get an embedding of $\mathbb{Z}^\alpha \cdot \omega$ into $\mathbb{Z}^\alpha$, contradicting the induction assumption for $\mathbb{Z}^\alpha$.

Let α be a limit ordinal, suppose that the lemma is verified for all $\beta < \alpha$, and suppose that f is an embedding of $\mathbb{Z}^\alpha$ into a proper initial segment of itself. Since $\mathbb{Z}^\alpha = \bigcup_{\beta<\alpha} \iota_{\beta,\alpha}\left[\mathbb{Z}^\beta\right]$, and since each embedding $\iota_{\beta,\alpha}$ is convex, there is a nonempty final segment of $\mathbb{Z}^\alpha$ whose image under f is contained in $\iota_{\beta,\alpha}[\mathbb{Z}^\beta]$ for some $\beta < \alpha$. This allows us to find an embedding of $\mathbb{Z}^{\beta+1}$ into $\mathbb{Z}^\beta$, again contradicting the induction assumption. □

We also use shuffle sums of linear orders. We recall that in the countable setting, an η_0-shuffle sum of a countable collection of linear orders $\{\mathcal{L}_i\}_{i\in I}$ (denoted $\sigma_0(\{\mathcal{L}_i\}_{i\in I})$) is the linear order obtained by partitioning η_0 into $|I|$ many dense, codense sets and replacing each point in the ith set by a copy of $\mathcal{L}_i$.

Definition 1.10. Let $\mathbb{Q}_1 \in \eta_1$, that is, let $\mathbb{Q}_1$ be a saturated linear ordering of size $\aleph_1$. A set $Z \subseteq \mathbb{Q}_1$ is *saturated in* $\mathbb{Q}_1$ if for all countable $A, B \subset \mathbb{Q}_1$, the interval $(A, B)_{\mathbb{Q}_1} \cap Z$ is nonempty. A standard construction shows that for any cardinal $\kappa \leq \aleph_1$, there is a partition of $\mathbb{Q}_1$ into sets $\langle Z_\alpha \rangle_{\alpha<\kappa}$, each of which is saturated in $\mathbb{Q}_1$.

Let $\kappa \leq \aleph_1$ be a cardinal and let $\langle \mathcal{L}_\alpha \rangle_{\alpha<\kappa}$ be a sequence of linear orderings. The η_1*-shuffle sum* of this sequence is obtained by replacing each point in Z_α by $\mathcal{L}_\alpha$. A back-and-forth argument shows that the order-type of the shuffle sum does not depend on the choice of the sets Z_α, nor does it depend on the ordering of the sequence $\langle \mathcal{L}_\alpha \rangle_{\alpha<\kappa}$. We can thus unambiguously define, for a *set* Λ of order-types such that $|\Lambda| \leq \aleph_1$, the order-type $\sigma_1(\Lambda)$ of the shuffle sum of the order-types in Λ.

Finally, we list results of ω-computability theory and ω-computable structure theory (stated in the ω_1-framework) which also hold in the ω_1-framework, with similar or easier proofs.

Fact 1.11.

(1) There is an ω_1-computable bijection between ω_1 and the universe H_{ω_1}. This bijection induces an ω_1-computable ordering of H_{ω_1} of order-type ω_1, denoted by $<_{\omega_1}$.
(2) There is a uniformly ω_1-computable list $\langle \mathcal{L}_\beta \rangle_{\beta<\omega_1}$ of ω_1-computable linear orderings such that for any ω_1-computable linear ordering $\mathcal{A}$ there is some $\beta < \omega_1$ such that $\mathcal{A} \cong \mathcal{L}_\beta$.
(3) For any ω_1-degree $\mathbf{d} \geq \mathbf{0}'$, there is an ω_1-degree $\mathbf{a}$ such that $\mathbf{a}' = \mathbf{d}$. In fact, there are incomparable ω_1-degrees $\mathbf{a}_1$ and $\mathbf{a}_2$ such that $\mathbf{a}_1' = \mathbf{d} = \mathbf{a}_2'$. Hence, there are non-$\omega_1$-computable low degrees.
(4) For any $n < \omega$ and any ω_1-degree $\mathbf{d} \geq \mathbf{0}^{(n)}$, there is an ω_1-degree $\mathbf{a}$ such that $\mathbf{a}^{(n)} = \mathbf{d}$. Moreover, provided $\mathbf{d} > \mathbf{0}^{(n)}$, for every ω_1-degree $\mathbf{a}_1$ with

$\mathbf{d} = \mathbf{a}_1^{(n)}$, there is an ω_1-degree $\mathbf{a}_2$ with $\mathbf{d} = \mathbf{a}_2^{(n)}$ and $\mathbf{a}_1^{(m)} \mid \mathbf{a}_2^{(m)}$ for any $m < n$.

2. Degree Spectra of Linear Orderings

In this section, we exhibit an order-type whose degree spectrum includes all hyperimmune ω_1-degrees but omits $\mathbf{0}$ (Subsection 2.1); a transfer theorem for all order-types (Subsection 2.2); for every finite n, an order-type whose degree spectrum is precisely the collection of non-low$_n$ ω_1-degrees (Subsection 2.3); and for each degree $\mathbf{d} \geq \mathbf{0}'$, an order-type of proper jump degree $\mathbf{d}$ (Subsection 2.4).

We recall the definition of the degree spectrum of an order-type.

Definition 2.1. For an order-type λ of size at most $\aleph_1$, we let $\mathrm{DegSpec}(\lambda)$, the *degree spectrum* of λ, be the collection of ω_1-Turing degrees of presentations of λ.

Recall (Remark 1.3) that we assume that the universe of any linear ordering is a subset of H_{ω_1}, and so every linear ordering indeed has a Turing degree.

We abuse notation slightly by writing $\mathrm{DegSpec}(\mathcal{L})$ for $\mathrm{DegSpec}(\mathrm{otp}(\mathcal{L}))$ for a linear ordering $\mathcal{L}$ of size at most $\aleph_1$.

A theorem of Knight [17] generalizes to the ω_1-context; for any order-type λ of size $\aleph_1$, an ω_1-Turing degree $\mathbf{d}$ is in the degree spectrum of λ if and only if it computes a presentation of λ.

2.1. **A Hyperimmune Spectrum.**

As mentioned above, R. Miller [24] demonstrated the existence of a countable, non-ω-computable order-type that has a presentation in every nonzero Δ^0_2 ω-degree. Miller built an order-type λ of the form $\sum_{i\in\omega}(\sigma_i + \kappa_i)$, where $\sigma_i = 1 + \eta + i + \eta + 1$ and κ_i was either ω or $c_i + \zeta$ for some $c_i < \omega$.

The purpose of the *separators* σ_i (the idea of which originates in [16]) was to divide λ into countably many intervals; the purpose of the *diagonalizers* κ_i was to diagonalize against the i^{th} computable linear order.

An inspection of Miller's proof shows that the linear ordering he constructed has a copy in every hyperimmune ω-degree. Recall that Rice [28] and Uspenskii [34] showed an ω-Turing degree is *hyperimmune* if and only if it computes a total function $f\colon \omega \to \omega$ such that for any total ω-computable function $g\colon \omega \to \omega$ there are infinitely many numbers n such that $f(n) > g(n)$.

Beyond ω, Chong and Wang [3] studied hyperimmune and hyperimmune-free α-degrees for various admissible ordinals α. Under our assumption that all reals are constructible, every subset of ω_1 is amenable and admissible. Under these conditions, Chong and Wang give a straightforward generalization of the countable concept: an ω_1-Turing degree $\mathbf{a}$ is *hyperimmune* if and only if it contains a set A such that for every computable list $\langle F_\alpha \rangle$ of pairwise disjoint countable subsets of ω_1, there is some $\alpha < \omega_1$ such that $F_\alpha \cap A \neq \emptyset$. Chong and Wang show that an ω_1-Turing degree is hyperimmune if and only if it computes a total function $f\colon \omega_1 \to \omega_1$ such that for any total ω_1-computable function $g\colon \omega_1 \to \omega_1$ there are uncountably many ordinals β such that $f(\beta) > g(\beta)$.

We build a linear order which has no ω_1-computable copy, but whose degree spectrum contains every hyperimmune ω_1-degree. This ordering $\mathcal{L}$ will be of the form $\sum_{\beta\in\omega_1}(\mathcal{S}_\beta + \mathcal{K}_\beta)$. The orderings $\mathcal{S}_\beta$ will serve as *separators*, denoting the location of the *diagonalizers* $\mathcal{K}_\beta$. We first discuss these building blocks of $\mathcal{L}$, and then give the construction defining $\mathcal{L}$.

Definition 2.2. Fix an enumeration $\langle q_i \rangle_{i<\omega}$ of the rational numbers $\mathbb{Q}$ and a computable enumeration $\langle r_\alpha \rangle_{\alpha<\omega_1}$ of the irrational numbers $\mathbb{I}$. We let $\mathcal{S}_\beta$ be obtained from $\mathbb{R}$ by omitting all irrational numbers but r_β, and by replacing the rational number q_i by $i+2$ many points.

Formally, for $r \in \mathbb{R}$, we define

$$\mathcal{C}_{r,\beta} := \begin{cases} 1 & \text{if } r = r_\beta, \\ i+2 & \text{if } r = q_i, \\ 0 & \text{otherwise,} \end{cases}$$

and let $\mathcal{S}_\beta := \sum_{r\in\mathbb{R}} \mathcal{C}_{r,\beta}$.

Each linear ordering $\mathcal{S}_\beta$ is countable, and the map $\beta \mapsto \mathcal{S}_\beta$ is computable.

Lemma 2.3. *Let $\beta, \gamma < \omega_1$ be distinct.*

(1) The linear order $\mathcal{S}_\beta$ is not isomorphic to any proper convex subset of itself.
(2) The linear order $\mathcal{S}_\beta$ is not isomorphic to any convex subset of $\mathcal{S}_\gamma$.

Proof. The point is that for all $i < \omega$ and all $\beta < \omega_1$, the suborder $\mathcal{C}_{q_i,\beta}$ is the unique maximal block of $\mathcal{S}_\beta$ of size $i+2$. Hence if $f\colon \mathcal{S}_\beta \to \mathcal{S}_\gamma$ is a convex embedding, then for all $i < \omega$ it must be that $f[\mathcal{C}_{q_i,\beta}]$ equals $\mathcal{C}_{q_i,\gamma}$. This implies that the range of f is $\mathcal{S}_\gamma$, and so also that $\beta = \gamma$. □

The diagonalizers $\mathcal{K}_\beta$ are built as sums of the linear orders $\mathbb{Z}^\alpha$ for $\alpha < \omega_1$. For $\beta \leq \omega_1$, we let $\mathcal{A}_\beta := \sum_{\alpha<\beta} \mathbb{Z}^\alpha$ and $\mathcal{B}_\beta := (\mathcal{A}_\beta)^*$; the latter is isomorphic to $\sum_{\alpha\in\beta^*} \mathbb{Z}^\alpha$ (with an abuse of notation). For $\beta < \gamma \leq \omega_1$, let $j_{\beta,\gamma}$ be the canonical initial segment embedding of $\mathcal{A}_\beta$ into $\mathcal{A}_\gamma$.

Lemma 2.4. *Let $\beta \leq \omega_1$.*

(1) There is no embedding of $\mathcal{A}_\beta$ into a proper initial segment of itself.
(2) If β is a limit ordinal, then there is no proper initial segment of $\mathcal{A}_\beta$ into which there is an embedding of $\mathcal{A}_\gamma$ for all $\gamma < \beta$.

Proof. Both parts follow from Lemma 1.9(2). □

It follows that if a linear order $\mathcal{L}$ is isomorphic to the sum $\mathcal{A}_\alpha + \mathcal{B}_\beta$ for some ordinals α and β, then there is a unique decomposition of $\mathcal{L}$ as a sum of linear orderings $\mathcal{L}_1 + \mathcal{L}_2$ such that $\mathcal{L}_1 \cong \mathcal{A}_\alpha$ and $\mathcal{L}_2 \cong \mathcal{B}_\beta$.

Lemma 2.5. *Let $\beta < \omega_1$.*

(1) For any limit ordinal $\delta \leq \omega_1$, the order $\mathcal{A}_\delta$ is isomorphic to the direct limit of the directed system $\langle \mathcal{A}_\beta, j_{\beta,\gamma} \rangle_{\beta<\gamma<\delta}$.
(2) For any nonempty initial segment $\mathcal{C}$ of $\mathcal{B}_{\omega_1}$, there is an embedding of $\mathcal{A}_{\beta+1}$ into $\mathcal{A}_\beta + \mathcal{C}$.
(3) There is an embedding of $\mathcal{A}_\beta + \mathcal{B}_\beta$ into $\mathcal{A}_{\beta+1}$ extending $j_{\beta,\beta+1}$.

Proof. (1) is immediate. For (2), it suffices to show that for all β, the order $\mathbb{Z}^\beta$ is embeddable in $\mathcal{C}$, which is immediate.

For (3), it suffices to show that there is an embedding of $\mathcal{B}_\beta$ into $\mathbb{Z}^\beta$. This is proved by induction. Suppose that f_β is an embedding of $\mathcal{B}_\beta$ into $\mathbb{Z}^\beta$. As $\mathcal{B}_{\beta+1} \cong \mathbb{Z}^\beta + \mathcal{B}_\beta$, we can extend f_β to an embedding of $\mathcal{B}_{\beta+1}$ into $\mathbb{Z}^\beta \cdot 2$, and hence into $\mathbb{Z}^{\beta+1}$. For a limit ordinal β, let $\langle \beta_n \rangle_{n<\omega}$ be an increasing and cofinal sequence in β; for $n < \omega$, let f_{β_n} be an embedding of $\mathcal{B}_{\beta_n}$ into $\mathbb{Z}^{\beta_n}$. If $j^*_{\beta_n,\beta_{n+1}}$ is the canonical final segment embedding of $\mathcal{B}_{\beta_n}$ into $\mathcal{B}_{\beta_{n+1}}$ (the analogue of $j_{\beta_n,\beta_{n+1}}$), we can inductively construct embeddings $g_{\beta_n} : \mathcal{B}_{\beta_n} \to \mathbb{Z}^{\beta_n+1}$ so that $g_{\beta_{n+1}} \circ j_{\beta_n,\beta_{n+1}}$ agrees with g_{β_n}. The limit of these maps is then an embedding of $\mathcal{B}_\beta$ into $\mathbb{Z}^\beta$. □

Our separators and building blocks do not interact:

Lemma 2.6. *For all $\alpha, \beta < \omega_1$, no nonempty initial or final segment of $\mathcal{S}_\beta$ is isomorphic to any convex subset of $\mathcal{A}_\alpha$ or $\mathcal{B}_\alpha$.*

Proof. For any $\alpha > 0$, every maximal block in $\mathbb{Z}^\alpha$ is infinite, whereas $\mathcal{S}_\beta$ contains no infinite blocks. Hence for any α, no nonempty initial or final segment of $\mathcal{S}_\beta$ is isomorphic to any convex subset of $\mathbb{Z}^\alpha$. The lemma follows. □

We are now ready to prove the main result of this subsection. We note that the construction below only relies on the properties of the orderings $\mathcal{S}_\beta$, $\mathcal{A}_\beta$, and $\mathcal{B}_\beta$ detailed in Lemma 2.3, Lemma 2.4, Lemma 2.5, and Lemma 2.6. In a sense, this is a modular approach to the construction, which we believe sheds light on Miller's construction as well.

Theorem 2.7. *There is a linear ordering $\mathcal{L}$ of size $\aleph_1$ such that* DegSpec($\mathcal{L}$) *contains every hyperimmune ω_1-degree, but does not contain* **0**.

Proof. The linear order $\mathcal{L}$ we construct will be $\sum_{\beta\in\omega_1}(\mathcal{S}_\beta + \mathcal{K}_\beta)$, where $\mathcal{K}_\beta$ is either $\mathcal{A}_{\omega_1}$ or $\mathcal{A}_\alpha + \mathcal{B}_{\omega_1}$ for some countable ordinal α. By Fact 1.11(2), we fix a sequence $\{\mathcal{L}_\beta\}_{\beta\in\omega_1}$ of all computable linear orderings. The purpose of $\mathcal{K}_\beta$ is to diagonalize against $\mathcal{L}_\beta$.

Lemma 2.6 implies that for all $\beta < \omega_1$ and $\gamma < \omega_1$, no nonempty initial or final segment of $\mathcal{S}_\beta$ is isomorphic to a convex subset of $\mathcal{K}_\gamma$. Lemma 2.3 and Lemma 2.6 now guarantee that if built according to our plan, for all $\beta < \omega_1$, there is a unique convex subset of $\mathcal{L}$ isomorphic to $\mathcal{S}_\beta$. We identify $\mathcal{S}_\beta$ with that convex subset of $\mathcal{L}$.

Construction: For each $\beta < \omega_1$, we need to determine the largest ordinal $\alpha = \alpha(\beta) \leq \omega_1$ such that $\mathcal{A}_\alpha$ should be an initial segment of $\mathcal{K}_\beta$. If $\alpha = \omega_1$ then $\mathcal{K}_\beta := \mathcal{A}_{\omega_1}$, and if $\alpha < \omega_1$ then $\mathcal{K}_\beta := \mathcal{A}_\alpha + \mathcal{B}_{\omega_1}$. The choice of $\alpha(\beta)$, of course, will not be done effectively since we want to ensure that otp($\mathcal{L}$) is not computable. However, we need to make this choice "as computably as possible" so that any sufficiently fast-growing function does have the ability to compute, uniformly in β, a copy of $\mathcal{K}_\beta$.

The choice of each $\alpha(\beta)$ is made independently, based only on $\mathcal{L}_\beta$. If $\mathcal{L}_\beta$ were to be isomorphic to $\mathcal{L}$, then $\mathcal{L}_\beta$ would have a unique convex subset $S = S(\beta)$ isomorphic to $\mathcal{S}_\beta$, a unique convex subset $T = T(\beta)$ isomorphic to $\mathcal{S}_{\beta+1}$, and would have $S(\beta) <_{\mathcal{L}_\beta} T(\beta)$. Furthermore, any isomorphism between $\mathcal{L}_\beta$ and $\mathcal{L}$ would have to extend the isomorphisms between S and $\mathcal{S}_\beta$, and T and $\mathcal{S}_{\beta+1}$; so the isomorphism would map $(S,T)_{\mathcal{L}_\beta}$ onto $\mathcal{K}_\beta$. Since S and T are countable, both subsets would be enumerated into $\mathcal{L}_\beta$ in their entirety by some countable stage.

Thus, at each stage $s < \omega_1$, we let $(S_s(\beta), T_s(\beta))$ be the $<_{\omega_1}$-least pair of convex subsets of $\mathcal{L}_\beta \restriction s$ such that $S_s(\beta)$ is seen (at stage s) to be isomorphic to $\mathcal{S}_\beta$, $T_s(\beta)$ is seen (at stage s) to be isomorphic to $\mathcal{S}_{\beta+1}$, and $S_s(\beta) <_{\mathcal{L}_\beta} T_s(\beta)$. We then let $\mathcal{I}_s(\beta) = (S_s(\beta), T_s(\beta))_{\mathcal{L}_\beta}$ be the $\mathcal{L}_\beta$-interval (not the $(\mathcal{L}_\beta \restriction s)$-interval) determined by these subsets. The plan is to ensure that if $\mathcal{I}_s(\beta)$ reaches a limit, then it is not isomorphic to $\mathcal{K}_\beta$. If such subsets $S_s(\beta)$ and $T_s(\beta)$ are not found, then $\mathcal{I}_s(\beta)$ is undefined.

We describe how to define $\alpha_s = \alpha_s(\beta)$, our stage s approximation for the ordinal $\alpha(\beta)$. This approximation will be nondecreasing and continuous. The sequences $\langle \mathcal{I}_s(\beta) \rangle_{s<\omega_1}$ and $\langle \alpha_s(\beta) \rangle_{s<\omega_1}$ will be ω_1-computable, uniformly in β.

We try to pick a point $x_s = x_s(\beta) \in \mathcal{I}_s(\beta)$ which will aid our diagonalization efforts. Once picked, we only change our choice of point if the ambient interval $\mathcal{I}_s(\beta)$ changes. That is:

- If $s = t+1$ is a successor stage, $\mathcal{I}_s = \mathcal{I}_t$ are both defined, and x_t is defined, then we let $x_s := x_t$;
- If s is a limit stage, there is some $t < s$ such that for all stages $r \in [t, s)$, $\mathcal{I}_r = \mathcal{I}_s$ are all defined, and x_t is defined, then $x_s := x_t$.

If $\mathcal{I}_s$ is defined, but x_s is not yet defined by the previous clause, and there is some $x \in \mathcal{I}_s \restriction s$ such that $\mathcal{A}_{\alpha_s}$ is seen, at stage s, to be embeddable into $(-\infty, x)_{\mathcal{I}_s}$, then we let x_s be the $<_{\omega_1}$-least such x; if there is no such x, then we leave x_s undefined.

If x_s is undefined, then we let $\alpha_{s+1} := \alpha_s$. If x_s is defined, then we let α_{s+1} be the supremum of the ordinals $\alpha < \omega_1$ such that at stage s, $\mathcal{A}_\alpha$ is seen to be embeddable into $(-\infty, x_s)_{\mathcal{I}_s}$. By induction on s, we can easily see that if x_s is defined, then $\mathcal{A}_{\alpha+1}$ is embeddable into $(-\infty, x_s)_{\mathcal{I}_s}$ for all $\alpha < \alpha_s$, and so $\alpha_{s+1} \geq \alpha_s$.

We let $\alpha(\beta) = \alpha_{\omega_1}(\beta) := \sup_{s<\omega_1} \alpha_s(\beta)$. This determines $\mathcal{K}_\beta$, and so completes the definition of the linear ordering $\mathcal{L}$.

Verification: Before we formally show that $\mathcal{L}$ is not isomorphic to $\mathcal{L}_\beta$ for any $\beta < \omega_1$, and so that $\mathbf{0} \notin \mathrm{DegSpec}(\mathcal{L})$, we explain what goes wrong if we follow a naive strategy for computing a copy of $\mathcal{L}$. For $s < \omega_1$, we let $\mathcal{K}_{\beta,s} = \mathcal{A}_{\alpha_s(\beta)} + \mathcal{B}_s$. Suppose that, uniformly in β, we want to enumerate a direct system of embeddings $f_{s,t}\colon \mathcal{K}_{\beta,s} \to \mathcal{K}_{\beta,t}$, whose direct limit will be $\mathcal{K}_\beta$. If $\alpha_{s+1}(\beta) = \alpha_s(\beta)$, then we add a copy of $\mathbb{Z}^s$ between $\mathcal{A}_{\alpha_s(\beta)}$ to $\mathcal{B}_s$ to get a copy of $\mathcal{K}_{\beta,s+1}$; in other words, $f_{s,s+1}$ is the "disjoint union" of $j_{\alpha_s,\alpha_{s+1}}$ and $j^*_{s,s+1}$. If $\alpha_{s+1}(\beta) > \alpha_s(\beta)$, then we want to "swallow" $\mathcal{K}_{\beta,s}$ in $\mathcal{A}_{\alpha_{s+1}(\beta)}$, and then add a copy of $\mathcal{B}_s$ to the right; in other words, we want $f_{s,s+1}$ be an embedding of $\mathcal{K}_{\beta,s}$ in $\mathcal{A}_{\alpha_{s+1}}$ extending $j_{\alpha_s,\alpha_{s+1}}$. The swallowing is necessary so that if $\alpha(\beta) = \omega_1$, then all copies of $\mathcal{B}_s$ disappear into copies of greater $\mathcal{A}_\alpha$'s and at the end we would get $\mathcal{K}_\beta = \mathcal{A}_{\omega_1}$. The problem is that Lemma 2.5 (3) only ensures that $\mathcal{K}_{\beta,s}$ is embeddable in a copy of $\mathcal{A}_{s+1}$, and it may be that $\alpha_{s+1}(\beta)$, while greater than $\alpha_s(\beta)$, is still smaller that $s+1$, and so our attempt to capture $\alpha(\beta)$ may "overshoot". This failure can be translated into a proof that $\mathcal{L}$ has no computable copy, and modified (by looking sufficiently far into the future) into a construction showing that any hyperimmune degree can compute a copy of $\mathcal{L}$.

Noncomputability: We now show that for each $\beta \in \omega_1$, we have $\mathcal{L} \ncong \mathcal{L}_\beta$, and so $\mathbf{0} \notin \mathrm{DegSpec}(\mathcal{L})$. Let $\beta < \omega_1$, and for a contradiction suppose that $f\colon \mathcal{L}_\beta \to \mathcal{L}$ is an isomorphism.

Let $S := S(\beta) := f^{-1}\mathcal{S}_\beta$ and $T := T(\beta) := f^{-1}\mathcal{S}_{\beta+1}$. As already noted, this implies $S <_{\mathcal{L}_\beta} T$, the set S is the unique convex subset of $\mathcal{L}$ isomorphic to $\mathcal{S}_\beta$, and T is the unique convex subset of $\mathcal{L}$ isomorphic to $\mathcal{S}_{\beta+1}$. Hence, for every pair (S', T') of subsets of $\mathcal{L}_\beta$ which precede (S, T) in the canonical ordering $<_{\omega_1}$ of H_{ω_1} such that $S' <_{\mathcal{L}_\beta} T'$, $S' \cong \mathcal{S}_\beta$ and $T' \cong \mathcal{S}_{\beta+1}$, either S' is not a convex subset of $\mathcal{L}_\beta$, or T' is not a convex subset of $\mathcal{L}_\beta$. It follows that for each pair $(S', T') <_{\omega_1} (S, T)$

there is some stage $s < \omega_1$ such that for all $t \geq s$, $(S', T') \neq (S_t(\beta), T_t(\beta))$. Since ω_1 is regular, for all but countably many stages s, we have $S_s(\beta) = S$ and $T_s(\beta) = T$. Let s_0 be the least stage such that for all $s \geq s_0$, $(S_s(\beta), T_s(\beta)) = (S, T)$. Let $\mathcal{I} = (S, T)_{\mathcal{L}_\beta} = f^{-1}\mathcal{K}_\beta$; s_0 is the least stage s such that for all $t \geq s$, $\mathcal{I}_s(\beta) = \mathcal{I}$. We show that there is some stage $s \geq s_0$ at which $x_s(\beta)$ is defined. For the sake of a contradiction, suppose that for no $s \geq s_0$ is $x_s(\beta)$ defined. Then for all $s \geq s_0$, $\alpha_s(\beta) = \alpha_{s_0}(\beta)$, and so $\alpha(\beta) = \alpha_{s_0}(\beta)$, and $\mathcal{K}_\beta = \mathcal{A}_{\alpha_{s_0}(\beta)} + \mathcal{B}_{\omega_1}$. But then $f^{-1} \restriction \mathcal{A}_{\alpha(\beta)}$ is an embedding of $\mathcal{A}_{\alpha(\beta)}$ into a proper initial segment of $\mathcal{I}$. This embedding is discovered at some countable stage, at which we would define $x_s(\beta)$.

So let $s_1 \geq s_0$ be the least stage $s \geq s_0$ at which $x_s(\beta)$ is defined. Let $x = x_{s_1}(\beta)$; then for all $s \geq s_1$, we have $x_s(\beta) = x$. The definition of $\alpha(\beta)$ implies that $\alpha(\beta)$ is the supremum of the ordinals α such that $\mathcal{A}_\alpha$ is embeddable into $\mathcal{I}(< x)$.

Now either $f(x) \in \mathcal{A}_{\alpha(\beta)}$ or $f(x) \in \mathcal{B}_{\omega_1}$; in either case, we reach a contradiction. If $f(x) \in \mathcal{B}_{\omega_1}$, then $\alpha(\beta) < \omega_1$; but by Lemma 2.5 (2), there is an embedding of $\mathcal{A}_{\alpha(\beta)+1}$ into $\mathcal{A}_{\alpha(\beta)} + \mathcal{B}_{\omega_1}(< f(x))$, and so into $\mathcal{I}(< x)$, contradicting the definition of $\alpha(\beta)$.

On the other hand, suppose that $f(x) \in \mathcal{A}_{\alpha(\beta)}$. If $\alpha(\beta)$ is a successor ordinal, then by definition of $\alpha(\beta)$, there is an embedding g of $\mathcal{A}_{\alpha(\beta)}$ into $\mathcal{I}(< x)$. Composing g with f gives an embedding of $\mathcal{A}_{\alpha(\beta)}$ into a proper initial segment of $\mathcal{A}_{\alpha(\beta)}$, which is impossible by Lemma 2.4 (1). If $\alpha(\beta)$ is a limit ordinal, then the same argument shows that for all $\gamma < \alpha(\beta)$, there is an embedding of $\mathcal{A}_\gamma$ into the proper initial segment $\mathcal{A}_{\alpha(\beta)}(< f(x))$, which is impossible by Lemma 2.4 (2).

Hyperimmune Degrees: Let $g\colon \omega_1 \to \omega_1$ be a function such that for any computable function $f\colon \omega_1 \to \omega_1$, there are uncountably many ordinals $\beta < \omega_1$ such that $g(\beta) > f(\beta)$. We show that g can compute, uniformly in $\beta < \omega_1$, a copy of $\mathcal{K}_\beta$. Hence DegSpec($\mathcal{L}$) contains every hyperimmune degree.

Fix $\beta < \omega_1$; we omit the argument β and so write α_s for $\alpha_s(\beta)$, etc. We may assume that for all s, $g(s) \geq s$.

We define a g-computable closed unbounded subset I of ω_1. For $s \in I$, we let $\mathcal{K}_{\beta,s} = \mathcal{A}_{\alpha_s} + \mathcal{B}_s$. We define a g-computable system of embeddings $f_{t,s}\colon \mathcal{K}_{\beta,t} \to \mathcal{K}_{\beta,s}$ for $t < s$ in I, where, of course, if $t < r < s$ are in I then $f_{t,s} = f_{t,r} \circ f_{r,s}$. We ensure that for $t < s$ in I, $f_{t,s} \restriction \mathcal{A}_{\alpha_t} = j_{\alpha_t,\alpha_s}$.

Let $s < \omega_1$, and suppose that we have already determined that $s \in I$, and that we have defined $f_{t,r}$ for $t < r \leq s$ in I. Now there are two possibilities:

- If $\alpha_{g(s)} > s$, then as $\alpha_s \leq s$, Lemma 2.5 (3) ensures that there is an embedding $f_{s,g(s)}$ of $\mathcal{K}_{\beta,s}$ into $\mathcal{A}_{\alpha_{g(s)}}$ extending $j_{\alpha_s,\alpha_{g(s)}}$. We let $g(s)$ be the next greatest element of I after s.
- If $\alpha_{g(s)} \leq s$, we let $s+1$ be in I. We let $f_{s,s+1} = j_{\alpha_s,\alpha_{s+1}} + j^*_{s,s+1}$. That is, $f_{s,s+1}$ embeds $\mathcal{A}_{\alpha_s}$ into $\mathcal{A}_{\alpha_{s+1}}$ and $\mathcal{B}_s$ into $\mathcal{B}_{s+1}$ canonically; and so $\mathcal{K}_{\beta,s+1} \setminus f[\mathcal{K}_{\beta,s}] = (f[\mathcal{A}_{\alpha_s}], f[\mathcal{B}_s])_{\mathcal{K}_{\beta,s+1}}$.

For bookkeeping, we let $J = \left\{s \in I \,:\, \alpha_{g(s)} > s\right\}$.

Suppose that $s \leq \omega_1$ is a limit point of I (and so $s \in I$). Let $\mathcal{K}_{\beta,<s}$ be the direct limit of the system $\langle \mathcal{K}_{\beta,t}, f_{t,r}\rangle_{r,t\in I, r<t<s}$, and for $t < s$ in I, let $f_{t,<s}$ be the limit of the maps $\langle f_{t,r}\rangle_{r\in I, t<r<s}$. As each map $f_{t,r}$ extends j_{α_t,α_r}, and as $\alpha_s = \sup_{t<s}\alpha_t$, we see that for all $t < s$ in I, $f_{\alpha,<s} \restriction \mathcal{A}_{\alpha_t} = j_{\alpha_t,\alpha_s}$. As each j_{α_t,α_r} is an initial segment embedding of $\mathcal{A}_{\alpha_r}$ into $\mathcal{K}_{\beta,r}$, we see that $\mathcal{A}_{\alpha_s}$ is an initial segment of $\mathcal{K}_{\beta,<s}$.

There are two possibilities:

- If $J \cap s$ is unbounded in s, then for all $t < s$ in I, there is some $r \in I$ such that $t < r < s$ and such that $f_{t,r}[\mathcal{B}_t] \subseteq \mathcal{A}_{\alpha_r}$. This implies that $\mathcal{K}_{\beta,<s}$ is the direct limit of the maps j_{α_t,α_r} for $t < r < s$ in I, that is, $\mathcal{K}_{\beta,<s} = \mathcal{A}_{\alpha_s}$.
- If $J \cap s$ is bounded in s, let $t_0 = \sup(J) \cap s$. In this case, for all $t, r \in I$ such that $t_0 \leq t < r < s$, we have $f_{t,r} = j_{\alpha_t,\alpha_r} + j^*_{t,r}$, and so $\mathcal{K}_{\beta,<s}$, being the direct limit of these maps, is $\mathcal{A}_{\alpha_s} + \mathcal{B}_s = \mathcal{K}_{\beta,s}$.

In either case, we can let, for $t < s$ in I, $f_{t,s} = f_{t,<s}$, where in the first case, the maps are composed with the identity inclusion of $\mathcal{K}_{\beta,<s}$ into $\mathcal{K}_{\beta,s} = \mathcal{K}_{\beta,<s} + \mathcal{B}_s$.

Now we argue that $\mathcal{K}_{\beta,<\omega_1}$, which is computable in g, uniformly in β, is isomorphic to $\mathcal{K}_\beta$. We have verified that if J is bounded below ω_1, then $\mathcal{K}_{\beta,<\omega_1} \cong \mathcal{A}_{\alpha(\beta)} + \mathcal{B}_{\omega_1}$, and that if J is cofinal in ω_1, then $\mathcal{K}_{\beta,<\omega_1} \cong \mathcal{A}_{\alpha(\beta)}$. Certainly if J is unbounded in ω_1 then $\alpha(\beta) = \omega_1$. We thus only need to show that if $\alpha(\beta) = \omega_1$, then J is cofinal in ω_1.

Assume that $\alpha(\beta) = \omega_1$, and suppose, for contradiction, that J is bounded below ω_1. Let $s_0 = \sup(J)$. Then $(s_0, \omega_1) \subseteq I$. Define a computable function $h\colon \omega_1 \to \omega_1$ by letting $h(\gamma)$ be the least stage $s < \omega_1$ such that $\alpha_s > \gamma$. By our assumption, there is some $s > s_0$ such that $g(s) > h(s)$, so $\alpha_{g(s)} \geq \alpha_{h(s)} > s$. As $s \in I$, it follows that $s \in J$, contradicting $s > s_0$.

This completes the proof. □

Remark 2.8. We further mention that the construction is flexible in that it is not important that $\mathcal{L}$ be an ω_1-sum of separators and diagonalizers. For example, we can obtain $\mathcal{L}$ from $\mathbb{R}$ by replacing the i^{th} rational number q_i by $\mathcal{S}_i$, and the α^{th} irrational number r_α by $\mathcal{K}_\alpha$. We just need the location of $\mathcal{A}_\alpha$ to be determined by the location of a countable uniformly computable set of $\mathcal{S}_\beta$'s.

2.2. **Transfer Theorems.** Within the ω-setting, there are several well-known and widely used theorems stating that if an order-type λ is $\mathbf{a}$-ω-computable (for some fixed theorem-dependent degree $\mathbf{a}$), then $\kappa \cdot \lambda$ is ω-computable (for some fixed theorem-dependent order-type κ). For example, the following theorem has been used to exhibit linear orders having spectra exactly the non-low$_n$ degrees for $n \geq 2$ (see [11]) and to exhibit linear orders having arbitrary α^{th} jump degree (see [5]).

Theorem 2.9 (Downey and Knight [5])**.** *If λ is $\mathbf{0}'$-ω-computable, then $(\eta_0+2+\eta_0)\cdot\lambda$ is ω-computable.*

Here, we show that there are no such simple transfer theorems of the above type (involving only multiplication of linear orders) in the ω_1-setting. The following theorem is an extension of Theorem 6.5 of Greenberg and Knight [15].

Theorem 2.10. *For any degree $\mathbf{a} > \mathbf{0}$, there is an $\mathbf{a}$-ω_1-computable order-type λ such that $\kappa \cdot \lambda$ is not ω_1-computable for any (non-empty) order-type κ.*

Moreover, the order-type λ can be chosen so that, for any non-empty order-type κ, the degree spectrum of $\kappa \cdot \lambda$ is the intersection of DegSpec(κ) *with the cone of degrees above* $\mathbf{a}$.

Let $\mathbb{I} := \mathbb{R} \setminus \mathbb{Q}$ be the collection of irrational real numbers. This is an uncountable computable set, and so is isomorphic to ω_1 by a computable bijection $h : \omega_1 \to \mathbb{I}$. For any set $S \subseteq \omega_1$, let $\mathcal{L}_S := \mathbb{Q} \cup h[S]$, with the ordering inherited from $\mathbb{R}$. Greenberg and Knight [15] observed that for any set $S \subseteq \omega_1$, an ω_1-degree $\mathbf{a}$ belongs to the set DegSpec$(\mathcal{L}_S)$ if and only if S is $\mathbf{a}$-c.e. This observation can be used to establish the theorem.

Proof of Theorem 2.10. Given an ω_1-degree $\mathbf{a}$, we fix a set $A \in \mathbf{a}$. Then the set $S := A \oplus (\omega_1 \setminus A)$ has the property that S is ω_1-c.e. in an ω_1-degree $\mathbf{b}$ if and only if $\mathbf{b} \geq \mathbf{a}$. We argue that $\lambda := \mathrm{otp}(\mathcal{L}_S)$ has the desired properties.

Let κ be any non-empty order-type. If $\mathbf{b} \in \mathrm{DegSpec}(\lambda) \cap \mathrm{DegSpec}(\kappa)$, then it is immediate that $\mathbf{b} \in \mathrm{DegSpec}(\kappa \cdot \lambda)$. For the reverse direction, we show that any linear order $\mathcal{B}$ in $\kappa \cdot \lambda$ computes both $\mathbf{a}$ and a presentation of κ.

Fix a presentation $\mathcal{B} \in \kappa \cdot \lambda$. Fix an order-preserving embedding $g : \mathbb{Q} \to \mathcal{B}$ by picking, for each rational $q \in \mathbb{Q}$, a point $g(q)$ in the qth copy of κ. Using g as a countable parameter, we show that $\mathcal{B}$ can enumerate the set S.

Indeed, for $x \in \omega_1$, let (L_x, R_x) be the cut of $\mathbb{Q}$ such that $(L_x, R_x)_{\mathbb{R}} = \{h(x)\}$ (where $h : \omega_1 \to \mathbb{I}$ is a fixed computable bijection). Then $x \in S$ if and only if $(g[L_x], g[R_x])_{\mathcal{B}}$ is non-empty. Since the cut (L_x, R_x) can be effectively obtained from x, this gives a $\Sigma^0_1(\mathcal{B})$ definition of S. By our choice of S, this implies $\mathcal{B} \geq_T \mathbf{a}$.

As $\mathbf{a} > \mathbf{0}$, it must be the case that S is non-empty. We fix $z \in S$ and consider the interval $(g[L_z], g[R_z])_{\mathcal{B}}$. It has order-type κ. As $g[L_z]$ and $g[R_z]$ are countable, it follows that $\mathcal{B} \restriction (g[L_z], g[R_z])_{\mathcal{B}}$ is a $\mathcal{B}$-computable presentation of κ.

Thus, an arbitrary presentation $\mathcal{B}$ of $\kappa \cdot \lambda$ computes both $\mathbf{a}$ and κ. $\square$

The proof of Theorem 2.10, or simply using the theorem with any computable order-type κ, yields the Greenberg-Knight result:

Theorem 2.11 (Greenberg and Knight [15])**.** *For any ω_1-degree $\mathbf{a}$, there is a linear ordering whose degree spectrum is the cone of degrees above $\mathbf{a}$ (including $\mathbf{a}$).*

Although multiplication does not work, transfer theorems do exist.

Definition 2.12. For a linear order $\mathcal{L}$, define an equivalence relation $\sim$ on subsets of $\mathcal{L}$ by

$$A_0 \sim A_1 \qquad \text{if and only if} \qquad \mathrm{dcl}_{A_0 \cup A_1}(A_0) = \mathrm{dcl}_{A_0 \cup A_1}(A_1).$$

It is easily checked that $\sim$ is an equivalence relation.

Define $\mathcal{L}^{\mathrm{c}}$ to be the smallest extension of $\mathcal{L}$ satisfying

$$|(\mathrm{dcl}_{\mathcal{L}}(A), \mathcal{L} - \mathrm{dcl}_{\mathcal{L}}(A))_{\mathcal{L}^{\mathrm{c}}}| = 1$$

for every at most countable $A \subseteq \mathcal{L}$. In other words, the linear ordering $\mathcal{L}^{\mathrm{c}}$ is the linear ordering formed from $\mathcal{L}$ by filling with one point every cut such that the set of points to the left of the cut has at most countable cofinality.

Define $\mathcal{L}^{\mathrm{t}}$ (termed the *transfer of* $\mathcal{L}$) to be the linear ordering

$$\mathcal{L}^{\mathrm{t}} := \sum_{x \in \mathcal{L}^{\mathrm{c}}} A_x,$$

where $A_x := 2$ if $x \in \mathcal{L}$ and $A_x := \eta_1$ if $x \in \mathcal{L}^{\mathrm{c}} - \mathcal{L}$.

Note that if $\mathcal{L}$ is computable, the linear orderings $\mathcal{L}^{\mathrm{c}}$ and $\mathcal{L}^{\mathrm{t}}$ are computable.

Lemma 2.13. *Fix an ω_1-degree $\mathbf{a}$. A linear ordering $\mathcal{L}$ is $\mathbf{a}'$-computable if and only if $\mathcal{L}^{\mathrm{t}}$ is $\mathbf{a}$-computable.*

Incidentally, the transition between $\mathcal{L}$ and $\mathcal{L}^{\mathrm{t}}$ is uniform in the indices in both directions.

Proof. ($\Longleftarrow$) Given an $\mathbf{a}$-computable presentation of $\mathcal{L}^{\mathrm{t}}$, let $\mathcal{K} := \mathrm{Succ}(\mathcal{L}^{\mathrm{t}})$. Then $\mathcal{K}$ is $\mathbf{a}'$-computable and has the appropriate order-type when given the induced order from $\mathcal{L}^{\mathrm{t}}$.

($\Longrightarrow$) By the universal property of η_1, we may assume that $\mathcal{L}$ is an $\mathbf{a}'$-computable subset of a computable presentation of η_1. We will, of course, approximate $\mathcal{L}$ in an $\mathbf{a}$-computable manner, building a linear ordering $\mathcal{K} \in \mathrm{otp}(\mathcal{L}^{\mathrm{t}})$ from this approximation.

When we see an element enter $\mathcal{L}$, we add an appropriate pair of elements into $\mathcal{K}$. When we see an element leave $\mathcal{L}$, since we cannot remove the corresponding pair from $\mathcal{K}$, we instead incorporate it into the copy of η_1 immediately to its left. Since the approximation at every stage is at most countable, there are at most countably many points in the current approximation to $\mathcal{L}$ which are to the left of the removed point — call this set A. So there is always a copy of η_1 to the immediate left of the removed pair — the copy of η_1 corresponding to the unique element of $\mathcal{L}^{\mathrm{c}}$ in $(\mathrm{dcl}_{\mathcal{L}}(A), \mathcal{L} - \mathrm{dcl}_{\mathcal{L}}(A))_{\mathcal{L}^{\mathrm{c}}}$.

Of course, we must also build these copies of η_1. Naively, one might hope to consider every countable subset of the current approximation to $\mathcal{L}$ and build a corresponding copy of η_1. Unfortunately, there may be uncountably many such subsets, so we cannot do this in a single stage. Instead, at every stage we consider a single countable subset of η_1. If this set is a subset of the current approximation to $\mathcal{L}$, then we build a copy of η_1 for it. Every countable subset of $\mathcal{L}$ will eventually always be a subset of the approximation, so as long as we arrange to consider every subset at uncountably many stages, every countable set of $\mathcal{L}$ will eventually be handled. We must also build a copy of η_1 if it does not already exist when we seek to incorporate a pair into it as described above.

Of course, since η_1 is an uncountable object, we cannot actually build an entire copy of it at a single stage. Instead, we declare what we call a *saturating interval*. At uncountably many later stages, we will add points to this saturating interval, causing it to grow into a copy of η_1.

If a point x leaves the approximation to $\mathcal{L}$, we must consider the effect on the saturating intervals we have built so far. If I is a saturating interval built on behalf of the countable set X, and x is not the largest element of X, then we do not need to adjust I; since $X \sim X - \{x\}$, I can continue to be the saturating interval which we build on behalf of $X - \{x\}$. If x is the largest point in X, however, then there is no longer a need for I. In this case, I must be the interval immediately to the right of the pair which corresponds to x. This pair will be merged with the saturating interval to its left, and we can merge I with the same interval.

Finally, we must concern ourselves with what happens at limit stages. We assume that the approximation to $\mathcal{L}$ at a limit stage is the limit infimum of the approximations at previous stages. Thus, the only points in the approximation at a limit stage are the points which were in for a terminal segment of previous stages. Hence, for pairs, there is nothing to do. For saturating intervals, however, we may need to cause more mergers.

For example, consider the following situation: The approximation to $\mathcal{L}$ at stage ω has order type ω^2. At stage ω, we have saturating intervals in order type $\omega + 1$, built on behalf of the "sets" $\emptyset$, ω, $\omega \cdot 2$, $\omega \cdot 3$, $\ldots$, and ω^2. At stage $\omega + \omega$, the approximation to $\mathcal{L}$ is empty.

Then at every stage $\omega + n$, every pair of saturating intervals is separated by countably many elements, and so will not merge. However, at stage $\omega + \omega$, there are no elements separating any of the saturating intervals. As η_1 and $\eta_1 \cdot \omega$ are not isomorphic, we will need to merge these saturating intervals.

In general, at a limit stage we will merge all saturating intervals which are not separated by a pair.

We will also define a sequence of functions F_s and G_s, which will assist us in tracking the relationship between $\mathcal{L}$ and $\mathcal{K}$. The function F_s will map the elements of $\mathcal{L}_s$ to their corresponding pair in $\mathcal{K}_s$. The function G_s will map a saturating interval in $\mathcal{K}_s$ to its corresponding at most countable subset in $\mathcal{L}_s$. It will be convenient to assume that these subsets are downward closed. So even if we make no changes to a saturating interval I between stages s and $s+1$, we will redefine $G_{s+1}(I)$ to be the downward closure (in $\mathcal{L}_{s+1}$) of $G_s(I)$. It will be the case that $G_s(I)$ is downward closed automatically at limit stages.

Preliminaries: Let $(\mathcal{L}_s)_{s<\omega_1}$ be an **a**-computable sequence of countable subsets of η_1 satisfying:

- $\mathcal{L}_0 = \emptyset$;
- $\mathcal{L}_s \triangle \mathcal{L}_{s+1} = \{z_s\}$ for some z_s;
- for s a limit ordinal, $\mathcal{L}_s = \liminf_{t<s} \mathcal{L}_t$; and
- $\mathcal{L} = \lim_s \mathcal{L}_s$.

We construct $\mathcal{K}$ as the union of countable linear orders $(\mathcal{K}_s)_{s<\omega_1}$. Each $\mathcal{K}_s$ will be partitioned into *saturating intervals* and *pairs*.

We also build a continuous sequence of order-preserving functions $(F_s)_{s<\omega_1}$, with F_s mapping elements of $\mathcal{L}_s$ to pairs in $\mathcal{K}_s$. The limit map $F := \lim_s F_s$ will map the elements of $\mathcal{L}$ to the corresponding pairs in $\mathcal{K}$. For $x \in \mathcal{L}_s$, we let $(F_s(x))_1$ and $(F_s(x))_2$ denote the left and right elements of the pair $F_s(x)$, respectively.

We also build a sequence of functions $(G_s)_{s<\omega_1}$, with G_s mapping the saturating intervals of $\mathcal{K}_s$ to downward closed subsets of $\mathcal{L}_s$. The map G_s will also be order-preserving in that if $I <_{\mathcal{K}_s} J$, then $G_s(I) \subset G_s(J)$.

We fix a computable enumeration $(A_s, B_s)_{s<\omega_1}$ of pairs from H_{ω_1} such that every pair occurs uncountably many times in the enumeration, and fix a computable enumeration $(Y_s)_{s<\omega_1}$ of H_{ω_1} such that every element occurs uncountably many times. These will be used in the creation of the saturating intervals.

Construction: At stage $s = 0$, we define $\mathcal{K}_0$, F_0, and G_0 to be empty.

At a successor stage $s+1$, we work in three steps, building intermediate orders $\mathcal{K}^1_{s+1}$ and $\mathcal{K}^2_{s+1}$ and intermediate functions G^1_{s+1} and G^2_{s+1}: First, we adjust F_s and the pairs in $\mathcal{K}_s$ for the change from $\mathcal{L}_s$ to $\mathcal{L}_{s+1}$; second, we create new saturating intervals as necessary; third, we work to build the saturating intervals into η_1.

(1) If $\mathcal{L}_{s+1} = \mathcal{L}_s \cup \{z_s\}$, then we add a new pair to be the image of z_s. More precisely, let

$$R := \mathrm{ucl}_{\mathcal{K}_s} \{(F_s(y))_1 : y \in \mathcal{L}_s \text{ and } z_s <_{\mathcal{L}_{s+1}} y\}$$

and let $Q := \mathcal{K}_s - R$. We choose two new elements a and b and define $\mathcal{K}^1_{s+1} := \mathcal{K}_s \cup \{a, b\}$. We define $<_{\mathcal{K}^1_{s+1}}$ by extending $<_{\mathcal{K}_s}$ with

$$Q <_{\mathcal{K}^1_{s+1}} a <_{\mathcal{K}^1_{s+1}} b <_{\mathcal{K}^1_{s+1}} R.$$

We make (a, b) a pair in $\mathcal{K}^1_{s+1}$ and define $F_{s+1} := F_s \cup \{(z_s, (a, b))\}$. All pairs and saturating intervals of $\mathcal{K}_s$ remain pairs and saturating intervals of $\mathcal{K}^1_{s+1}$, respectively. For every saturating interval $I \subseteq \mathcal{K}_s$, we define $G^1_{s+1}(I) := \mathrm{dcl}_{\mathcal{L}_{s+1}}(G_s(I))$.

If instead $\mathcal{L}_s = \mathcal{L}_{s+1} \cup \{z_s\}$, then we merge the pair $F_s(z_s)$ with the saturating interval to its left. More precisely, let $(a,b) := F_s(z_s)$ and let

$$Q := \{y : y \in \mathcal{L}_s \text{ and } y <_{\mathcal{L}_s} z_s\}.$$

If there are saturating intervals $I, J \subseteq \mathcal{K}_s$ with $G_s(I) = Q$ and $G_s(J) = Q \cup \{z_s\}$, then we make $I \cup \{a,b\} \cup J$ a saturating interval of $\mathcal{K}^1_{s+1}$ with $G^1_{s+1}(I \cup \{a,b\} \cup J) = Q$. If I exists but not J, we make $I \cup \{a,b\}$ a saturating interval of $\mathcal{K}^1_{s+1}$ with $G^1_{s+1}(I \cup \{a,b\}) = Q$. If J exists but not I, then we make $\{a,b\} \cup J$ a saturating interval of $\mathcal{K}^1_{s+1}$ with $G^1_{s+1}(\{a,b\} \cup J) = Q$. If neither I nor J exists, we make $\{a,b\}$ a saturating interval of $\mathcal{K}^1_{s+1}$ with $G^1_{s+1}(\{a,b\}) = Q$.

We define $F_{s+1} := F_s \restriction \mathcal{L}_{s+1}$. We do not make (a,b) a pair in $\mathcal{K}^1_{s+1}$. All other pairs and saturating intervals of $\mathcal{K}_s$ other than I and J remain pairs and saturating intervals of $\mathcal{K}^1_{s+1}$, respectively. For any saturating interval $H \subseteq \mathcal{K}_s$ other than I and J, we define $G^1_{s+1}(H) := G_s(H) - \{z_s\}$.

(2) If there is no saturating interval $I \subseteq \mathcal{K}^1_{s+1}$ with $G^1_{s+1}(I) = \mathrm{dcl}_{\mathcal{L}^1_{s+1}}(Y_s)$, let

$$Q := \{(F_{s+1}(y))_2 : y \in Y_s\},$$
$$R := \{(F_{s+1}(y))_1 : y \in \mathcal{L}_{s+1} \text{ and } Y_s <_{\mathcal{L}_{s+1}} y\}.$$

We choose a new element c and define $\mathcal{K}^2_{s+1} := \mathcal{K}^1_{s+1} \cup \{c\}$. We define $<_{\mathcal{K}^2_{s+1}}$ by extending $<_{\mathcal{K}^1_{s+1}}$ with

$$Q <_{\mathcal{K}^2_{s+1}} c <_{\mathcal{K}^2_{s+1}} R.$$

We make $\{c\}$ a saturating interval in $\mathcal{K}^2_{s+1}$ with $G^2_{s+1}(\{c\}) = \mathrm{dcl}_{\mathcal{L}^1_{s+1}}(Y_s)$. All pairs and saturating intervals of $\mathcal{K}^1_{s+1}$ remain pairs and saturating intervals of $\mathcal{K}^2_{s+1}$, respectively.

Otherwise, we define $\mathcal{K}^2_{s+1} := \mathcal{K}^1_{s+1}$. All pairs and saturating intervals of $\mathcal{K}^1_{s+1}$ remain pairs and saturating intervals of $\mathcal{K}^2_{s+1}$, respectively.

For every saturating interval $I \subseteq \mathcal{K}^1_{s+1}$, we define $G^2_{s+1}(I) := G^1_{s+1}(I)$, noting these are downward closed subsets.

(3) If there is some saturating interval $I \subseteq \mathcal{K}^2_{s+1}$ with $A_s, B_s \subseteq I$ and $A_s <_{\mathcal{K}^2_{s+1}} B_s$ and $(A_s, B_s)_{\mathcal{K}^2_{s+1}} = \emptyset$, we choose a new element d and define $\mathcal{K}_{s+1} := \mathcal{K}^2_{s+1} \cup \{d\}$. We define $<_{\mathcal{K}_{s+1}}$ by extending $<_{\mathcal{K}^2_{s+1}}$ with

$$A_s <_{\mathcal{K}_{s+1}} d <_{\mathcal{K}_{s+1}} B_s.$$

We make $I \cup \{d\}$ a saturating interval in $\mathcal{K}_{s+1}$ with $G_{s+1}(I \cup \{d\}) := G^2_{s+1}(I)$. All other pairs and saturating intervals of $\mathcal{K}^2_{s+1}$ remain pairs and saturating intervals of $\mathcal{K}_{s+1}$, respectively. For every other saturating interval $J \subseteq \mathcal{K}^2_{s+1}$, we define $G_{s+1}(J) := G^2_{s+1}(J)$.

At a limit stage s, we work in two steps, building an intermediate function G'_s: First we define the pairs and saturating intervals as the limits of the previous stages. Then we merge saturating intervals where necessary.

Before doing so, we define $\mathcal{K}_s := \bigcup_{t<s} \mathcal{K}_t$ and $F_s := \lim_{t<s} F_t$, noting the limit exists because $\mathcal{L}_s = \liminf_{t<s} \mathcal{L}_t$.

(1) We make (a,b) a pair in $\mathcal{K}_s$ if there is a stage $s_0 < s$ such that (a,b) is a pair in $\mathcal{K}_t$ for every t with $s_0 < t < s$.

By Claim 2.13.1, for every t with $s_0 < t < s$ and every saturating interval $I \subseteq \mathcal{K}_{s_0}$, there is a unique saturating interval $I_t \subseteq \mathcal{K}_t$ with $I \cap I_t \neq \emptyset$, and further this unique saturating interval satisfies $I \subseteq I_t$.

Thus for every $s_0 < s$ and every saturating interval $I \subseteq \mathcal{K}_{s_0}$, the set $I'_s := \bigcup_{s_0<t<s} I_t$ is convex. We let $G'_s(I'_s) := \liminf_{t<s} G_t(I_t)$, observing this is downward closed.

We observe that if $I'_s = J'_s$, then $G'_s(I'_s) = G'_s(J'_s)$ because, fixing I, the collection of J such that $I'_s = J'_s$ is a directed set. Thus, the choice of the stage s_0 and starting interval I is unimportant.

(2) As discussed earlier, there may be I and J such that $I'_s \neq J'_s$ but $G'_s(I'_s) = G'_s(J'_s)$. Note that in this case, there can be no $y \in \mathcal{L}_s$ with $F(y) = (a,b)$ and $I'_s <_{\mathcal{L}_s} a <_{\mathcal{L}_s} b <_{\mathcal{L}_s} J'_s$, because then y would be in $G'_s(J'_s) \setminus G'_s(I'_s)$. Also the converse holds, so if there is no such y, then $G'_s(I'_s) = G'_s(J'_s)$.

For every saturating interval $I \subset \mathcal{K}_t$ for some $t < s$, we make

$$I_s = \bigcup_{G'_s(J'_s)=G'_s(I'_s)} J'_s$$

a saturating interval in $\mathcal{K}_s$. We define $G_s(I_s) := G'_s(I'_s)$.

This completes the construction.

We let $\mathcal{K} := \mathcal{K}_{\omega_1}$, $F := F_{\omega_1}$ and $G := G'_{\omega_1}$. We note that sets in the range of G may be uncountable, unlike sets in the range of G_s for $s < \omega_1$; also we do not perform the final step of combining saturating intervals at stage ω_1 (we argue in Claim 2.13.4 that it is unnecessary).

Verification: Clearly $\mathcal{K}$ is **a**-computable, F is an order-preserving bijection from $\mathcal{L}$ to the pairs in $\mathcal{K}$, and G is an order-preserving map from the saturating intervals to the downward closed subsets of $\mathcal{L}$. Also, by the action of Step 3 at successor stages, every saturating interval in $\mathcal{K}$ has order type η_1.

Claim 2.13.1. For every $t \leq s$ and every saturating interval $I \subseteq \mathcal{K}_t$, there is a unique saturating interval $I_s \subseteq \mathcal{K}_s$ with $I \cap I_s \neq \emptyset$. Furthermore, $I \subseteq I_s$ and $G_s(I_s)$ is contained in the downward closure of $G_t(I)$ in η_1 (recalling that $\mathcal{L} \subseteq \eta_1$).

Proof. Immediate by construction and induction on s. □

Claim 2.13.2. If $I \subseteq \mathcal{K}$ is a saturating interval, then there is an at most countable $Y \subseteq \mathcal{L}$ with $Y \sim G(I)$.

Proof. Fix a saturating interval $J \subseteq \mathcal{K}_s$ such that $J \subset I$. By regularity, there is a stage $t > s$ such that $\mathcal{L}_t \subseteq \mathcal{L}_{t'}$ for all $t' > t$. Let J_t be the saturating interval of $\mathcal{K}_t$ containing J. Then $G_t(J_t) \subseteq G(I)$ by construction, and $G(I)$ is contained in the downward closure of $G_t(J_t)$. Hence, the set $G_t(J_t)$ suffices as a choice for Y. □

Claim 2.13.3. At every stage s, the map G_s is injective.

Proof. This follows by induction on s: At limit stages, this is by explicit construction. At successor stages, this is by construction and the inductive hypothesis. □

Claim 2.13.4. For every $Y \in [\mathcal{L}]^{<\omega_1}$, there is precisely one saturating interval $I \subseteq \mathcal{K}$ with $G(I) \sim Y$.

Proof. Let s_0 be a stage such that $Y \subseteq \mathcal{L}_s$ for all $s \geq s_0$, and let $s_1 > s_0$ be a stage such that $Y = Y_{s_1}$. Then there is a saturating interval $J \subseteq \mathcal{K}_{s_1+1}$ with $G_{s_1+1}(J) \sim Y$ (which is created if it did not already exist). For every $t > s_1$, let J_t be the unique saturating interval in $\mathcal{K}_t$ with $J \subseteq J_t$. By Claim 2.13.1, $G_t(J_t) \sim Y$ for all t. Thus the saturating interval $J'_{\omega_1} \subseteq \mathcal{K}$ has $G(J'_{\omega_1}) \sim Y$.

Towards uniqueness, assume there were two such intervals I_0 and I_1. Let s be a stage such that there are saturating intervals $J_0, J_1 \subset \mathcal{K}_s$ with $J_0 \subset I_0$ and $J_1 \subset I_1$, and such that $\mathcal{L}_s \subseteq \mathcal{L}_t$ for all $t > s$. Then by the argument in Claim 2.13.2, $G_s(J_0) \sim G_s(J_1)$. But since these sets are downward closed in $\mathcal{L}_s$, we would have $G_s(J_0) = G_s(J_1)$, contrary to Claim 2.13.3. □

Claim 2.13.5. If $I, J \subseteq \mathcal{K}$ are saturating intervals with $G(I) \subset G(J)$, then $I <_{\mathcal{K}} J$.

Furthermore, if $y \in \mathcal{L}$ with $G(I) <_{\mathcal{L}} y$ and $y \in G(J)$, then $I <_{\mathcal{K}} (F(y))_1 <_{\mathcal{K}} (F(y))_2 <_{\mathcal{K}} J$.

Proof. By construction, this is true for any saturating intervals $I', J' \subseteq \mathcal{K}_s$ with $I' \subset I$ and $J' \subset J$. Thus it is true for I and J. □

Thus we can map $x \in \mathcal{L}^{\mathrm{c}}$ to $A_x \subseteq \mathcal{K}$ by sending $x \in \mathcal{L}$ to $F(x)$ and $x \in \mathcal{L}^{\mathrm{c}} - \mathcal{L}$ to $G^{-1}\{y \in \mathcal{L} \mid y < x\}$, and this map is order-preserving and its image covers $\mathcal{K}$.

This completes the proof of Lemma 2.13. □

2.3. **A Nonlow$_n$ Spectrum.** For $n \geq 2$, there are countable linear orderings whose degree spectrums consist of the nonlow$_n$ ω-degrees [11]. For $n = 1$, though, while it is known (see [13]) that the collection of nonlow ω-degrees is a degree spectrum, it is yet unknown if it is the degree spectrum of a linear order. We show that this problem has a solution in the ω_1-context: For every n, including $n = 1$, there is an order-type of size $\aleph_1$ whose degree spectrum consists of the nonlow$_n$ ω_1-degrees, that is, of the ω_1-Turing degrees $\mathbf{a}$ such that $\mathbf{a}^{(n)} > \mathbf{0}^{(n)}$.

We begin with the case $n = 1$. The order-type whose degree spectrum is the nonlow degrees will be the η_1-shuffle sum of linear orders coding a family $\mathcal{F}$ of sets which is Σ^0_2 in every nonlow ω_1-degree, but not Σ^0_2.

As in Section 2.1, let $\mathcal{A}_\beta := \sum_{\alpha<\beta} \mathbb{Z}^\alpha$ and $\mathcal{B}_\beta := \mathcal{A}^*_\beta$.

Lemma 2.14. *Let $S \subseteq \omega_1$ and $\mathbf{a}$ be an ω_1-Turing degree. There is a sequence of uniformly $\mathbf{a}$-computable linear orders $\langle \mathcal{L}_i \rangle_{i<\omega_1}$ such that*

$$\mathcal{L}_i \cong \begin{cases} \mathcal{A}_{\omega_1} & \text{if } i \in S, \\ \mathcal{A}_{\omega_1} + \mathcal{B}_{\omega_1} & \text{otherwise,} \end{cases}$$

if and only if the set S is $\Pi^0_2(\mathbf{a})$.

Moreover, the passage between an $\mathbf{a}$-computable index for the sequence of ω_1-computable linear orders and a $\Pi^0_2(\mathbf{a})$-index for S is effective.

Proof. ($\Longrightarrow$) Let $\langle \mathcal{L}_i \rangle_{i<\omega_1}$ be a uniformly $\mathbf{a}$-computable sequence of linear orders. Then the collection of $i < \omega_1$ such that $\mathrm{cf}(\mathcal{L}_i) = \omega_1$ is $\Pi^0_2(\mathbf{a})$, as $\mathrm{cf}(\mathcal{L}_i) = \omega_1$ if and only if every countable subset of L_i is strictly bounded in $\mathcal{L}_i$. It is easy to see that $\mathrm{cf}(\mathcal{A}_{\omega_1}) = \omega_1$ and that $\mathrm{cf}(\mathcal{A}_{\omega_1} + \mathcal{B}_{\omega_1}) = 1$.

($\Longleftarrow$) Fix a $\Pi^0_2(\mathbf{a})$ set S. We can, uniformly in $\mathbf{a}$, enumerate sets U_i such that for all i, the set U_i is uncountable if and only if $i \in S$. Fixing i, at stage s we define $\mathcal{C}_s := \mathcal{A}_s + \mathcal{B}_s$ and an embedding $f^i_{s,s+1}$ of $\mathcal{C}_s$ into $\mathcal{C}_{s+1}$ extending the initial segment embedding $j_{s,s+1}$ of $\mathcal{A}_s$ into $\mathcal{A}_{s+1}$. If a new number is enumerated into U_i (i.e., we see new evidence that $i \in S$), then we let $f_{s,s+1}$ embed $\mathcal{C}_s$ into $\mathcal{A}_{s+1}$ (i.e., we move past work built for $\mathcal{B}$ into $\mathcal{A}$); otherwise, we let $f_{s,s+1} = j_{s,s+1} + j^*_{s,s+1}$ (i.e., we continue building $\mathcal{A}$ and $\mathcal{B}$ separately). We let $\mathcal{L}_i$ be the direct limit of the system $\left\langle \mathcal{C}_s, f^i_{s,t} \right\rangle_{s \le t < \omega_1}$. The arguments of the previous section show that if U_i is uncountable, then all copies of $\mathcal{B}_s$ are "swallowed" and we get $\mathcal{L}_i \cong \mathcal{A}_{\omega_1}$; otherwise, we get $\mathcal{L}_i \cong \mathcal{A}_{\omega_1} + \mathcal{B}_{\omega_1}$. □

As is done in the countable framework, we say that a set $\mathcal{F}$ of subsets of ω_1 is ω_1-c.e. in some degree $\mathbf{a}$ if there is a uniformly $\mathbf{a}$-c.e. sequence of sets $\langle F_i \rangle_{i<\omega_1}$ such that $\mathcal{F} = \{F_i : i < \omega_1\}$. Similarly, we use the notion of a collection of sets being Σ^0_2 in a degree $\mathbf{a}$.

Lemma 2.15. *There is a family $\mathcal{F}$ of sets which is Σ^0_2 in a degree $\mathbf{a}$ if and only if $\mathbf{a}$ is nonlow. In fact, fixing a degree $\mathbf{c}$, there is a family $\mathcal{F}$ of sets which is Σ^0_2 in a degree $\mathbf{a}$ if and only if $\mathbf{a}$ is nonlow over $\mathbf{c}$.*

Proof. As in the countable framework, for any ω_1-degree $\mathbf{d}$, a set is $\Sigma^0_2(\mathbf{d})$ if and only if it is ω_1-c.e. in $\mathbf{d}'$. Hence, we are looking for a family $\mathcal{F}$ of sets which is ω_1-c.e. in $\mathbf{a}'$ for every $\mathbf{a}$ with $\mathbf{a}' > \mathbf{0}'$ but is not ω_1-c.e. in $\mathbf{0}'$.

The construction of $\mathcal{F}$ is the relativization to $\emptyset'$ of Wehner [35] of a family of sets which is c.e. in every nonzero ω-Turing degree but is not c.e. The change of setting to ω_1 does not change any of the details. Namely, we let

$$\mathcal{F} := \left\{ \{\alpha\} \oplus A \,:\, A \text{ is countable, and } A \neq W^{\emptyset'}_\alpha \right\}.$$

The Recursion Theorem shows that $\mathcal{F}$ is not ω_1-c.e. in $\mathbf{0}'$; but $\mathcal{F}$ is ω_1-c.e. in every degree $\mathbf{a} > \mathbf{0}'$, because $\mathbf{a}$ can code, element by element, a set $W \in \mathbf{a}$ which is not Σ^0_2, to escape equality with a given $W^{\emptyset'}_\alpha$. □

We introduce the order-types that will be used to code the sets in $\mathcal{F}$.

Definition 2.16. Again fix an enumeration $\langle q_i \rangle_{i<\omega}$ of the set of rational numbers $\mathbb{Q}$. Let $\mathbb{I}$ be the set of irrationals. For $q = q_i$, let $\mathcal{P}_q = i + 2$.

For $X \subseteq \mathbb{I}$ and $r \in \mathbb{R}$, define

$$\mathcal{Q}_{X,r} := \begin{cases} \mathcal{P}_r & \text{if } r \in \mathbb{Q}, \\ \mathcal{A}_{\omega_1} + \mathcal{B}_{\omega_1} & \text{if } r \in X, \\ \mathcal{A}_{\omega_1} & \text{if } r \in \mathbb{I} \setminus X, \end{cases}$$

and let $\mathcal{Q}_X := \sum_{r\in\mathbb{R}} \mathcal{Q}_{X,r}$.

Let $\mathcal{P} := \sum_{q\in\mathbb{Q}} \mathcal{P}_q$. For $X \subseteq \mathbb{I}$, let f_X be the natural embedding of $\mathcal{P}$ into $\mathcal{Q}_X$; for $q \in \mathbb{Q}$, f_X maps the copy of $\mathcal{P}_q$ in $\mathcal{P}$ to $\mathcal{Q}_{X,q}$. The range of f_X consists of those points in $\mathcal{Q}_X$ which are contained in finite maximal blocks of size larger than one.

Furthermore, the argument of Lemma 2.3 shows that if $X, Y \subseteq \mathbb{I}$ and $X \neq Y$, then $\mathcal{Q}_X$ is not isomorphic to any convex subset of $\mathcal{Q}_Y$.

We see that the linear ordering $\mathcal{Q}_X$ indeed "jump-codes" the set X.

Lemma 2.17. *For any $X \subseteq \mathbb{I}$ and ω_1-degree $\mathbf{a}$, the set X is $\Sigma^0_2(\mathbf{a})$ if and only if $\mathbf{a} \in \mathrm{DegSpec}(\mathcal{Q}_X)$. Furthermore, the equivalence is uniform: From a $\Sigma^0_2(\mathbf{a})$-index for X we can effectively pass to an $\mathbf{a}$-computable index for a linear ordering isomorphic to $\mathcal{Q}_X$, and vice versa.*

Proof. Suppose first that X is $\Sigma_2^0(\mathbf{a})$. Taking an effective bijection between ω_1 and $\mathbb{I}$, by Lemma 2.14, there is a uniformly $\mathbf{a}$-computable sequence $\langle \mathcal{L}_r \rangle_{r\in\mathbb{I}}$ of linear orderings such that if $r \in X$ then $\mathcal{L}_r \cong \mathcal{A}_{\omega_1} + \mathcal{B}_{\omega_1}$, and if $r \notin X$ then $\mathcal{L}_r \cong \mathcal{A}_{\omega_1}$. We then see that $\sum_{r\in\mathbb{R}} \mathcal{D}_r$, where

$$\mathcal{D}_r := \begin{cases} \mathcal{P}_r & \text{if } r \in \mathbb{Q}, \\ \mathcal{L}_r & \text{if } r \in \mathbb{I}, \end{cases}$$

is $\mathbf{a}$-computable and is isomorphic to $\mathcal{Q}_X$.

For the other direction, suppose that $\mathcal{L}$ is $\mathbf{a}$-computable, and that $g\colon \mathcal{Q}_X \to \mathcal{L}$ is an isomorphism. We first note that if we did not insist on uniformity, then the conclusion that X is $\Sigma_2^0(\mathbf{a})$ follows from Lemma 2.14 as follows. Since $g \circ f_X$ and $\mathcal{P}$ are countable, we can fix them as parameters. For $r \in \mathbb{I}$, let $C_r := \bigcup_{q<r} \mathcal{P}_q$ and $D_r := \bigcup_{q>r} \mathcal{P}_q$ be the indicated subsets of $\mathcal{P}$, noting that the pair (C_r, D_r) can be obtained effectively from r. Let $\mathcal{L}_r := ((g \circ f_X)[C_r], (g \circ f_X)[D_r])_{\mathcal{L}}$. Then $\mathcal{L}_r = g[\mathcal{Q}_{X,r}]$ and so $\langle \mathcal{L}_r \rangle_{r\in\mathbb{I}}$ is a sequence which witnesses, by Lemma 2.14, that X is $\Sigma_2^0(\mathbf{a})$.

However, this argument is nonuniform, as it required fixing the parameter $g \circ f_X$. To obtain uniformity, we will prove that $\mathbf{a}'$ can find this parameter. The argument of the previous paragraph and of the easy direction of Lemma 2.14 then shows that, given this parameter, the ω_1-degree $\mathbf{a}'$ can enumerate X: For each r, the ω_1-degree $\mathbf{a}'$ can obtain an $\mathbf{a}$-computable index for $\mathcal{L}_r$ and can then enumerate those r for which it discovers a maximal element in $\mathcal{L}_r$.

To show that $g \circ f_X$ can be obtained from $\mathcal{L}$ in a $\Delta_2^0(\mathbf{a})$-fashion, we unfortunately cannot use the characterization of $g \circ f_X$ as the unique isomorphism between $\mathcal{P}$ and the set of points in $\mathcal{L}$ contained in maximal finite blocks of size greater than one. This is because, in general, the computation of the maximal block containing an element takes two jumps rather than one jump. However, there are $\mathbf{a}'$-computable properties whose conjunction is satisfied only by $g \circ f_X$. For $q \in \mathbb{Q}$, let A_q and B_q be the subsets of $\mathcal{P}$ (for the copy we fixed above) such that $\mathcal{P} = A_q + \mathcal{P}_q + B_q$. Since the copy of $\mathcal{P}$ is fixed, this decomposition (note that $\mathcal{P}_q$ is a subset of $\mathcal{P}$, not an order-type, so it is unique within $\mathcal{P}$) is effective in q. We claim that $g \circ f_X$ is the unique embedding h of $\mathcal{P}$ into $\mathcal{L}$ such that for all $q \in \mathbb{Q}$,

(1) $h[\mathcal{P}_q]$ is a convex subset of $\mathcal{L}$; and
(2) $(h[A_q], h[\mathcal{P}_q])_{\mathcal{L}}$ and $(h[\mathcal{P}_q], h[B_q])_{\mathcal{L}}$ are both empty.

Both conditions are $\Pi_1^0(\mathbf{a})$, since it is $\Pi_1^0(\mathbf{a})$ to tell, given countable $C, D \subset \mathcal{L}$, whether $(C, D)_{\mathcal{L}}$ is empty or not. Certainly $g \circ f_X$ satisfies both conditions for all $q \in \mathbb{Q}$. To show that this is the only embedding of $\mathcal{P}$ into $\mathcal{L}$ which satisfies both conditions for all $q \in \mathbb{Q}$, we show that f_X is the only embedding of $\mathcal{P}$ into $\mathcal{Q}_X$ which satisfies the corresponding conditions for all $q \in \mathbb{Q}$.

Suppose that $h\colon \mathcal{P} \to \mathcal{Q}_X$ is an embedding, that for all $q \in \mathbb{Q}$, the set $h[\mathcal{P}_q]$ is a convex subset of $\mathcal{Q}_X$, and that for all $q \in \mathbb{Q}$, both $(h[A_q], h[\mathcal{P}_q])_{\mathcal{Q}_X}$ and $(h[\mathcal{P}_q], h[B_q])_{\mathcal{Q}_X}$ are empty. We first show that $h[\mathcal{P}] \subseteq f_X[\mathcal{P}]$. In other words, we show if $r \in \mathbb{I}$ and $q \in \mathbb{Q}$ then $h[\mathcal{P}_q] \cap \mathcal{Q}_{X,r}$ is empty. If not, then as $h[\mathcal{P}_q]$ is a finite convex subset of $\mathcal{Q}_X$ and the maximal blocks of $\mathcal{Q}_{X,r}$ are of size one or infinite, we must have $h[\mathcal{P}_q] \subset \mathcal{Q}_{X,r}$, and the initial segment of $\mathcal{Q}_X$ consisting of the points to the left of $h[\mathcal{P}_q]$ contains a greatest element x. Now A_q does not contain a greatest element, so $h[A_q]$ cannot contain x; so $h[A_q] <_{\mathcal{Q}_X} x <_{\mathcal{Q}_X} h[\mathcal{P}_q]$, contradicting the assumption on h. Now a similar argument shows that if $i < j$

then $h[\mathcal{P}_{q_i}]$ cannot intersect $\mathcal{Q}_{X,q_j}$. If $q, r \in \mathbb{Q}$ and $h[\mathcal{P}_q] \cap \mathcal{Q}_{X,r}$ is nonempty, then as $\mathcal{Q}_{X,r}$ is a maximal block of $\mathcal{Q}_X$ and $h[\mathcal{P}_q]$ is convex in $\mathcal{Q}_X$, we must have $h[\mathcal{P}_Q] \subseteq \mathcal{Q}_{X,r}$. This shows that if $i > j$ then $h[\mathcal{P}_{q_i}]$ does not intersect $\mathcal{Q}_{X,q_j}$. Hence for all $q \in \mathbb{Q}$, $h[\mathcal{P}_q] = \mathcal{Q}_{X,q}$, which shows that $h = f_X$. □

Theorem 2.18. *There is an order-type whose degree spectrum consists of the non-low ω_1-degrees. In fact, fixing a degree* $\mathbf{c}$*, there is an order-type whose degree spectrum consists of the ω_1-degrees nonlow over* $\mathbf{c}$*.*

Proof. Fix a family $\mathcal{F}$ as in Lemma 2.15; by fixing an effective bijection between H_{ω_1} and $\mathbb{I}$, we may assume that every element of $\mathcal{F}$ is a subset of $\mathbb{I}$. We show that the η_1-shuffle sum

$$\lambda := \sigma_1\left(\{\mathcal{Q}_X : X \in \mathcal{F}\}\right)$$

(recall Definition 1.10) has presentations in exactly the non-low ω_1-degrees. By Lemma 2.15, it is sufficient to show that a degree $\mathbf{a}$ computes a presentation of λ if and only if $\mathcal{F}$ is Σ^0_2 in $\mathbf{a}$.

Let $\mathbf{a}$ be an ω_1-Turing degree. Suppose first that $\mathcal{F}$ is Σ^0_2 in $\mathbf{a}$. Then the uniformity guaranteed by Lemma 2.17 shows that there is a sequence $\langle \mathcal{L}_\alpha \rangle_{\alpha<\omega_1}$ of uniformly $\mathbf{a}$-computable linear orders such that

$$\{\operatorname{otp}(\mathcal{L}_\alpha) \,:\, \alpha < \omega_1\} = \{\operatorname{otp}(\mathcal{Q}_X) \,:\, X \in \mathcal{F}\}\,.$$

From the sequence $\langle \mathcal{L}_\alpha \rangle$ we can easily build a presentation of λ, noting that a computable presentation $\mathbb{Q}_1$ of η_1 can be split into a partition of ω_1-many uniformly computable subsets, each saturated in $\mathbb{Q}_1$.

For the converse, suppose that $\mathcal{L}$ is an $\mathbf{a}$-computable presentation of λ. With oracle $\mathbf{a}'$, we enumerate the sets in $\mathcal{F}$. To do so, with this oracle, we enumerate all the countable functions $g \circ f_X$, where $X \in \mathcal{F}$ and g is a convex embedding of $\mathcal{Q}_X$ into $\mathcal{L}$. The $\Delta^0_2(\mathbf{a})$-conditions on an embedding $h \colon \mathcal{P} \to \mathcal{L}$ to be one of these functions are the conditions (1) and (2) of the proof of Lemma 2.17, together the following condition:

(3) For all $r \in \mathbb{I}$, the interval $(h[C_r], h[D_r])_{\mathcal{L}}$ is *scattered* (i.e., does not contain a copy of $\mathbb{Q}$). Here, again, $C_r := \bigcup_{q<r} \mathcal{P}_q$ and $D_r := \bigcup_{q>r} \mathcal{P}_q$.

Condition (3), together with the previous conditions, implies that $h[\mathcal{P}]$ must be contained in a single convex copy of some $\mathcal{Q}_X$ inside $\mathcal{L}$. Otherwise, fix some convex copy $\mathcal{K}$ of some $\mathcal{Q}_X$ in $\mathcal{L}$ which intersects $h[\mathcal{P}]$. Again, if $q \in \mathbb{Q}$ and $h[\mathcal{P}_q] \cap \mathcal{K} \neq \emptyset$ then $h[\mathcal{P}_q] \subset \mathcal{K}$. If it is not the case that $h[\mathcal{P}] \subset \mathcal{K}$, say, without loss of generality, that there are some $s, q \in \mathbb{Q}$ such that $s < q$, $h[\mathcal{P}_q] \subset \mathcal{K}$ and $h[\mathcal{P}_s] \cap \mathcal{K} = \emptyset$, then let r be the greatest lower bound of the rationals q such that $h[\mathcal{P}_q] \subset \mathcal{K}$. Now condition (2) implies that $r \in \mathbb{I}$; but the interval $(h[C_r], h[D_r])_{\mathcal{L}}$ must embed η_1, and so the rationals, contradicting (3). Then the argument proving Lemma 2.17 shows that $h = g \circ f_X$ where $g \colon \mathcal{Q}_X \to \mathcal{K}$ is an isomorphism.

Condition (3) is $\Pi^0_1(\mathbf{a})$, the universal quantification being over both irrational numbers and potential embeddings of $\mathbb{Q}$ into the intervals $(h[C_r], h[D_r])_{\mathcal{L}}$. Hence condition (3) can also be verified by $\mathbf{a}'$. The method, from the proof of Lemma 2.17, of enumerating X with oracle $\mathbf{a}'$ from $g \circ f_X$, is now applied to each of these maps, giving the desired $\mathbf{a}'$-computable enumeration of $\mathcal{F}$. □

We can now use the result for $n = 1$ to extend it to all finite ordinals.

Theorem 2.19. *For any degree* $\mathbf{a}$ *and any nonzero* $n < \omega$*, there is an order-type whose degree spectrum is* $\{\mathbf{b} : \mathbf{b} > \mathbf{a} \text{ and } \mathbf{b}^{(n)} > \mathbf{a}^{(n)}\}$.

In particular, for any nonzero $n < \omega$*, there is an order-type whose degree spectrum consists of exactly the non-low*$_n$ *degrees.*

Proof. We induct on n, simultaneously for all degrees $\mathbf{a}$, beginning with the case $n = 1$.

First, we relativize the proof of Theorem 2.18 to $\mathbf{a}$, obtaining a linear order $\mathcal{L}$ with presentations in every degree $\mathbf{b}$ with $\mathbf{b} > \mathbf{a}$ and $\mathbf{b}' > \mathbf{a}'$. Furthermore, the linear order $\mathcal{L}$ does not have a presentation in any degree $\mathbf{b}$ with $\mathbf{b} \geq \mathbf{a}$ and $\mathbf{b}' = \mathbf{a}'$.

Next, in order to handle degrees $\mathbf{b}$ with $\mathbf{b} \not\geq \mathbf{a}$, using Theorem 2.11, we fix a linear order $\mathcal{K}$ whose degree spectrum is the cone above $\mathbf{a}$. Then the degree spectrum of $\mathcal{L} + 1 + \mathcal{K}$ is the intersection of the degree spectra of $\mathcal{L}$ and $\mathcal{K}$, and so is as desired.

For $n > 1$, let $\mathcal{L}$ be a linear order whose degree spectrum consists of the degrees $\mathbf{b} > \mathbf{a}'$ such that $\mathbf{b}^{(n-1)} > \mathbf{a}^{(n)}$ (by the inductive hypothesis applied to $\mathbf{a}'$). Then the transfer $\mathcal{L}^{\mathrm{t}}$ has presentations in every degree $\mathbf{b}$ with $\mathbf{b} > \mathbf{a}$ and $\mathbf{b}^{(n)} > \mathbf{a}^{(n)}$. Furthermore, $\mathcal{L}^{\mathrm{t}}$ does not have a presentation in any degree $\mathbf{b}$ with $\mathbf{b} \geq \mathbf{a}$ and $\mathbf{b}^{(n)} = \mathbf{a}^{(n)}$. As in the case $n = 1$, the order $\mathcal{L}^{\mathrm{t}} + 1 + \mathcal{K}$ is as desired. □

2.4. **Arbitrary Finite Jump Degrees.** The results of the previous section allow us to obtain results about the finite jump degrees of linear orders.

Definition 2.20. Fix a structure $\mathcal{A}$, a natural number $n < \omega$, and a degree $\mathbf{a}$. The structure $\mathcal{A}$ has n^{th} *jump degree* $\mathbf{a}$ if $\mathbf{a}$ is the least element of the set

$$\{\mathbf{d}^{(n)} : \mathbf{d} \in \mathrm{DegSpec}(\mathcal{A})\}.$$

When $n = 0$, we say that $\mathcal{A}$ has *degree* $\mathbf{a}$.

For $n > 0$, the structure $\mathcal{A}$ has *proper* n^{th} *jump degree* $\mathbf{a}$ if $\mathcal{A}$ has n^{th} jump degree $\mathbf{a}$, but does not have any $(n-1)^{\text{st}}$ jump degree.

Thus, Theorem 2.11 can be restated as saying that every ω_1-degree is the degree of some linear ordering. Of course, as already noted, this contrasts rather sharply with the countable setting, where Richter [29] showed if a linear ordering has degree, that ω-degree must be $\mathbf{0}$. Furthermore, Knight [17] showed that if a countable linear ordering has a first jump degree, then this jump-degree must be $\mathbf{0}'$; whereas Downey and Knight [5] showed that for all $n \geq 2$, every degree $\mathbf{a} \geq \mathbf{0}^{(n)}$ is the proper n^{th} jump-degree of a countable linear ordering. In the uncountable setting, for every $n < \omega$, all possible (proper) jump degrees are realized.

Theorem 2.21. *Fix a finite ordinal* $n < \omega$*. For every* ω_1*-degree* $\mathbf{b} \geq \mathbf{0}^{(n)}$*, there is an order-type with proper* n^{th} *jump degree* $\mathbf{b}$.

Proof. For $n = 0$, this is Theorem 2.11.

For $n = 1$, from Fact 1.11, we obtain an ω_1-degree $\mathbf{a}$ with $\mathbf{a}' = \mathbf{b}$. We then relativize the proof of Theorem 2.7 to $\mathbf{a}$, obtaining a linear order $\mathcal{L}$. Then $\mathcal{L}$ has a presentation in every ω_1-degree $\mathbf{c}$ with $\mathbf{c} > \mathbf{a}$ and $\mathbf{c} \in \Delta^0_2(\mathbf{a})$. Notably, there are such ω_1-degrees $\mathbf{c}$ that are low over $\mathbf{a}$. Furthermore, the linear ordering $\mathcal{L}$ does not have a presentation in $\mathbf{a}$. As in the proof of Theorem 2.19, we take $\mathcal{L} + 1 + \mathcal{K}$, where $\mathcal{K}$ is a linear ordering such that $\mathrm{DegSpec}(\mathcal{K})$ is the cone above $\mathbf{a}$.

For $n > 1$, from Fact 1.11 we obtain an ω_1-degree $\mathbf{a}$ with $\mathbf{a}^{(n)} = \mathbf{b}$. From Theorem 2.19 with $\mathbf{a}$ and $n - 1$, we obtain a linear ordering $\mathcal{L}$. It is easy to verify that the linear order $\mathcal{L}$ suffices. □

3. ω_1-Computable Categoricity

In this section, we characterize the ω_1-computably categorical and uniformly ω_1-computably categorical linear orders. We recall the appropriate definitions.

Definition 3.1. Fix a cardinal $\kappa \in \{\omega, \omega_1\}$.

A κ-computable order-type λ is *κ-computably categorical* if for all κ-computable $\mathcal{A}, \mathcal{B} \in \lambda$ there is a κ-computable isomorphism $f : \mathcal{A} \cong \mathcal{B}$.

A κ-computable order-type λ is *uniformly κ-computably categorical* if there is a κ-computable function mapping a pair of indices of two κ-computable presentations $\mathcal{A}, \mathcal{B} \in \lambda$ to an index of a κ-computable isomorphism between them.

If $\mathcal{L}$ is a κ-computable linear order, then we say that $\mathcal{L}$ is *(uniformly) κ-computably categorical* if its order-type is (uniformly) κ-computably categorical.

As mentioned in the introduction, in the countable framework, Dzgoev [9] (see also [12]) and Remmel [27] independently showed that a computable linear ordering is computably categorical if and only if it has finitely many adjacencies. In some sense, this characterization is an artifact, stemming from the fact that finite subsets can determine only finitely many intervals. An examination of the proof of the Dzgoev-Remmel theorem shows that a more "correct" characterization is that an ω-computable linear ordering $\mathcal{L}$ is computably categorical if and only if there is a finite set $C \subseteq \mathcal{L}$ such that every $\mathcal{L}$-interval determined by C is either finite or has order-type η_0.

Before we give the correct generalization of the Dzgoev-Remmel characterization to ω_1, we show that the naive generalization – containing only countably many adjacencies (or, equivalently, countably many finite intervals) – is neither a sufficient nor a necessary condition for an ω_1-computable linear ordering to be computably categorical.

Example 3.2. The order-type $2 \cdot \rho$ is ω_1-computably categorical. To see this, fix computable presentations $\mathcal{A}$ and $\mathcal{B}$ of $2 \cdot \rho$. We may fix the "dense" countable subsets $2 \cdot \eta$ of $2 \cdot \rho$ in both $\mathcal{A}$ and $\mathcal{B}$ as a parameter. Then for any point in $\mathcal{A}$ or $\mathcal{B}$, we can determine whether it is the "left" or "right" point of its pair simply by waiting until both have shown up in the same interval determined by the copy of $2 \cdot \eta$.

Example 3.3. The order-type $\eta \cdot \omega_1$ is not computably categorical. To see this, we construct computable presentations $\mathcal{A}$ and $\mathcal{B}$ of $\eta \cdot \omega_1$ meeting the requirement

$$\mathcal{R}_e : \text{The function } \Phi_e \text{ is not an isomorphism from } \mathcal{A} \text{ to } \mathcal{B}.$$

for all $e \in \omega_1$.

In order to satisfy $\mathcal{R}_e$, we wait for Φ_e to be completely defined on some copy of η in $\mathcal{A}$, where the image of this copy in $\mathcal{B}$ is greater than the *restraint*. We then add an extra point to $\mathcal{B}$ within the image and move the *restraint* (for $\mathcal{R}_j$ with $j > e$) to a point in $\mathcal{B}$ greater than the image of this copy.

As a step towards characterizing the computably categorical linear orderings, we treat the uniform case.

Theorem 3.4. *An order-type λ is uniformly ω_1-computably categorical if and only if λ is finite or $\lambda = \eta_1$.*

Remark 3.5. We note that not only is the order-type η_1 uniformly ω_1-computably categorical, the effective back-and-forth argument demonstrating uniform ω_1-computable categoricity shows that if $\mathcal{A}$ and $\mathcal{B}$ are computable presentations of η_1, we can effectively extend any countable partial embedding $\psi : \mathcal{A} \to \mathcal{B}$ to an isomorphism between $\mathcal{A}$ and $\mathcal{B}$. This is uniform given ψ and ω_1-computable indices for $\mathcal{A}$ and $\mathcal{B}$.

Proof of Theorem 3.4. Every finite order-type is clearly uniformly ω_1-computably categorical. An effective back-and-forth argument of length ω_1 shows that η_1 is uniformly ω_1-computably categorical. This establishes one direction of the theorem.

In order to prove the other direction, let λ be an infinite, uniformly ω_1-computably categorical order-type, and let $\mathcal{L}$ be a computable presentation of λ. We show that $\mathcal{L}$ is $\aleph_1$-saturated. To do this, given countable subsets A and B of $\mathcal{L}$ such that $A <_{\mathcal{L}} B$, we "force" $\mathcal{L}$ to enumerate a point between A and B.

This is done by building an auxiliary ω_1-computable linear ordering $\mathcal{K}$. We ensure that $\mathcal{K}$ is isomorphic to $\mathcal{L}$ by defining a Δ^0_2-computable isomorphism $F : \mathcal{K} \to \mathcal{L}$. From the Recursion Theorem[1], we obtain an ω_1-computable index for $\mathcal{K}$. The fact that λ is uniformly ω_1-computably categorical yields an ω_1-computable index for an isomorphism $\Phi\colon \mathcal{K} \to \mathcal{L}$. We will exploit that Φ is (must be) an isomorphism to force $\mathcal{L}$ to enumerate a point between A and B.

The auxiliary order $\mathcal{K}$ is used as follows. At some stage s, we observe countable subsets C and D of $\mathcal{L} \upharpoonright s$ with $C <_{\mathcal{L}} D$ and $(C, D)_{\mathcal{L}\upharpoonright s} = \emptyset$. Ideally, we would be able to add a point z between $\Phi^{-1}(C)$ and $\Phi^{-1}(D)$, maintain the isomorphism F by taking an embedding of $(\mathcal{K} \upharpoonright s) \cup \{z\}$ into $\mathcal{L}$ and adding additional points as necessary, and wait for $\mathcal{L}$ to add a point in the $\mathcal{L}$-interval $(C, D)_{\mathcal{L}}$. If $\mathcal{L}$ fails to add a point to $(C, D)_{\mathcal{L}}$, then Φ is necessarily not an isomorphism; and if $\mathcal{L}$ adds a point to $(C, D)_{\mathcal{L}}$, then we have made progress towards $\aleph_1$-saturation.

Unfortunately, we cannot always guarantee the existence of such an embedding. If $\mathcal{L} \upharpoonright s$ is nonscattered, then such an embedding is ensured. If $\mathcal{L} \upharpoonright s$ is scattered, then there may not be such an embedding. Thus, before executing the above strategy, we work towards guaranteeing that $\mathcal{L}$ is nonscattered.

Again, we utilize the auxiliary order $\mathcal{K}$. If $\mathcal{L}_s$ is infinite and scattered, there is necessarily an infinite block B in $\mathcal{L}_s$. We then add a point between any adjacency of $\Phi^{-1}(B)$ and redefine the isomorphism F to accommodate these points. If $\mathcal{L}$ fails to add these extra points, then Φ is necessarily not an isomorphism; and if $\mathcal{L}$ adds these points, we have made progress towards being nonscattered.

The construction is thus split into two phases: a *descattering* phase and a *saturating* phase. If $\mathcal{L}$ enumerates points every time when "forced", then it will be $\aleph_1$-saturated; if it does not, then Φ will not be an isomorphism, contrary to hypothesis.

The final aspect of the construction is the definition of the auxiliary isomorphism $F\colon \mathcal{K} \to \mathcal{L}$. At stage s, we approximate F by an isomorphism F_s from $\mathcal{K}_s$ to $\mathcal{L}_s$. However, action at stage s may mean that F_{s+1} does not extend F_s. The question, then, is what to do at limit stages. Here we use the regularity of ω_1. Making the sequence $\langle \mathcal{K}_s \rangle$ continuous at limit stages, the fact that Φ is an isomorphism from $\mathcal{K}$ to $\mathcal{L}$ implies that for a closed and unbounded set of stages s, the map $\Phi \upharpoonright s$ is an isomorphism between $\mathcal{K}_s$ and $\mathcal{L}_s$, already observed at stage s. If we only act at

[1]Note that it is crucial that we ensure $\mathcal{K}$ is isomorphic to $\mathcal{L}$ in all circumstances, in particular whether or not Φ is an isomorphism, in order for the Recursion Theorem to apply.

such stages, then at limits of stages at which we act, the map $\Phi \upharpoonright s$ will be the required F_s, with no effort required on our part.

Construction: Since $\mathcal{L}$ is infinite, we may assume that $\mathcal{L}_\omega$ is infinite. We define an increasing and continuous sequence $\langle \mathcal{K}_s \rangle_{\omega \leq s < \omega_1}$ of countable linear orderings; and for each s with $\omega \leq s \leq \omega_1$, an isomorphism $F_s \colon \mathcal{K}_s \to \mathcal{L}_s$. We start with $\mathcal{K}_\omega := \mathcal{L}_\omega$ and $F_\omega := \mathrm{id}_{\mathcal{K}_\omega}$.

Let $s < \omega_1$ be infinite, and suppose that $\mathcal{K}_s$ and F_s are already defined. We first define an embedding $\hat{F}_s$ of $\mathcal{K}_s$ into $\mathcal{L}_s$. After $\hat{F}_s$ is defined, we let $\mathcal{K}_{s+1}$ be an extension of $\mathcal{K}_s$ to a countable linear ordering such that we can extend $\hat{F}_s$ to an isomorphism $F_{s+1} \colon \mathcal{K}_{s+1} \to \mathcal{L}_{s+1}$; this will conclude stage s.

We define $\hat{F}_s$. Let Φ_s be the function Φ, restricted to the inputs x such that $\Phi(x)$ converges before stage s. At stage s, we check if Φ_s is an isomorphism from $\mathcal{K}_s$ to $\mathcal{L}_s$. If not, then we let $\hat{F}_s := F_s$. This means that in this case, F_{s+1} will extend F_s.

Suppose that $\Phi_s \colon \mathcal{K}_s \to \mathcal{L}_s$ is an isomorphism. There are two cases, depending on whether $\mathcal{L}_s$ is scattered or not.

- *Descattering:* If $\mathcal{L}_s$ is scattered, we let B be the $<_{\omega_1}$-least infinite block of $\mathcal{L}_s$. Since the order-type of B is either ω, ω^*, or ζ, there is a self-embedding f_s of $\mathcal{L}_s$ (which we can take to be the identity outside B, though this is unimportant) such that for some adjacent $a <_{\mathcal{L}} b$ in B, $f_s(a)$ and $f_s(b)$ are not adjacent in $\mathcal{L}_s$. We pick some such embedding f_s. We let $\hat{F}_s := f_s \circ \Phi_s$.
- *Saturating*: If $\mathcal{L}_s$ is nonscattered, we let (C_s, D_s) be the $<_{\omega_1}$-least pair of countable sets $C, D \subseteq \mathcal{L}_s$ such that $C <_{\mathcal{L}} D$ and $(C, D)_{\mathcal{L}_s}$ is empty. Since $\mathcal{L}_s$ is nonscattered, there is a self-embedding f_s of $\mathcal{L}_s$ such that $(f_s[C_s], f_s[D_s])_{\mathcal{L}_s}$ is nonempty (add a point to $\mathcal{L}_s$ between C_s and D_s and embed the result into $\mathcal{L}_s$). We let $\hat{F}_s := f_s \circ \Phi_s$.

To complete the construction, we need to define F_s for limit stages s, since we already stipulated that $\mathcal{K}_s := \bigcup_{t<s} \mathcal{K}_t$ for limit s. Let J be the set of stages t such that Φ_t is an isomorphism from $\mathcal{K}_t$ to $\mathcal{L}_t$. Let s be a limit stage. If $J \cap s$ is bounded below s, then (by induction) for all $r < t$ in the interval $(\sup(J \cap s), s)$, we have $F_r \subset F_t$. It then follows that $F_s := \bigcup_{t \in (\sup(J\cap s), s)} F_t$ is an isomorphism between $\mathcal{K}_s$ and $\mathcal{L}_s$. If $J \cap s$ is unbounded below s, then we let $F_s := \Phi_s$.

Verification: Let $\mathcal{K} := \mathcal{K}_{\omega_1}$. We first show that $\mathcal{K}$ and $\mathcal{L}$ are isomorphic. Certainly if J is unbounded in ω_1, then Φ is an isomorphism from $\mathcal{K}$ to $\mathcal{L}$. On the other hand, if J is bounded below ω_1, then $F := \bigcup_{s > \sup(J)} F_s$ is an isomorphism between $\mathcal{K}$ and $\mathcal{L}$. Hence $\mathcal{K}$ and $\mathcal{L}$ are isomorphic.

As $\mathcal{K}$ and $\mathcal{L}$ are isomorphic by hypothesis, the map Φ is an isomorphism. Since ω_1 is regular and the sequences $\langle \mathcal{K}_s \rangle$ and $\langle \mathcal{L}_s \rangle$ are continuous, the set J is unbounded in ω_1.

Next, we show that $\mathcal{L}$ is nonscattered. Suppose, for a contradiction, that $\mathcal{L}$ is scattered. Hence for all s, the order $\mathcal{L}_s$ is scattered. Now we observe that if $s < t$ are both in J, then $B_s \neq B_t$. For let $a, b \in B_s$ be adjacent in B_s such that $f_s(a)$ and $f_s(b)$ are not adjacent in $\mathcal{L}_s$. The definition $\hat{F}_s = f_s \circ \Phi_s$ implies that $\Phi^{-1}(a)$ and $\Phi^{-1}(b)$ are not adjacent in $\mathcal{K}_{s+1}$. Since Φ_t is an isomorphism of $\mathcal{K}_t$ with $\mathcal{L}_t$, and Φ_t extends Φ_s, we see that a and b are not adjacent in $\mathcal{L}_t$, and so that B_s is not a block of $\mathcal{L}_t$; so $B_t \neq B_s$.

Now the fact that J is unbounded in ω_1 shows that $\mathcal{L}$ is nonscattered. For if $\mathcal{L}$ is scattered, then it contains an infinite block. Let B be the $<_{\omega_1}$-least infinite block of $\mathcal{L}$. Being an infinite block of $\mathcal{L}$ is a Π^0_1 property; this, and the regularity of ω_1 implies that for all but countably many $s \in J$, B is the $<_{\omega_1}$-least infinite block of $\mathcal{L}_s$, i.e., $B_s = B$. This contradicts the fact that J is unbounded and the fact that $s < t$ in J implies $B_s \neq B_t$.

Let s_0 be the least stage such that $\mathcal{L}_{s_0}$ is nonscattered. We now show that $\mathcal{L}$ is $\aleph_1$-saturated. The proof is similar. First we observe that if $s_0 \leq s < t$ and $s, t \in J$, then $(C_s, D_s) \neq (C_t, D_t)$. For the definition $\hat{F}_s = f_s \circ \Phi_s$ and the property of f_s imply that the interval $\left(\Phi^{-1}C_s, \Phi^{-1}D_s\right)_{\mathcal{K}_{s+1}}$ is nonempty, and so as Φ_t is an isomorphism from $\mathcal{K}_t$ to $\mathcal{L}_t$, the interval $(C_s, D_s)_{\mathcal{K}_t}$ is nonempty. We can then show that no pair (C, D) can be the $<_{\omega_1}$-least pair of countable subsets $C <_{\mathcal{L}} D$ such that $(C, D)_{\mathcal{L}}$ is empty, as this would contradict that J is unbounded in ω_1; again, the property defining the pair (C, D) is Π^0_1. Hence $\mathcal{L}$ is $\aleph_1$-saturated, which completes the proof. □

We turn to the main result of this section, the characterization of ω_1-computably categorical linear orderings. During earlier work on this subject, trying to generalize the Remmel-Dzgoev criterion, Knight conjectured that a linear ordering $\mathcal{L}$ is computably categorical if and only there is a countable subset Q of $\mathcal{L}$ and a number n such that every Q-interval of $\mathcal{L}$ is either empty, contains exactly n points, or is $\aleph_1$-saturated. While not correct, this conjecture does contain an important ingredient which is correct: If $\mathcal{L}$ is computably categorical, then there is some countable subset Q of $\mathcal{L}$ such that every Q-interval is either finite or has order-type η_1.

The added ingredient is effectiveness. An ordering $\mathcal{L}$ with a countable subset Q can be computably categorical, witnessed by Q, even if $\mathcal{L}$ contains finite Q-intervals of different sizes. However, for each n, we need to effectively enumerate those cuts of Q that define intervals that may have size n. This added ingredient sheds light on the countable case as well. The characterization below of ω_1-computable categoricity is a correct characterization of ω-computable categoricity if we replace "countable" by "finite". The special properties of the cardinal ω make the effectiveness condition redundant in the countable case. The uncountable case allows us to recover this important aspect of the criterion, which is invisible if one only sees the countable case.

The effectiveness condition of Theorem 3.6 implies another difference between countable and uncountable linear orderings. Given the theorem (and relativizing it), it is easy to construct an order-type λ of size $\aleph_1$ with computable presentations which is not computably categorical but is *relatively computably categorical above* **d**: There is a degree **d** such that any two presentations $\mathcal{L}_1, \mathcal{L}_2 \geq \mathbf{d}$ of λ are $(\mathcal{L}_1 \oplus \mathcal{L}_2)$-computably isomorphic. There are no such countable order-types: If λ is a countable order-type with computable elements that is not ω-computably categorical, then for every ω-Turing degree **d** there are **d**-computable presentations $\mathcal{L}_1$ and $\mathcal{L}_2$ of λ which are not isomorphic by any **d**-computable isomorphism.

Theorem 3.6. *An ω_1-computable linear order $\mathcal{L}$ is ω_1-computably categorical if and only if there are a countable set $Q \subset \mathcal{L}$ and a collection $\{V_n : 0 < n < \omega\}$ of pairwise disjoint ω_1-c.e. sets of cuts of Q with the following properties:*

(1) Every Q-interval of $\mathcal{L}$ is either finite or has order-type η_1.
(2) For any cut (Q_1, Q_2) of Q, if the Q-interval $(Q_1, Q_2)_{\mathcal{L}}$ has size $n > 0$, then $(Q_1, Q_2) \in V_n$.

Note that since the c.e. sets V_n are pairwise disjoint, it follows that if $(Q_1, Q_2) \in V_n$ then the interval $(Q_1, Q_2)_{\mathcal{L}}$ is either empty, has size n, or is $\aleph_1$-saturated.

Proof. ($\Longleftarrow$) Let $\mathcal{L}$ be an ω_1-computable linear order, equipped with sets Q and $\{V_n\}$ as described in the theorem. To show that $\mathcal{L}$ is ω_1-computably categorical, let $\mathcal{K}$ be a computable linear order which is isomorphic to $\mathcal{L}$, and let $g\colon \mathcal{L} \to \mathcal{K}$ be an arbitrary (not necessarily effective) isomorphism. We define a computable isomorphism $f\colon \mathcal{L} \to \mathcal{K}$ by starting with $g \restriction Q$. We extend f to a map on $\mathcal{L}$ by defining f on every Q-interval. Let $A := (Q_1, Q_2)_{\mathcal{L}}$ be a Q-interval; let $B := g[A] = (g[Q_1], g[Q_2])_{\mathcal{K}}$. If A is empty, we do not need to define f on A. If A is nonempty, we wait for a stage s at which either $A_s := (Q_1, Q_2)_{\mathcal{L}\restriction s}$ is infinite, or $(Q_1, Q_2) \in V_n$ at stage s, $|A_s| = n$, and $B_s := (g[Q_1], g[Q_2])_{\mathcal{K}\restriction s}$ also has size n for some positive $n < \omega$; at least one of the two has to happen. Here we use the fact that since the collection of sets $\{V_n\}$ is countable, the sequence $\langle V_n \rangle$ is uniformly c.e.

Now in the latter case, we define f to be the order-preserving bijection between A_s and B_s. In the former case, we know that both A and B are $\aleph_1$-saturated, so a computable isomorphism between A and B can be built uniformly from our indices $(Q_1, Q_2)_{\mathcal{L}}$ and $(g[Q_1], g[Q_2])_{\mathcal{K}}$ for A and B. Furthermore, if we first see that $|A_s| = n = |B_s|$ and $(Q_1, Q_2) \in V_n$, we define f on A_s, and then more points are added to A, it must be that A and B have order-type η_1. The map $f \restriction A_s$ can be uniformly extended to a computable isomorphism between A and B.

($\Longrightarrow$) Let $\mathcal{L}$ be an ω_1-computable, ω_1-computably categorical linear order. We want to find sets Q and V_n as in the theorem. We attempt to emulate the proof of Theorem 3.4. To force $\mathcal{L}$ into the desired form, we construct an auxiliary computable linear ordering $\mathcal{K}$ isomorphic to $\mathcal{L}$. Since the computable categoricity of $\mathcal{L}$ may fail to be uniform, this time we need to guess which computable function is the computable isomorphism between $\mathcal{K}$ and $\mathcal{L}$. Let $\langle \Phi_j \rangle_{j<\omega_1}$ list all partial computable functions. The *guess* R_j guesses that Φ_j is an isomorphism from $\mathcal{K}$ to $\mathcal{L}$. As in the previous proof, we build $\mathcal{K}$ as the union of an increasing, continuous, computable sequence $\langle \mathcal{K}_s \rangle$ of countable linear orders. For this discussion, for $j < \omega_1$, let J_j be the collection of stages s such that $\Phi_{j,s}$ is an isomorphism from $\mathcal{K}_s$ to $\mathcal{L}_s$ (in the actual construction, we will change this definition a little). We assume that the guess R_j is correct at stages $s \in J_j$. If at a given stage, more than one guess appears correct, we, of course, act for the strongest one, which is the one with least index.

All the guesses together collaborate on constructing embeddings $F_s\colon \mathcal{K}_s \to \mathcal{L}_s$. We need to ensure that in the contradictory outcome in which every guess R_j is wrong (i.e., if every set J_j is bounded), that $\langle F_s \rangle$ reaches a limit that is an isomorphism between $\mathcal{K}$ and $\mathcal{L}$. The important point is that even though each J_j may be bounded below ω_1, the union $\bigcup_{j<\omega_1} J_j$ may be unbounded in ω_1. This is why we cannot just let $F_s := \Phi_{j,s}$ for the least j such that $s \in J_j$. We utilize the priority ordering between the various guesses to force the convergence of $\langle F_s \rangle$ to a limit F. The strongest guess R_0 may assume that $F_s := \Phi_{0,s}$ for all $s \in J_0$ (though this will turn out to be unimportant); at other stages, guesses weaker than R_0 are not allowed to change the last definition made by R_0. That is, if $s \notin J_0$, then F_s must extend F_{t+1}, where $t := \sup(J_0 \cap s)$; recall that J_0 is closed, so $t \in J_0$.

Now consider R_1. If $s \in J_1 \setminus J_0$, and $t = \sup(J_0 \cap s)$, then inductively, $F_s \colon \mathcal{K}_s \to \mathcal{L}_s$ extends F_{t+1}; the guess R_1 needs to extend F_s to an isomorphism $F_{s+1} \colon \mathcal{K}_{s+1} \to \mathcal{L}_{s+1}$, which does not necessarily extend F_s, but must still extend F_{t+1}. In other words, for $S := \operatorname{dom} F_{t+1}$, the guess R_1 must define $F_{s+1} \restriction S = F_{t+1}$, and then define F_{s+1} on every S-interval of $\mathcal{K}_{s+1}$. Hence, R_1 must implement the strategies from the proof of Theorem 3.4 on each S-interval separately. This indicates how the set Q eventuates; at first approximation, in the example that R_1 is correct and R_0 is not, we can let $Q := \Phi_{1,s}[S]$. If neither R_0 not R_1 are correct, but R_2 is correct, then R_2 will have to take into consideration the restraints on F_s imposed by both R_0 and R_1, and so on.

This approximation to the definition of Q is not quite correct because of an annoying fact. For simplicity, consider again R_1's point of view at a stage $s \in J_1 \setminus J_0$. On an interval A of $\mathcal{K}_s$ determined by a cut of $S := \operatorname{dom} F_{t+1}$, R_1 would like to define F_{s+1} to be some modification of $\Phi_{1,s}$. However, there is no reason why $\Phi_{1,s}$ would map A to the corresponding interval in $\mathcal{L}_s$. In other words, it need not be the case that $\Phi_{1,s}$ extends F_{t+1}. Let (S_1, S_2) be a cut of S; for $r \geq s$, let $A_r := (S_1, S_2)_{\mathcal{K}_r}$, $B_r := (F_{t+1}[S_1], F_{t+1}[S_2])_{\mathcal{L}_r}$, and $C_r := (\Phi_1[S_1], \Phi_1[S_2])_{\mathcal{L}_r}$. If, for example, R_1's task is to add points to A_s en route to making sure that C_{ω_1} is not scattered, then in the previous proof, we used a self-embedding f of C_s to define F_{s+1} on the extension A_{s+1} of A_s; here we will need to ensure that B_s, not C_s, has the requisite self-embedding that would enable us to add the necessary points to A_s. In the scattered case, we will ensure that C_s and B_s are isomorphic by $F_s \circ (\Phi_{1,s})^{-1}$, so we can translate the self-embedding of C_s to B_s to justify adding the point to A_s. If B_s is nonscattered, then, of course, we can get an embedding of any extension of A_s into B_s.

All of this relates to the definition of F_s at limit stages s. The difficulty arises in the situation in which a single guess acts unboundedly below s. Again for simplicity, suppose that J_0 is bounded below s but that J_1 is unbounded in s. As we saw above, we cannot define $F_s := \Phi_{1,s}$ as this would violate the restraint, imposed by R_0, to keep F_s extending F_{t+1}. Using the notation above, we need to define F_s on A_s, and we cannot appeal to $\Phi_{1,s}$ in doing so. We will need to modify the strategies for descattering, etc., to ensure that we can define an isomorphism from A_s to B_s if B_s is nonscattered; in the scattered case, it will turn out that we cannot guarantee that A_s and B_s are isomorphic, merely the obvious fact that A_s is embeddable into B_s; thus for limit s, F_s may not be onto $\mathcal{L}_s$.

The crux of the issue, then, is how to descatter infinite intervals so that $F_r \restriction A_r$, for $r < s$, reaches a limit which will be an isomorphism from A_s to B_s; and, what was not covered in the previous proof, how to deal with finite intervals. Both are delicate. Consider finite intervals first. In the notation above, suppose that $s \in J_1$ and that $|C_s| = n$. We would like to enumerate the cut $(\Phi_1[S_1], \Phi_1[S_2])$ defining C_s into V_n, and then ensure that if $C_{\omega_1} \neq C_s$ then C_{ω_1} is infinite. If $B_s = C_s$, that is, if $(\Phi_1[S_1], \Phi_1[S_2]) = (F_{t+1}[S_1], F_{t+1}[S_2])$, then we have a simple strategy for doing so; we note, though, that this strategy requires immediate action at a stage $r \geq s$ at which B_r grows, even if $r \notin J_1$. At the least stage $r \geq s$ at which points are added to $C_r = B_r$, we can add points to A_r so that $\Phi_1 \restriction A_s$ cannot be extended to an isomorphism from A_r to C_r. This may require changing F, that is, $F_{r+1} \restriction A_{r+1}$

may not extend $F_r \restriction A_r$. This is fine, except that we cannot repeat this action infinitely often, or else $F_u \restriction A_u$ will not stabilize before some future limit stages. If $C_s = B_s$ then we do not need to repeat this redefinition of $F \restriction A$; we have created an interval Y in A_r such that for all $u > r$, the Φ_1-image of Y in C_u must contain more points than the F_u-image of Y in B_u, or C_u is infinite. However, if $C_s \neq B_s$, and we promise to only change F on A once, then our opponent may rectify the damage our action did by enumerating points in different locations in B_u and C_u, allowing them to recover an isomorphism between C_u and A_u extending $\Phi_{1,s}$.

The solution is to consider not only A_s, but also the interval $\Phi_1^{-1}F[A_s]$, and $F^{-1}\Phi_1[A_s]$, and $\Phi_1^{-1}F[\Phi_1^{-1}F[A_s]]$, and so on, that is, all the intervals of $\mathcal{K}_s$ which are conjugate to A_s by the action of $\Phi_1^{-1} \circ F$. At stage s, we require that all of these intervals are isomorphic, and so all contain exactly n points. To make sure that these conjugates of A_s are well-defined, we need to ensure that $\Phi_1^{-1} \circ F$ and all of its iterates and inverses restrict to automorphisms of S; this requires us to extend the set S, and so the set $Q = \Phi_1[S]$, from the original attempt $\operatorname{dom} F_{t+1}$ above, by closing under the action of $\Phi_1^{-1} \circ F$. Once we have done this, we will see that if we change F on A_s at some stage $r \geq s$, and then the opponent rectifies the isomorphism, then the opponent's action allows us to change F on some conjugate of A_s, as the C-interval (the Φ_1-image) of one conjugate of A_s is the B-interval (the F_s-image) of another such conjugate. If we ensure that some conjugate of A_s eventually becomes infinite, then Φ_1 will ensure that eventually, A_s too will become infinite. That is, action for some interval brings benefit to all of its conjugates. Recall, however, that to ensure limits stabilize, we limit ourselves to one F-change on any A-interval; so we need, for instance, to ensure that we don't "burn out" all of A's conjugates at the first step. The downside of the shared benefit is shared responsibility: We need to act for all conjugates of an interval in tandem, ensuring that if one acts, then others wait peacefully.

The same ideas allow us to act for infinite, nonscattered intervals, in a way which is compatible with stabilizing limits at limit stages. In this case, we again work with all conjugates of the given interval, carrying out the same action in all of them. We then argue that this concerted action ensures that adjacencies that we tried to fill in A_s in fact become infinite intervals in future A_r for $r \in J_1$. This means that when taking further action for a block in A_r, we may fix the unique element of the block which has been previously subjected to an F-change.

These are the ideas behind the construction; we are ready for the formalities.

Construction: We define an increasing, continuous, and computable sequence $\langle \mathcal{K}_s \rangle_{s<\omega_1}$ of countable linear orderings. For each $s < \omega_1$, we define an embedding $F_s \colon \mathcal{K}_s \to \mathcal{L}_s$. If s is a successor ordinal, then F_{s+1} will actually be an isomorphism between $\mathcal{K}_s$ and $\mathcal{L}_s$. We start with $\mathcal{K}_0 := \mathcal{L}_0$ being the empty ordering, and F_0 being the empty function.

At any stage s, for $j < s$, we decide if the guess R_j *requires attention* at stage s. We let I_j be the collection of stages at which R_j requires attention. If $\Phi_{j,s}$ is an isomorphism between $\mathcal{K}_s$ and $\mathcal{L}_s$, then R_j will require attention at stage s. We will also ensure that each I_j is closed; that is, if s is a limit of stages at which R_j requires attention, then R_j also requires attention at stage s. At any stage s, the guess R_j *receives attention* if j is least such that $s \in I_j$. Again, only R_j with $j < s$ may require attention at stage s, so $I_j \subseteq (j, \omega_1)$.

For $s < \omega_1$ and $j \leq s$, let

$$r_{j,s} := \sup \left\{ t+1 \,:\, t \in \bigcup_{i<j} (I_i \cap s) \right\};$$

so $r_{j,s} \leq s$. The map $F_{r_{j,s}}$ is the restraint imposed on R_j at stage s. We will ensure (Claim 3.6.2) that F_s extends $F_{r_{j,s}}$ for all j. If R_j receives attention at stage s, then it is R_j's task to define F_{s+1}; the guess R_j must let F_{s+1} also extend $F_{r_{j,s}}$. (For completeness, for $j \geq s$, we let $r_{j,s} := s$.)

Suppose that $\mathcal{K}_s$ and F_s are defined, and that $\Phi_{j,s}$ is an isomorphism from $\mathcal{K}_s$ to $\mathcal{L}_s$. Then $\Phi_{j,s}^{-1} \circ F_s$ is a self-embedding of $\mathcal{K}_s$. Let $N_{j,s} := \bigcap_{n<\omega} \left(\Phi_{j,s}^{-1} \circ F_s\right)^n [\mathcal{K}_s]$. So $N_{j,s}$ is the largest subset of $\mathcal{K}_s$ restricted to which $\Phi_{j,s}^{-1} \circ F_s$ is an automorphism. For brevity, we let $h_{j,s} := \left(\Phi_{j,s}^{-1} \circ F_s\right) \restriction N_{j,s}$. Dually, we let $M_{j,s} := F_s[N_{j,s}] = \Phi_{j,s}[N_{j,s}]$ be the largest subset of $\mathcal{L}_s$ restricted to which the self-embedding $F_s \circ \Phi_{j,s}^{-1}$ of $\mathcal{L}_s$ is an automorphism; we let $g_{j,s} := \left(F_s \circ \Phi_{j,s}^{-1}\right) \restriction M_{j,s}$.

If $\Phi_{j,s} \colon \mathcal{K}_s \to \mathcal{L}_s$ is an isomorphism and $j < s$, and further $\mathcal{K}_{r_{j,s}} \subseteq N_{j,s}$, then we let $S_{j,s}$ be the smallest subset of $N_{j,s}$ containing $\mathcal{K}_{r_{j,s}}$ which is closed under the action of $h_{j,s}$; the point is that $\mathcal{K}_{r_{j,s}} = \operatorname{dom} F_{r_{j,s}}$. In other words, we let $S_{j,s} := \bigcup_{n\in\mathbb{Z}} (h_{j,s})^n [\mathcal{K}_{r_{j,s}}]$. We let J_j be the collection of stages s such that $j < s$, $\Phi_{j,s}$ is an isomorphism from $\mathcal{K}_s$ to $\mathcal{L}_s$, and $\mathcal{K}_{r_{j,s}} \subseteq N_{j,s}$.

If $t \in J_j$, $s \geq t$, and (S_1, S_2) is a cut of $S_{j,t}$, we let

$$\begin{aligned} A_s(j,t,S_1,S_2) &:= (S_1,S_2)_{\mathcal{K}_s}, \\ B_s(j,t,S_1,S_2) &:= (F_t[S_1], F_t[S_2])_{\mathcal{L}_s}, \text{ and} \\ C_s(j,t,S_1,S_2) &:= (\Phi_j[S_1], \Phi_j[S_2])_{\mathcal{L}_s}. \end{aligned}$$

We often write A_s for $A_s(j,t,S_1,S_2)$, and similarly write B_s and C_s. We have $C_t = \Phi_j[A_t]$ and $B_t \supseteq F_t[A_t]$; if $A_t \subseteq N_{j,s}$ then $B_t = F_t[A_t]$. If $r \in [t,s)$ then $A_r = A_s \cap \mathcal{K}_r$, and similarly for B_r and C_r.

For $n \in \mathbb{Z}$, we let

$$A_{n,s}(j,t,S_1,S_2) := A_s\left(j,t,(h_{j,t})^n[S_1],(h_{j,t})^n[S_2]\right),$$

and again usually omit the quadruple (j,t,S_1,S_2), which is understood from the context. So $A_{0,s} = A_s$. We call the intervals $A_{n,s}$ the *j,t-conjugates* of A_s. If $A_t \subseteq N_{j,t}$, then $A_{n,t} = (h_{j,t})^n[A_t]$, so in this case, the j,t-conjugates of A_t are precisely the elements of the orbit of A_t under the action of $h_{j,t}$ on the subsets of $N_{j,t}$. We let $B_{n,s}$ and $C_{n,s}$ be the intervals corresponding to $A_{n,s}$; we have $B_{n,s} = C_{n+1,s}$ for all $n \in \mathbb{Z}$. An easy argument shows that either for all n, $A_{n,s} = A_s$ (in which case for all n, $B_{n,s} = C_{n,s} = B_s = C_s$), or the intervals $\{A_{n,s} \,:\, n \in \mathbb{Z}\}$ are pairwise disjoint (and the intervals $\{B_{n,s}\ n \in \mathbb{Z}\}$ are also pairwise disjoint).

Let $s < \omega_1$ and $j < s$. Suppose that $J_j \cap [r_{j,s}, s]$ is nonempty. Claim 3.6.3 says that the sets $S_{j,t}$ and the functions $F_t \restriction S_{j,t}$ do not depend on the choice of $t \in J_j \cap [r_{j,s}, s]$. It follows that for any cut (S_1, S_2) of $S_{j,t}$, the interval $A_s = A_s(j,t,S_1,S_2)$ and the related intervals B_s and C_s do not depend on the choice of $t \in J_j \cap [r_{j,s}, s]$. It also follows that the functions $h_{j,t} \restriction S_{j,t}$ do not depend on the choice of $t \in J_j \cap [r_{j,s}, s]$, and so, the j,t-conjugates $A_{n,s}$ of A_s and $B_{n,s}$ of B_s also do not depend on the choice of t. We thus simply refer to the j-conjugates of A_s.

We also write $S_{j,s}$ for $S_{j,t}$ for any $t \in J_j \cap [r_{j,s}, s]$, even if $s \notin J_j$. We thus say that $S_{j,s}$ *is defined* if $J_j \cap [r_{j,s}, s]$ is nonempty.

Let $j < s < \omega_1$, and suppose that $S_{j,s}$ is defined. Let A_s be a nonempty $S_{j,s}$-interval of $\mathcal{K}_s$. We say that R_j *diagonalizes on* A_s*, with* m *points* (at stage s), if R_j receives attention at stage s, $m = |B_{s+1}| = |C_{s+1}| > |A_s|$, and $F_{s+1} \restriction A_{s+1}$ does not extend $F_s \restriction A_s$. This happens because R_j adds points to A_s so that $\Phi_{j,s}$ cannot be extended to an isomorphism between A_{s+1} and C_{s+1}.

Suppose that A_s is finite, and that there is some $t \in J_j \cap [r_{j,s}, s]$ such that A_t is nonempty; we let $t^{\mathtt{fin}}_{j,s}(A_s)$ be the least such t. We say that R_j *has an opportunity to diagonalize on* A_s *(with* m *points)* if $m = |B_{s+1}| = |C_{s+1}| > |A_s|$, and:

- R_j did not diagonalize on A_r (with any number of points) at any stage in $[t^{\mathtt{fin}}_{j,s}(A_s), s)$; and
- for any j-conjugate A'_s of A_s, R_j did not diagonalize on A'_r with m points at any stage in $[t^{\mathtt{fin}}_{j,s}(A_s), s)$.

First, we assume that $\mathcal{K}_s$ and F_s are already defined, and we show how to define $\mathcal{K}_{s+1}$ and $\mathcal{L}_{s+1}$. A guess R_j requires attention at stage s if $j < s$ and one of the following holds:

- $\Phi_{j,s}$ is an isomorphism from $\mathcal{K}_s$ to $\mathcal{L}_s$; or
- $I_j \cap s$ is unbounded in s; or
- $S_{j,s}$ is defined, and R_j has an opportunity to diagonalize on some finite $S_{j,s}$-interval at stage s.

If no guess requires attention at stage s, then we let $\mathcal{K}_{s+1}$ be an extension of $\mathcal{K}_s$ such that there is some isomorphism $F_{s+1}\colon \mathcal{K}_{s+1} \to \mathcal{L}_{s+1}$ extending F_s. Otherwise, let R_j be the guess which receives attention at stage s. We act similarly, letting F_{s+1} extend F_s to an isomorphism from $\mathcal{K}_{s+1}$ to $\mathcal{L}_{s+1}$, if $S_{j,s}$ is not defined. The reason for this action is merely to impose restraint on weaker guesses.

Assume that $S_{j,s}$ is defined. We will let F_{s+1} agree with F_s on $S_{j,s}$. To define $\mathcal{K}_{s+1}$ and F_{s+1}, we will define, for every nonempty $S_{j,s}$-interval A_s of $\mathcal{K}_s$, an isomorphism $F_{s+1} \restriction A_{s+1}$ between A_{s+1} (which we define) and the corresponding B_{s+1} (which the opponent plays). Exactly how to do this depends on the order-type of B_s.

Claim 3.6.1 says that even if $s \notin J_j$, F_s and F_t agree on $S_{j,s} = S_{j,t}$ for all $t \in J_j \cap [r_{j,s}, s]$. The definition of B_s then shows that $F_s \restriction A_s$ is an embedding of A_s into B_s. By Claim 3.6.4, unless B_s is nonscattered (and so infinite), $F_s \restriction A_s$ is in fact an isomorphism between A_s and B_s.

0. Unless specified below, we will extend *simply*, that is, let A_{s+1} be any extension of A_s for which there is an isomorphism $F_{s+1} \restriction A_{s+1}$ from A_{s+1} to B_{s+1} extending F_s.

1. Suppose that R_j has an opportunity to diagonalize on A_s with m points at stage s (so B_s is finite), and that A_s is the $<_{\omega_1}$-least such interval among its j-conjugates.

As $|A_s| < m = |C_{s+1}|$, the map $\Phi_{j,s} \restriction A_s$ is not onto C_s. This shows that we can extend A_s to an ordering A_{s+1} of size m such that $\Phi_{j,s}$ cannot be extended to an isomorphism between A_{s+1} and C_{s+1}. To see this, let $t = t^{\mathtt{fin}}_{j,s}(A_s)$; we required that A_t is nonempty. As $t \in J_j$, we know that $\Phi_{j,t}$ is an isomorphism of $\mathcal{K}_t$ and $\mathcal{L}_t$, and so $A_t \subseteq \operatorname{dom} \Phi_{j,t}$; as $\Phi_{j,s}$ extends $\Phi_{j,t}$, we see that $A_s \cap \operatorname{dom} \Phi_{j,s}$ is nonempty.

Since $|C_{s+1}| > |A_s|$, there is some cut (D, E) of A_t such that $(D, E)_{A_s}$ is smaller than $(\Phi_j[D], \Phi_j[E])_{C_{s+1}}$. Since A_t is nonempty, $(D, E)_{A_s}$ is not the only interval of A_s, so we can add points to A_{s+1} elsewhere, so that A_{s+1} contains m points, but $(D, E)_{A_{s+1}} = (D, E)_{A_s}$. Then $\Phi_{j,t} \upharpoonright A_t$ cannot be extended to an isomorphism between A_{s+1} and C_{s+1}. We let $F_{s+1} \upharpoonright A_{s+1}$ be the unique isomorphism.

It is important that at stage s, we do not let R_j diagonalize on any j-conjugate of A_s with m points other than A_s itself; this is why we demanded that A_s be the $<_{\omega_1}$-least such interval among its j-conjugates. So if A'_s is a j-conjugate of A_s, distinct from A_s, and at stage s, R_j has the opportunity to diagonalize on A'_s with m points, then we do not let R_j do so, but rather extend A'_s simply as in ($\mathbf{0}$) above.

2. If $s \in J_j$ and B_s is infinite and scattered, we treat all of the conjugates $A_{n,s}$ of A_s in one step. For definiteness, assume that A_s is $<_{\omega_1}$-least among its j-conjugates. As mentioned above, Claim 3.6.4 says that in this case, for all n, $F_s \upharpoonright A_{n,s}$ is an isomorphism from $A_{n,s}$ to $B_{n,s}$. Since $s \in J_j$, for all n, $\Phi_{j,s} \upharpoonright A_{n,s}$ is an isomorphism from $A_{n,s}$ to $C_{n,s} = B_{n-1,s}$. Hence each $B_{n,s}$ is contained in $M_{j,s}$, and $g_{j,s} \upharpoonright B_{n,s}$ is an isomorphism from $B_{n,s}$ to $B_{n+1,s}$.

We let $t^{\mathtt{inf}}_{j,s}(A_s)$ be the least $t \in J_j \cap [r_{j,s}, s]$ such that A_t is infinite.

Let $D_{0,s}$ be the $<_{\omega_1}$-least *maximal* infinite block of $B_{0,s}$. For $n \in \mathbb{Z}$, let $D_{n,s} = (g_{j,s})^n[D_{0,s}]$; so $D_{n,s}$ is a maximal infinite block of $B_{n,s}$. By Claim 3.6.6, there are self-embeddings $f_{n,s}$ of $D_{n,s}$ with the following four properties.

- Coherence: The functions $f_{n,s}$ are coherent with respect to $h_{j,s}$: For all n and m, $f_{n+m,s} = (h_{j,s})^m \circ f_{n,s} \circ (h_{j,s})^{-m}$.
- Fixed Points: For all n, the set $E_{n,s} := \{a \in D_{n,s} : f_{n,s}(a) = a\}$ is a finite (possibly empty) convex subset of $D_{n,s}$. Note that the coherence of the functions $f_{n,s}$ shows that for all n and m, $E_{n+m,s} = (h_{j,s})^m[E_{n,s}]$.
- Historical Responsibility: For all n, if $a \in D_{n,s}$ and there is some stage $u \in I_j \cap [t^{\mathtt{inf}}_{j,s}(A_s), s)$ such that $a \in B_{n,u}$ and $F^{-1}_{u+1}(a) \neq F^{-1}_u(a)$, then $a \in E_{n,s}$. Dually, if $x \in (F_s)^{-1}D_{n,s}$ and there is some stage $u \in I_j \cap [t^{\mathtt{inf}}_{j,s}(A_s), s)$ such that $x \in A_{n,s}$ and $F_{u+1}(x) \neq F_u(x)$, then $x \in F_s^{-1}E_{n,s}$.
- Interpolation: For all n, $a \in D_{n,s} \setminus E_{n,s}$ and $b \in D_{n,s}$ distinct from a, $f_{n,s}(a)$ and $f_{n,s}(b)$ are not adjacent in $D_{n,s}$.

We fix such maps $f_{n,s}$, and extend them to all of $B_{n,s}$ by the identity on $B_{n,s} \setminus D_{n,s}$; this is a self-embedding of $B_{n,s}$ since $D_{n,s}$ is a convex subset of $B_{n,s}$. For all $n \in \mathbb{Z}$, we let $A_{n,s+1}$ be an extension of $A_{n,s}$ and $F_{s+1} \upharpoonright A_{n,s+1}$ an extension of $f_{n,s} \circ (F_s \upharpoonright A_{n,s})$ to an isomorphism from $A_{n,s+1}$ to $B_{n,s+1}$.

3. If $s \in J_j$ and B_s is nonscattered, we let (X_s, Y_s) be the $<_{\omega_1}$-least pair of countable subsets (X, Y) of B_s such that $X <_{B_s} Y$ and $(X, Y)_{B_s}$ is empty. We let f be a self-embedding of B_s such that $(f[X_s], f[Y_s])_{B_s}$ is nonempty. We let $(f[X_s], f[Y_s])_{B_s}$ be nonempty. We let A_{s+1} be an extension of $f \circ (F_s \upharpoonright A_s)$ to an isomorphism from A_{s+1} to B_{s+1}.

This completes the instructions for stage s, given $\mathcal{K}_s$ and F_s. At limit stages s, we need to define F_s; we already stipulated that $\mathcal{K}_s = \bigcup_{t<s} \mathcal{K}_t$.

There are four cases.

A. Suppose that $r_{s,s} < s$. So between stages $r_{s,s}$ and s, no guess requires attention. Then our instructions show that for all $t < t'$ in $(r_{s,s}, s)$, $F_{t'}$ extends F_t. In this case we let $F_s = \bigcup_{t\in(r_{s,s},s)} F_t$.

B. If (A) fails, we let j be the least ordinal $j \le s$ such that $r_{j,s} = s$. Suppose that j is a limit ordinal. We consider the analogue of the Dekker nondeficiency set relative to the set $\{r_{i,s} : i < j\}$, which under the assumptions is unbounded in s. We let T be the collection of stages $r_{i,s}$ where $i < j$ and for all $i' < i$, $r_{i',s} < r_{i,s}$. Then T is unbounded in s. For all $i < j$, if $r_{i,s} \in T$, then for all $t \in (r_{i,s}, s)$ we have $r_{i,t} = r_{i,s}$, and so by Claim 3.6.2, F_t extends $F_{r_{i,s}}$. Hence we can let $F_s = \bigcup_{t\in T} F_t$.

C. If both (A) and (B) fail, then there is some (unique) $j < s$ such that $r_{j,s} < s$ but $I_j \cap s$ is unbounded in s. Suppose that $J_j \cap [r_{j,s}, s)$ is empty. Let $t \in I_j \cap [r_{j,s}, s)$. Since R_j receives attention at stage s, the instructions show that F_{t+1} extends F_t. By Claim 3.6.2, if t' is the next element of I_j beyond t, then $F_{t'}$ extends F_{t+1}, as $r_{i+1,t'} = t + 1$. Hence, for all $t < t'$ in $I_j \cap [r_{j,s}, s)$, $F_t \subseteq F_{t+1} \subseteq F_{t'} \subseteq F_{t'+1}$. We thus let $F_s = \bigcup_{t\in I_j\cap[r_{j,s},s)} F_t = \bigcup_{t\in I_j\cap[r_{j,s},s)} F_{t+1}$.

D. Otherwise, we again take $j < s$ such that $r_{j,s} < s$ but $I_j \cap s$ is unbounded in s; and now we suppose that $J_j \cap [r_{j,s}, s)$ is nonempty. As discussed above, the set $S_{j,t}$ and the function $F_t \restriction S_{j,t}$ do not depend on the choice of $t \in J_j \cap [r_{j,s}, s)$. We then let F_s extend this map. To define the rest of F_s, we need to define F_s on any nonempty $S_{j,s}$-interval A_s of $\mathcal{K}_s$.

Let A_s be an $S_{j,s}$-interval of $\mathcal{K}_s$. If B_s is nonscattered, then we let $F_s \restriction A_s$ be any embedding of A_s into B_s. Suppose that B_s is scattered. Let $w = \min(J_j \cap [r_{j,s}, s))$. As we mentioned above, for all $t \in [w, s)$, $F_t \restriction A_t$ is an embedding of A_t into B_t (and in fact, is an isomorphism). By Claim 3.6.5, for all $x \in A_s$, $F_t(x)$ takes on at most three values as t ranges over $I_j \cap [w, s)$; so we can let $F_s(x)$ be the limit $\lim_{t\in I_j\cap[w,s)} F_t(x)$. It is easy to see that $F_s \restriction A_s$ is order-preserving.

This completes the construction of $\mathcal{K}$.

Promises Were Made: To carry out the construction, we relied on various facts about the construction itself, which we now establish. Thus, the correctness of the following claims, along with the construction itself, are really described and verified together, by simultaneous induction on stage s of the construction. We begin by showing that restraints are respected.

Claim 3.6.1. Let $s < \omega_1$ and $j < s$. Let $t \in J_j \cap [r_{j,s}, s)$. Then F_s and F_t agree on $S_{j,t}$.

Proof. We note that $r_{j,s} < s$ implies that for all $u \in [r_{j,s}, s]$, $r_{j,u} = r_{j,s}$.

Suppose first that s is a successor stage. If $s = t + 1$, then as $r_{j,s} < s$, and $t \in J_j$, we see that R_j receives attention at stage $s - 1$. The instructions then ensure that F_s and F_{s-1} agree on $S_{j,s-1} = S_{j,t}$. Suppose then that $t < s-1$. Since $r_{j,s} \le t$, we have $r_{j,s} = r_{j,s-1}$, so $r_{j,s-1} \le t$. By induction, F_{s-1} and F_t agree on $S_{j,t}$. If F_s extends F_{s-1} then we are done. Similarly, if R_j receives attention at stage $s - 1$, then R_j is instructed to let F_s and F_{s-1} agree on $S_{j,s-1} = S_{j,t}$; so again F_s and F_t agree on $S_{j,t}$.

We suppose, then, that F_s does not extends F_{s-1}, but that R_j does not receive attention at stage $s-1$. Let R_i be the guess which receives attention at stage $s-1$; then $S_{i,s-1}$ is defined. Since $r_{j,s} < s$, we cannot have $i < j$; so $i > j$.

The fact that $t \in J_j$ implies that $r_{i,s-1} > t$. Let $u \in J_i \cap [r_{i,s-1}, s-1]$. Since $S_{i,u}$ contains $\mathcal{K}_{r_{i,s-1}}$, it contains $S_{j,t}$. By induction, F_{s-1} and F_u agree on $S_{i,u}$, and so agree on $S_{j,t}$. Also by induction, F_u and F_t agree on $S_{j,t}$. At stage $s-1$, R_i is instructed to let F_s agree with F_{s-1} on $S_{i,u}$, and so on $S_{j,t}$. In all, we see that F_s and F_t agree on $S_{j,t}$.

Suppose that s is a limit stage. In cases (A), (B) and (C) of the definition of F_s, we let F_s be the union of F_u for some u in a set cofinal in s. In these cases, as for all $u \in (t,s)$, F_u and F_t agree on $S_{j,t}$, we have F_s and F_t agree on $S_{j,t}$. In case (D), suppose that R_i defined F_s, that is, $r_{i,s} < s$ but $I_i \cap s$ is unbounded in s. Since $r_{j,s} < s$, we must have $j \leq i$. If $i = j$, then R_j lets F_s agree with F_t on $S_{j,t}$. Suppose then that $i > j$. Then an argument paralleling the previous paragraph concludes the proof. Let $u \in J_i \cap [r_{i,s}, s)$. Since $t \in J_j$, $r_{i,s} > t$. By induction, F_u and F_t agree on $S_{j,t}$; $S_{i,u}$ contains $S_{j,t}$, as it contains $\mathcal{K}_{r_{i,s}}$; and by definition, F_s agrees with F_u on $S_{i,u}$. Hence in this case as well, F_s and F_t agree on $S_{j,t}$. □

Claim 3.6.2. For all $s, j < \omega_1$, F_s extends $F_{r_{j,s}}$.

Proof. Of course, if $r_{j,s} = s$ then we are done. Hence, we assume that $r_{j,s} < s$. As before, this means that for all $t \in [r_{j,s}, s]$, $r_{j,t} = r_{j,s}$.

First, suppose that s is a successor stage. Then $r_{j,s-1} = r_{j,s}$. By induction, F_{s-1} extends $F_{r_{j,s}}$. If F_s extends F_{s-1}, then the claim holds at s. Suppose that F_s does not extend F_{s-1}. Let R_i be the guess which receives attention at stage $s-1$; then $S_{i,s-1}$ is defined. Since $r_{j,s} < s$, $i \geq j$. Hence $r_{i,s-1} \geq r_{j,s-1} = r_{j,s}$. This means that $\mathcal{K}_{r_{j,s}}$ is contained in $S_{i,s-1}$, so $F_{s-1} \upharpoonright S_{i,s-1}$ extends $F_{r_{j,s}}$. At stage $s-1$, R_i is instructed to let F_s agree with F_{s-1} on $S_{i,s-1}$; so F_s extends $F_{r_{j,s}}$.

Next, suppose that s is a limit ordinal. Again we consider the cases defining F_s. If $F_s = \bigcup_{t \in T} F_t$, where T is cofinal in T, then F_s extends $F_{r_{j,s}}$, as by induction, F_t extends $F_{r_{j,s}}$ for $t \in [r_{j,s}, s)$. In case (D), let i be least such that $r_{i+1,s} = s$, so the guess R_i is responsible for defining F_s. Since $r_{j,s} < s$, we have $i \geq j$. Let $t \in J_i \cap [r_{i,s}, s)$. F_s agrees with F_t on $S_{i,s} = S_{i,t}$. Since $r_{i,s} \geq r_{j,s}$, the set $S_{i,s}$ contains $\mathcal{K}_{r_{j,s}}$. By induction, F_t extends $F_{r_{j,s}}$. So $F_t \upharpoonright S_{i,t}$ extends $F_{r_{j,s}}$, and so F_s extends $F_{r_{j,s}}$. □

Claim 3.6.3. Let $s < \omega_1$ and $j < s$. Let $t \in J_j \cap s$. Suppose that $r_{j,s} \leq t$ and that $\Phi_{j,s}$ is an isomorphism from $\mathcal{K}_s$ to $\mathcal{L}_s$. Then $s \in J_j$, $S_{j,s} = S_{j,t}$, and F_s and F_t agree on $S_{j,t}$.

Proof. $r_{j,s} \leq t$ implies that $r_{j,s} = r_{j,t}$. Let $r = r_{j,s}$. By Claim 3.6.1, F_s and F_t agree on $S_{j,t}$. Certainly $\Phi_{j,s}^{-1}$ and $\Phi_{j,t}^{-1}$ agree on $F_s[S_{j,t}] = F_t[S_{j,t}]$ as $\Phi_{j,s}$ extends $\Phi_{j,t}$. Hence $\Phi_{j,s}^{-1} \circ F_s$ and $\Phi_{j,t}^{-1} \circ F_t$ agree on $S_{j,t}$. The function $\Phi_{j,t}^{-1} \circ F_t$ restricts to an automorphism of $S_{j,t}$. It follows that $S_{j,t} \subseteq N_{j,s}$. Since $\mathcal{K}_r \subseteq S_{j,t}$, we see that $\mathcal{K}_r \subseteq N_{j,s}$. It follows that $s \in J_j$. As $h_{j,s}$ and $h_{j,t}$ agree on $\mathcal{K}_r$, it follows that $S_{j,s} = S_{j,t}$. □

Claim 3.6.4. Let $s < \omega_1$ and $j < s$. Suppose that $S_{j,s}$ is defined, and that no guess stronger than R_j receives attention at stage s. Then for any $S_{j,s}$-interval A_s, if B_s is nonscattered then $F_s \upharpoonright A_s$ is an isomorphism between A_s and B_s.

Proof. The definition of B_s, and the fact that F_s agrees with F_t on $S_{j,s} = S_{j,t}$ for all $t \in J_j \cap [r_{j,s}, s]$, shows that for any $S_{j,s}$-interval A_s, $B_s \cap \operatorname{range} F_s = F_s[A_s]$. Thus, if F_s is onto $\mathcal{L}_s$, then for any $S_{j,s}$-interval A_s, $F_s \upharpoonright A_s$ is onto B_s. By design, if s is a successor ordinal, then F_s is onto $\mathcal{L}_s$.

Suppose then that s is a limit ordinal, and consider the cases defining F_s. We claim that in cases (A), (B) and (C), F_s is the union of maps F_u where u ranges over a set T of *successor* stages cofinal in s; this would imply that in these cases too, F_s is onto $\mathcal{L}_s$. In case (A) we can let T be the collection of all successor ordinals in $(r_{s,s}, s)$. In case (C) we let T be the set of successors of ordinals in $I_i \cap [r_{i,s}, s)$, where $r_{i,s} < s$ but $I_i \cap s$ is unbounded in s. In case (B) let i be the least such that $r_{i,s} = s$. In the construction, we let T' be the collection of stages $r_{k,s}$ where $k < i$ and for all $k' < k$, $r_{k',s} < r_{k,s}$; and we let F_s be the union of F_u for $u \in T'$. While T' may contain limit ordinals, we show that T' contains a cofinal subset T consisting of successor ordinals. For $k < i$, let $\alpha_{k,s} = \sup_{t\in I_k\cap s}(t+1) = \max(I_k \cap s) + 1$. For all $k < i$, $r_{k,s} = \sup_{k'<k} \alpha_{k',s}$. Let T be the set of stages $\alpha_{k,s}$ such that $\alpha_{k,s} > r_{k,s}$. Certainly T consists of successor ordinals. The fact that $s = \sup_{k<i} \alpha_{k,s}$ shows that T is unbounded in s. And if $\alpha_{k,s} \in T$ then $r_{k+1,s} = \alpha_{k,s}$ and $r_{k+1,s} \in T'$. Thus $T \subseteq T'$.

We discuss case (D). Let $i < s$ such that $r_{i,s} < s$ but $I_i \cap s$ is unbounded in s. Hence $s \in I_i$. By the assumptions of this claim, we must have $j \le i$. Let A_s be an $S_{i,s}$-interval, and suppose that B_s is scattered. Let $w := \min(J_i \cap [r_{i,s}, s))$. Since for all $t \in [w, s)$, B_t is scattered, by induction, for all $t \in [w, s)$, $F_t[A_t] = B_t$. By Claim 3.6.5, for each $a \in B_s$, $F_t^{-1}(a)$ takes at most three values as t ranges over $I_j \cap [w, s)$. This shows that $a \in \operatorname{range} F_s$, so $F_s \restriction A_s$ is an isomorphism between A_s and B_s. Hence if $i = j$ then we are done.

Suppose that $j < i$. In this case we show that every $S_{j,t}$-interval is the union of $S_{i,s}$-intervals. As we have seen before, $r_{i,s} > t$ and so $S_{i,s}$ contains $S_{j,t}$; by Claim 3.6.1, F_s and F_t agree on $S_{j,t}$. This means that if A_s is an $S_{j,t}$-interval of $\mathcal{K}_s$, A'_s is an $S_{i,s}$-interval of $\mathcal{K}_s$, and $A_s \cap A'_s$ is nonempty, then $A'_s \subseteq A_s$, and so $B'_s \subseteq B_s$. Let A_s be an $S_{j,t}$-interval, and suppose that B_s is scattered. Let $a \in B_s$; we show that $a \in \operatorname{range} F_s$. If $a \in F_s[S_{i,s}]$ then certainly $a \in \operatorname{range} F_s$. Otherwise, there is a unique $S_{i,s}$-interval A'_s such that $a \in B'_s$. It follows that $B'_s \subseteq A'_s$, so B'_s is scattered. Hence by the previous paragraph, $B'_s \subseteq \operatorname{range} F_s$, so $a \in \operatorname{range} F_s$ as required. □

Claim 3.6.5. Let $s < \omega_1$ and $j < s$. Suppose that $S_{j,s}$ is defined. Let A_s be an $S_{j,s}$-interval such that B_s is scattered. Then:

(1) For all $x \in A_s$, as t varies over stages $t \in I_j \cap [r_{j,s}, s]$ such that $S_{j,t}$ is defined and $x \in A_t$, $F_t(x)$ obtains at most three values.
(2) For all $a \in B_s$, as t varies over stages $t \in I_j \cap [r_{j,s}, s]$ such that $S_{j,t}$ is defined and $a \in B_t$, $F_t^{-1}(a)$ obtains at most three values.

(Note that in (2), if $t \in I_j \cap [r_{j,s}, s]$ and $S_{j,t}$ is defined, and $a \in B_t$, then as B_t is scattered, by Claim 3.6.4, $a \in \operatorname{range} F_t$.)

Proof. Let $w := \min(J_j \cap [r_{j,s}, s])$. Of course, if $w = s$ there is nothing to prove. So we assume that $w < s$. We first note that if $t \in I_j \cap [w, s]$ and $t' := \min(I_j \cap (t, s])$ is the successor of t in I_j, then $F_{t'}$ extends F_t as $r_{j+1,t'} = t + 1$. Also, if $u \in (w, s]$ is a limit ordinal, then the claim holds at u by induction. It suffices, then, to show:

(1) For all $x \in A_s$ there are at most two stages $t \in I_j \cap [w, s)$ such that $x \in A_t$ and $F_{t+1}(x) \neq F_t(x)$.
(2) For all $a \in B_s$ there are at most two stages $t \in I_j \cap [w, s)$ such that $b \in A_t$ and $F_{t+1}^{-1}(x) \neq F_t^{-1}(x)$.

Both follow directly from our instructions, as $t \in I_j \cap [w, s)$ and $w \geq r_{j,s}$ implies that R_j receives attention at stage t. Let $x \in A_s$ and $a \in B_s$. By the definition of having an opportunity to diagonalize on a finite interval, there is at most one stage $t \in I_j \cap [w, s)$ such that B_t is finite and at which $F_{t+1} \restriction A_{t+1}$ does not extend $F_t \restriction A_t$. Also, the "historical responsibility" property of the functions $f_{n,t}$ shows that there is at most one stage $t \in I_j \cap [w, s)$ such that B_t is infinite and such that $F_{t+1}^{-1}(a) \neq F_t^{-1}(x)$; for if t is the least such stage, then at every stage $u \in I_j \cap (t, s)$, as $t \geq t_{j,u}^{\mathtt{inf}}(A_u)$, this property ensures that $f_{n,u}(a) = a$, where $F_{u+1} \restriction A_{u+1}$ extends $f_{n,u} \circ (F_u \restriction A_u)$, so $F_{u+1}^{-1}(a) = F_u^{-1}(a)$. Similarly, there is at most one stage $t \in I_j \cap [w, s)$ such that B_t is infinite and $F_{t+1}(x) \neq F_t(x)$. This completes the proof of the claim.

We note that it is not necessarily the case that for $x \in A_s$, the set $\{F_t(x) : t \in [w, s] \ \&\ x \in A_t\}$ contains at most three elements. This is because $F_t(x)$ could change often on the interval $[u, v)$, where u is the least stage such that $x \in A_u$ and $v := \min(I_j \cap (u, s))$ is u's successor in I_j. It is true that $F_t(x)$ takes at most finitely many values on this interval, but we do not need this fact. However, we do see that if s is a limit point of I_j, then $F_t(x)$ reaches a limit below s for all $x \in A_s$, so in case (D) of the definition of F_s, we could write $F_s(x) = \lim_{t \to s} F_t(x)$, and not restrict to taking the limit on the stages in I_j. □

We turn to proving that the embeddings used in case (2) of defining F_{s+1} always exist. The following claim is the last promise we need to verify.

Claim 3.6.6. Let $j < \omega_1$ and suppose that $s \in J_s$. Let A_s be an $S_{j,s}$-interval, and suppose that B_s is infinite and scattered. Then there are self-embeddings $f_{n,s}$ of $D_{n,s}$ as required for defining $F_{s+1} \restriction A_{n,s+1}$.

Proof. If $t_{j,s}^{\mathtt{inf}}(A_s) < s$, let $t \in J_j \cap [t_{j,s}^{\mathtt{inf}}(A_s), s)$, and let $u := \min(J_j \cap (t, s])$ be t's successor in J_j. If $v \in I_j \cap (t, u)$ then at stage v, R_j is instructed to let $F_{v+1} \restriction A_{v+1}$ extend $F_v \restriction A_v$. It follows that $F_u \restriction A_u$ extends $F_{t+1} \restriction A_{t+1}$.

We show that for all $n \in \mathbb{Z}$, and all distinct $a < b \in D_{n,t}$ which are not both in $E_{n,t}$, the interval $(a, b)_{B_{n,u}}$ is infinite. We prove, by induction on $m \geq 0$, that each such interval contains at least m points; the base case is vacuous. Assume we showed this for $m \geq 0$. Let $n \in \mathbb{Z}$ and let $a < b \in D_{n,t}$, not both in $E_{n,t}$. Let $x := \Phi_{j,t}^{-1}(a)$ and $y := \Phi_{j,t}^{-1}(b)$; so $x < y$ are elements of $A_{n+1,t}$; and $g_{j,t}(a) = F_t(x)$, and $g_{j,t}(b) = F_t(y)$ are elements of $D_{n+1,t}$. Let $a' := F_{t+1}(x)$ and $b' := F_{t+1}(y)$. Since $\Phi_{j,u}$ extends $\Phi_{j,t}$, and $a' = F_u(x)$, $b' = F_u(y)$, we see that $a' = g_{j,u}(a)$ and $b' = g_{j,u}(b)$.

The coherence property of the functions $f_{n,t}$ shows that not both of $g_{j,t}(a)$ and $g_{j,t}(b)$ are in $E_{n+1,t}$. The definition of F_{t+1} shows that $a' = f_{n+1,t}(g_{j,t}(a))$ and $b' = f_{n+1,t}(g_{j,t}(b))$. The interpolation property of the functions $f_{n,t}$ shows that there is some $c' \in (a', b')_{D_{n+1,t}}$. Now since the function $f_{n+1,t}$ is injective, the definition of the set $E_{n+1,t}$ implies that either a' or b' are not elements of $E_{n+1,t}$. By induction, either the interval $(a', c')_{B_{n+1,u}}$ or the interval $(c', b')_{B_{n+1,u}}$ contains at least m points; so the interval $(a', b')_{B_{n+1,u}}$ contains at least $m+1$ points. Since $g_{j,u}$ is an isomorphism from $B_{n,u}$ to $B_{n+1,u}$, we see that $(a, b)_{B_{n,u}}$ also contains at least $m + 1$ points, as required.

We note that this proof works if the conjugates $B_{n,s}$ are all identical, and also if they are pairwise disjoint.

Now let U_s be the set of $x \in \bigcup_n A_{n,s}$ such that there is some $t \in I_j \cap [t^{\texttt{inf}}_{j,s}(A_s), s)$ such that $F_{t+1}(x) \neq F_t(x)$; and dually, let V_s be the set of $a \in \bigcup_n B_{n,s}$ such that there is some $t \in I_j \cap [t^{\texttt{inf}}_{j,s}(A_s), s)$ such that $F^{-1}_{t+1}(a) \neq F^{-1}_t(a)$; noting, of course, that for all n and m, $t^{\texttt{inf}}_{j,s}(A_{n,s}) = t^{\texttt{inf}}_{j,s}(A_{m,s})$ as this stage is in J_j. We claim that V_s is invariant under $g_{j,s}$, that $V_s = \Phi_{j,s}[U_s] = F_s[U_s]$, and that for all n, $V_s \cap D_{n,s}$ is at most a singleton.

For let $a \in V_s$; let t witness this fact. So $b = f_{n,t}(a) \neq a$ (where $b \in B_{n,t}$). Let $a' := g_{j,t}(a)$. The coherence of $f_{m,t}$ shows that $b' := f_{n+1,t}(a') = g_{j,t}(b)$, so $b' \neq a'$. Let $x := \Phi^{-1}_{j,t}(a)$; we define $F_{t+1}(x) := b'$, so $b' = g_{j,t+1}(a)$. The fact that $F_t(x) = b'$ and $b' \neq a'$ means that for all $u \in J_j \cap (t, s)$ we have $b' \in U_u$, so inductively $F_u(x) = F_{u+1}(x) = b'$; so $b' = g_{j,s}(a)$. This shows that $g_{j,s}(a) \in V_s$ as well. An identical argument shows that $(g_{j,s})^{-1}(a) \in V_s$; so V_s is invariant under $g_{j,s}$.

This argument also shows that if $a \in V_s$, witnessed by t, then $\Phi^{-1}_{j,t}(a) \in U_s$. If $a \in B_{n,s}$ and $a \notin V_s$, then for all $t \in J_j \cap [t^{\texttt{inf}}_{j,s}(A_s), s)$ such that $a \in B_{n,t}$, the coherence property shows that $F_{t+1}(x) = F_t(x)$ for $x = \Phi^{-1}_{j,t}(a)$; so $x \notin U_s$. Hence $U_s = \Phi^{-1}_{j,s} V_s$. Since V_s is invariant under $g_{j,s}$, we also have $V_s = F_s[U_s]$.

Suppose, for a contradiction, that $n \in \mathbb{Z}$, and that $a, b \in U_s \cap D_{n,s}$ and $a < b$. Let t_a witness that $a \in U_s$ and t_b witness that $b \in U_s$. Without loss of generality, $t_b \geq t_a$. Then $b \in D_{n,t_b}$. Since $(a,b)_{B_{n,s}}$ is finite, so is $(a,b)_{B_{n,t_b}}$. Since D_{n,t_b} is a *maximal* block, $a \in D_{n,t_b}$ as well. Since $b \notin E_{n,t_b}$, the argument above shows that the interval $(a,b)_{B_{n,u}}$, where $u = \min(J_j \cap (t_b, s))$, is infinite, contradicting that $(a,b)_{B_{n,s}}$ is finite.

This tells us how to define the functions $f_{n,s}$. The order-type of $D_{0,s}$ is either ζ, ω or ω^*. If $\operatorname{otp}(D_{0,s}) = \zeta$, then we let $f_{0,s}$ be a self-embedding of $D_{0,s}$ which fixes the unique element of $V_s \cap D_{0,s}$, if that element exists, moves every other element, and satisfies the interpolation property; so $E_{0,s} = V_s \cap D_{0,s}$. If $\operatorname{otp}(D_{0,s}) = \omega$, then we let $f_{0,s}$ be a self-embedding of $D_{0,s}$ which fixes the initial segment of $D_{0,s}$ determined by the unique element of $V_s \cap D_{0,s}$, and moves every other element; this initial segment is, of course, finite; we can again define $f_{0,s}$ to satisfy interpolation. The case $\operatorname{otp}(D_{0,s}) = \omega^*$ is symmetrical. We then define $f_{n,s}$ for $n \neq 0$ so that coherence holds. The fact that V_s is invariant under $g_{j,s}$, and that $V_s = F_s[U_s]$, shows that this definition of $f_{n,s}$ satisfies the historical responsibility property. □

The Correct Guess: We show that some guess is correct, and is eventually able to act as it wishes. We first note that the arguments in cases (A) or (B) for defining F_s for limit stages s show that if for all $j < \omega_1$, $r_{j,\omega_1} < \omega_1$, that is, if for all $j < \omega_1$, I_j is bounded below ω_1, then we can define an isomorphism F_{w_1} from $\mathcal{K} = \mathcal{K}_{\omega_1}$ to $\mathcal{L} = \mathcal{L}_{\omega_1}$ by taking the union of maps F_t where t ranges over some set cofinal in ω_1. This isomorphism is Δ^0_2. The assumption that $\mathcal{L}$ is computably categorical then implies that there is a computable isomorphism from $\mathcal{K}$ to $\mathcal{L}$.

On the other hand, let $j < \omega_1$, and suppose that I_j is unbounded in ω_1. Then J_j is also unbounded in ω_1. For suppose otherwise; let $t := \max J_j$. The guess R_j requires attention at stage $s > t$ only if $r_{j,s} \leq t$ and R_j has the opportunity, at stage s, to diagonalize on some finite $S_{j,t}$-interval A_s such that A_t is nonempty. Since $\mathcal{K}_t$ is countable, there are only countably many nonempty $S_{j,t}$-intervals of $\mathcal{K}_t$. For each cut (S_1, S_2) of $S_{j,t}$ such that $A_t(j, t, S_1, S_2)$ is nonempty, there are at most countably many stages $s > t$ at which R_j diagonalizes on $A_s(j,t,S_1,S_2)$. This is because if t is such a stage then the size of A_{t+1} is strictly greater than the size of A_t. So R_j *receives* attention at most countably many times after stage t. If $r_{j,\omega_1} > t$, then after stage r_{j,ω_1}, R_j never requires attention, so I_j is bounded below ω_1. Otherwise, R_j receives attention at every stage $s \in I_j \cap [t, \omega_1)$, so again I_j is bounded below ω_1.

Certainly, if J_j is unbounded in ω_1, then Φ_j is an isomorphism from $\mathcal{K}$ to $\mathcal{L}$. We have established, therefore, that in either case, $\mathcal{K}$ and $\mathcal{L}$ are computably isomorphic. Let j be the least index such that Φ_j is a computable isomorphism from $\mathcal{K}$ to $\mathcal{L}$. The minimality of j shows that for all $i < j$, J_i is bounded below ω_1; we just argued that this implies that for all $i < j$, I_i is bounded below ω_1. Hence $r_{j,\omega_1} < \omega_1$.

We show that J_j is unbounded in ω_1. Let $r = r_{j,\omega_1}$. We know that the set H of stages $s \geq r$ such that $\Phi_{j,s}$ is an isomorphism from $\mathcal{K}_s$ to $\mathcal{L}_s$ is closed and unbounded in ω_1. Claim 3.6.2 implies that to show that J_j contains a final segment of H, it is sufficient to show that $J_j \cap [r, \omega_1)$ is nonempty. Suppose, for a contradiction, that $J_j \subseteq r$. Let s be the least limit point of H. As $H \subseteq I_j$, case (C) shows that F_s is the union of maps F_t where $t \in H \cap [r, s)$, and that F_s is onto $\mathcal{L}_s$. Hence $N_{j,s} = \mathcal{K}_s$, so $\mathcal{K}_r$ is contained in $N_{j,s}$; so $s \in J_j$ after all, for the desired contradiction.

We have thus established the existence of $j < \omega_1$ such that $r_{j,\omega_1} < \omega_1$ but J_j is unbounded in ω_1. The guess R_j is the "correct guess" with which we work to establish the structure theorem for $\mathcal{L}$.

Enumerating Finite Intervals: From now, we fix j such that $r_{j,\omega_1} < \omega_1$ but J_j is unbounded in ω_1. Let $r := r_{j,\omega_1}$. Let $S := S_{j,\omega_1} = S_{j,s}$ for all $s \in J_j \setminus r$, and let $Q := \Phi_j[S] = F_s[S]$ for such s.

The arguments of the proof of Theorem 3.4 show that every infinite Q-interval of $\mathcal{L}$ is $\aleph_1$-saturated. In slightly more detail, let B_{ω_1} be a Q-interval of $\mathcal{L}$. To show that B_{ω_1} is nonscattered, assume otherwise. For $n \in \mathbb{Z}$, let D_n be the $<_{w_1}$-least maximal infinite block of the j-conjugate B_{n,ω_1} of B_{ω_1}. For sufficiently late $s \in J_1$, for all n, D_n is the $<_{\omega_1}$-least maximal infinite block of $B_{n,s}$. Let $s \in J_1$ be sufficiently late. Then if $A_{n,s}$ is $<_{\omega_1}$-least among its j-conjugates, then $D_{n,s} = D_n$, and at stage s, we add points to $A_{n+1,s}$ to ensure that $D_{n,s}$ is in fact not a convex subset of B_{n,ω_1}; this follows from the fact that $E_{n,s} \neq D_{n,s}$, as $E_{n,s}$ is finite. This is a contradiction, and so B_{ω_1} is nonscattered. Then, an argument identical to the one in Theorem 3.4 shows that B_{ω_1} is $\aleph_1$-saturated.

It remains to deal with finite intervals. For $n > 0$, at stage $s \in J_1 \setminus r$ we enumerate a cut (Q_1, Q_2) of Q into V_n if the interval $B_s = (Q_1, Q_2)_{\mathcal{L}_s}$ contains exactly n points. Certainly if $(Q_1, Q_2)_{\mathcal{L}}$ has size $n > 0$ then $(Q_1, Q_2) \in V_n$. We need to show that the sets V_n are pairwise disjoint. That is, we show that if $u \in J_j$, $u \geq r$, B_u is finite and nonempty, and $s > u$ is also in J_j, then either B_s is infinite, or $B_s = B_u$. Fix such u, s and B_u, and suppose, for contradiction, that B_s is finite but that $B_s \neq B_u$. Let $t = t^{\mathtt{fin}}_{j,\omega_1}(A_{\omega_1})$ be the least stage in $J_j \setminus r$ such that B_t is nonempty.

The proof bifurcates into two cases. Either the j-conjugates $B_{n,s}$ of B_s are all identical, or they are pairwise disjoint. First, suppose they are identical. In this case, we first show that if $v \geq t$ and at stage v, R_j diagonalizes on A_v, then B_w is infinite, where $w = \min J_j \cap (v, \omega_1)$. For at stage v, we ensure that $\Phi_{j,v} \restriction A_v$ cannot be extended to the unique isomorphism $F_{v+1} \upharpoonright A_{v+1}$ from A_{v+1} to $C_{v+1} = B_{v+1}$; so $\Phi_{j,v}$ and F_{v+1} disagree on some element of A_v. If B_w is finite, then $F_w \upharpoonright A_w = \Phi_{j,w} \upharpoonright A_w$ is an isomorphism between A_w and B_w. However, $\Phi_{j,w}$ extends $\Phi_{j,v}$, and $F_w \upharpoonright A_w$ extends $F_{v+1} \upharpoonright A_{v+1}$, because the instructions don't allow R_j to change F on A between stages v and w; we also use Claim 3.6.2. This is impossible.

This implies that R_j did not diagonalize on B_v at any stage $v \in [t, s)$. However, let v be the least stage in $[u, s)$ such that $B_{v+1} = B_s$; so $B_{v+1} \neq B_v$. Then R_j has the opportunity to diagonalize on A_v at stage v, because it did not do so at an earlier stage, and $|A_v| = |B_v| < |B_{v+1}| = |C_{v+1}|$ (as $C_{v+1} = B_{v+1}$). This is a contradiction.

Suppose now that the intervals $B_{n,s}$ are pairwise disjoint. Let $m = |B_s|$. Since $s \in J_j$, $m = |B_{n,s}|$ for every j-conjugate $B_{n,s}$ of B_s. We first show that there is some stage $v \in [u, s)$ at which R_j diagonalizes on some conjugate $A_{n,u}$ with m points. Suppose otherwise. As for all $n \in \mathbb{Z}$, $|B_{n,u}| = |B_u| < m$, certainly R_j does not diagonalize on any $A_{n,v}$ with m points at any stage $v \in [t, u)$. For each $m' < m$, there is at most one n such that R_j diagonalizes with m' points on $A_{n,v}$ (at any stage $v \in [t, s)$). Certainly R_j does not diagonalize on any conjugate $A_{n,v}$ with more than m points at any stage before s. Let k be the maximal integer such that R_j diagonalized on $A_{k,v}$ at some $v \in [t, s)$. This is well-defined as the conjugates $A_{n,s}$ are pairwise disjoint. Let v_0 be the least stage $v < s$ at which $|B_{l,v+1}| = m$ for some $l \geq k$. Let l be the least integer $l \geq k$ such that $|B_{l,v_0+1}| = m$. Let v_1 be the least stage $v < s$ at which $|B_{l+1,v+1}| = m$; so $v_1 \geq v_0$. Then $|B_{l,v_1+1}| = |B_{l+1,v_1+1}| = m$ and $|A_{l+1,v_1}| = |B_{l+1,v_1}| < m$. So at stage v_1, R_j has the opportunity to diagonalize on A_{l+1,v_1} with m points, which is impossible.

Let $v \in [u, s)$ be a stage at which R_j diagonalizes on some $A_{n,v}$ with m points. At stage v, we ensure that $\Phi_{j,v} \upharpoonright A_{n,v}$ cannot be extended to an isomorphism of $A_{n,v+1}$ and $C_{n,v+1}$. However, $A_{n,v+1} = A_{n,s}$ and $C_{n,v+1} = C_{n,s}$, and $\Phi_{j,s} \upharpoonright A_{n,s}$ is an extension of $\Phi_{j,v} \upharpoonright A_v$ to precisely such an isomorphism. This yields the desired contradiction, with which we conclude the proof of Theorem 3.6. □

4. The Successor Relation

The *successor* (or *adjacency*) relation is central to understanding countable linear orders, both classically and effectively. For example, Hausdorff's analysis of universal (nonscattered) countable linear orders relies on his derivative operation of identifying adjacent points. Effectively, we mentioned the Remmel-Dzgoev characterization of computably categorical linear orderings in terms of their successor relation. Moses [25] showed that a computable linear ordering $\mathcal{L}$ is 1-decidable if and only if the successor relation on $\mathcal{L}$ is computable. This is one reason why the complexity of the successor relation on computable linear orderings was studied intensively, in particular in the theorem of Downey, Lempp and Wu mentioned above. Their result states that the Turing degrees of the successor relation of computable presentations of a computable order-type are closed upwards in the c.e. degrees, as long as, of course, the order-type has infinitely many adjacencies. In this section, we show that the Downey-Lempp-Wu theorem can fail for uncountable linear orderings and consider the consequences of this failure.

Recall (Definition 1.2) that for a linear order $\mathcal{L}$, the set of adjacent pairs in $\mathcal{L}$ is denoted $\mathrm{Succ}(\mathcal{L})$.

Definition 4.1. Let λ be an ω_1-computable order-type. Define

$$\mathrm{DegSpec}_{\mathrm{Succ}}(\lambda) := \{\deg_{\mathrm{T}}(\mathrm{Succ}(\mathcal{L})) \,:\, \mathcal{L} \text{ is a computable presentation of } \lambda\}\,.$$

Since the successor relation $\mathrm{Succ}(\mathcal{L})$ has a $\Pi^0_1(\mathcal{L})$-definition, for any ω_1-computable order-type λ, the set $\mathrm{DegSpec}_{\mathrm{Succ}}(\lambda)$ consists only of ω_1-c.e. degrees. We start by demonstrating that the natural analogue of the Downey, Lempp, and Wu theorem (the assumption that λ contains uncountably many adjacencies) fails in the uncountable setting. We then provide a sufficient condition for upward closure.

Example 4.2. The ω_1-computable order-type $2\cdot\rho$ has uncountably many adjacencies and satisfies $\mathrm{DegSpec}_{\mathrm{Succ}}(2\cdot\rho) = \{\mathbf{0}\}$. For let $\mathcal{L}$ be a computable presentation of $2\cdot\rho$; let $f\colon 2\cdot\mathbb{R} \to \mathcal{L}$ be an isomorphism, and let $Q := f[2\cdot\mathbb{Q}]$. Then x, y in $\mathcal{L}$ are adjacent if and only if they lie in the same Q-interval. Since we can fix Q as a countable parameter, this gives an algorithm for computing $\mathrm{Succ}(\mathcal{L})$.

Of course, the connection between Example 3.2 and Example 4.2 is immediate. The sufficient condition we offer for upwards closure (Theorem 4.4) is also related to the condition for ω_1-computable categoricity. Here, the difference is that any level of density (rather than only $\aleph_1$-saturation) suffices as the successor relation is empty within any dense interval (regardless of whether or not it is saturated). Nonetheless, again the crucial hypothesis is the existence of something like a copy of the rational numbers, the intervals relative to which behave in a uniform way. The linear ordering $2\cdot\mathbb{R}$ is "ρ-like": It contains a countable subset Q such that every Q-interval is finite.

Definition 4.3. A linear order $\mathcal{L}$ is *weakly separable* if it contains a countable subset Q such that every Q-interval is either finite or dense.

Theorem 4.4. *If λ is a computable order-type which is not weakly separable, then the spectrum $\mathrm{DegSpec}_{\mathrm{Succ}}(\lambda)$ is closed upwards in the ω_1-c.e. degrees.*

Proof. Let $\mathcal{L}$ be an ω_1-computable presentation of λ. Let W be an ω_1-c.e. set which computes $\mathrm{Succ}(\mathcal{L})$. Let $\langle W_s\rangle_{s<\omega_1}$ be an ω_1-computable enumeration of W. We build an ω_1-computable presentation $\mathcal{K} \in \lambda$ such that $\mathrm{Succ}(\mathcal{K}) \equiv_{\mathrm{T}} W$.

As in previous constructions, we let $\mathcal{L}_s := \mathcal{L} \upharpoonright s$ and build $\mathcal{K}$ as the union of an increasing, ω_1-computable sequence $\langle \mathcal{K}_s\rangle$ of countable linear orderings. To ensure that $\mathcal{K}$ is isomorphic to $\mathcal{L}$, we construct a Δ^0_2-isomorphism $F\colon \mathcal{K} \to \mathcal{L}$ as the limit of an ω_1-computable sequence of isomorphisms $F_s\colon \mathcal{K}_s \to \mathcal{L}_s$. Of course, we cannot make $\mathcal{K}$ and $\mathcal{L}$ computably isomorphic, else we would have $\mathrm{Succ}(\mathcal{K}) \equiv_{\mathrm{T}} \mathrm{Succ}(\mathcal{L})$.

To get W to compute $\mathrm{Succ}(\mathcal{K})$, we will ensure that W computes F. To get $\mathrm{Succ}(\mathcal{K})$ to compute W, we will ensure that the complement of W is c.e. in $\mathrm{Succ}(\mathcal{K})$. We define an *enumeration functional* Φ; axioms enumerated into Φ at stage s will name countably many successor pairs in $\mathcal{K}_s$, and declare that if all of these pairs are indeed successor pairs in $\mathcal{K}$, then some number x is enumerated into the $\mathrm{Succ}(\mathcal{K})$-c.e. set $\Phi(\mathrm{Succ}(\mathcal{K}))$. At stage s, we let $\Phi(\mathrm{Succ}(\mathcal{K}))[s]$ be the result of applying Φ_s, the functional as enumerated up to stage s, on the collection of adjacencies in $\mathcal{K}_s$. For all $i < \omega_1$, requirement R_i states that $i \in \Phi(\mathrm{Succ}(\mathcal{K}))$ if and only if $i \notin W$.

Informally, we describe the strategy for meeting a requirement R_i. As long as $i \notin W_s$, we take the $<_{\omega_1}$-least available successor pair (a, b) in $\mathcal{K}_s$, and with the information that $(a, b) \in \mathrm{Succ}(\mathcal{K}_s)$ we enumerate i into $\Phi(\mathrm{Succ}(\mathcal{K}))[s]$. If later we see that i enters W_t, we want to enumerate a new element into $\mathcal{K}_t$ between a and b.

We need to, in advance, pick the pair (a,b) so that adding such an element will still allow us to embed $\mathcal{K}_t$ into $\mathcal{L}_t$, possibly by changing F. Not surprisingly, the choice of (a,b) depends on whether $\mathcal{L}_s$ is scattered or nonscattered; in the scattered case, we will in fact need to use all the pairs in some infinite block.

Meanwhile, if i does not enter W, we need to maintain the adjacency of the pair (a,b). Of course $\mathcal{L}$ may force us to enumerate an element between a and b, by enumerating an element between $F_s(a)$ and $F_s(b)$. In this case, we just need to pick another pair; this will reach a limit. However, we need to actively prevent weaker requirements R_j for $j > i$ from enumerating elements between a and b. This is done by imposing restraint; weaker requirements are not allowed to change $F_s(a)$ and $F_s(b)$. This accumulated restraint gives a requirement R_i a countable set on which it is not allowed to change F; it needs to work in the intervals determined by this countable set, and find adjacencies in one of them. This is where the assumption on the structure of $\mathcal{L}$ comes into use.

Construction: For $j < \omega_1$, by recursion, we let $I_{j,s}$ be the set of stages less than s at which requirement R_j *requires attention* (as defined below). We define

$$r_{j,s} := \sup\Big\{t+1 : t \in \bigcup_{i<j} I_{i,s}\Big\}.$$

Let $s < \omega_1$, and suppose that $\mathcal{K}_s$ and F_s are recursively defined. A requirement R_j *requires attention* at stage s if $j < s$, $\Phi(\mathrm{Succ}(\mathcal{K}))(j) = W(j)\,[s]$, and there is some $\mathcal{K}_{r_{j,s}}$-interval of $\mathcal{K}_s$ which is infinite and not dense. The strongest requirement which requires attention receives it. If no requirement requires attention at stage s, or if R_j receives attention but $\Phi(\mathrm{Succ}(\mathcal{K}))(j) \neq W(j)\,[s]$, then we simply let $\mathcal{K}_{s+1}$ and F_{s+1} be extensions of $\mathcal{K}_s$ and F_s such that $F_{s+1} \colon \mathcal{K}_{s+1} \to \mathcal{L}_{s+1}$ is an isomorphism.

Otherwise, let R_j receive attention at stage s. If $j \notin W_s$, we let (S_1, S_2) be the $<_{\omega_1}$-least cut of $\mathcal{K}_{r_{j,s}}$ such that $A_s := (S_1,S_2)_{\mathcal{K}_s}$ is infinite and not dense. If A_s is scattered, let T_s be the $<_{\omega_1}$-least infinite block of A_s. If A_s is nonscattered, let T_s be the $<_{\omega_1}$-least subset $\{a,b\}$ of A_s such that a and b are adjacent in A_s. In either case, enumerate a new axiom into Φ, enumerating j into $\Phi(\mathrm{Succ}(\mathcal{K}))[s+1]$. The use of this computation is the collection of all successor pairs in $\mathcal{K}_{r_{j,s}} \cup T_s$. We again let $\mathcal{K}_{s+1}$ and F_{s+1} be extensions so that $F_{s+1} \colon \mathcal{K}_{s+1} \to \mathcal{L}_{s+1}$ is an isomorphism.

If $j \in W_s$, we need to change $\mathcal{K}_{s+1}$ to extract j from $\Phi(\mathrm{Succ}(\mathcal{K}))$. Let $t < s$ be the stage at which the computation $j \in \Phi(\mathrm{Succ}(\mathcal{K}))[s]$ was defined. (Note that at most one such computation can apply to the current oracle $\mathrm{Succ}(\mathcal{K})[s]$ at any stage.) Say $A_t = (S_1,S_2)_{\mathcal{K}_t}$. Then T_t is still a convex subset of $A_s = (S_1,S_2)_{\mathcal{K}_s}$, as otherwise j would already be extracted from $\Phi(\mathrm{Succ}(\mathcal{K}))$. We can find a self-embedding f of A_s such that for some adjacent $a,b \in T_t$, $f(a)$ and $f(b)$ are not adjacent in A_s; this is either because T_t is an infinite block of A_s, or A_s is nonscattered. As A_s is a convex subset of $\mathcal{K}_s$, we extend f to a self-embedding of $\mathcal{K}_s$ by being the identity outside A_s. We then extend $\mathcal{K}_s$ and $F_s \circ f$ to $\mathcal{K}_{s+1}$ and an isomorphism $F_{s+1} \colon \mathcal{K}_{s+1} \to \mathcal{L}_{s+1}$. This definition ensures the enumeration of some point between some successor pair of T_t, and so $j \notin \Phi(\mathrm{Succ}(\mathcal{K}))[s+1]$.

At limit stages, we define $\mathcal{K}_{<s} := \bigcup_{t<s} \mathcal{K}_t$, and define $F_{<s} := \lim_{t \to s} F_t$ to be the limit embedding of $\mathcal{K}_{<s}$ into $\mathcal{L}_s$. We then let $\mathcal{K}_s$ and F_s be an extension of $\mathcal{K}_{<s}$ and $F_{<s}$ to an isomorphism from $\mathcal{K}_s$ to $\mathcal{L}_s$. To see that $F_{<s}$ is well-defined, we use arguments analogous to arguments in the proof of Theorem 3.6, using Claim 4.4.2.

Suppose that $\langle F_t \rangle_{t<s}$ is not increasing on some final segment of s. One of two cases must hold. Suppose first that there is some limit $j \leq s$ such that for all $i < j$, $r_{i,s} < s$ but $s = \sup_{i<j} r_{i,s}$. In this case, we follow case (B) of the construction of Theorem 3.6; in this case, $F_{<s}$ is the union of maps F_t where t ranges over the Dekker nondeficiency subset of $\{r_{i,s} : i < j\}$. Otherwise, there is some $j < s$ such that $r_{j,s} < s$ but $I_{j,s}$ is unbounded in s. This case is analogous to case (C) of Theorem 3.6, $F_{<s}$ is the union of F_t for $t \in I_{j,s} \cap [r_{j,s}, s)$; the point is that since R_j requires attention at cofinally many stages before stage s, it must be that $j \notin W_s$, so at each such stage it sets $F_{t+1} \supseteq F_t$; we again appeal to Claim 4.4.2.

Verification: First, we show that restraints are respected. The following two claims are proved by simultaneous induction on s, and verify the promises made during the construction.

Claim 4.4.1. Fix $i, j, s < \omega_1$ with $i < j < s$, $i \notin W_s$, and $j \in \Phi(\mathrm{Succ}(\mathcal{K}))[s]$. Then $i \in \Phi(\mathrm{Succ}(\mathcal{K}))[s]$. Moreover, the computation $i \in \Phi(\mathrm{Succ}(\mathcal{K}))[s]$ was defined before the stage at which the computation $j \in \Phi(\mathrm{Succ}(\mathcal{K}))[s]$ was defined. It follows that the use of the computation $i \in \Phi(\mathrm{Succ}(\mathcal{K}))[s]$ is contained in the use of $j \in \Phi(\mathrm{Succ}(\mathcal{K}))[s]$.

Proof. Let $t < s$ be the stage at which the computation $j \in \Phi(\mathrm{Succ}(\mathcal{K}))[s]$ was defined. Then there is some infinite, nondense $\mathcal{K}_{r_{j,t}}$-interval of $\mathcal{K}_t$. Since $r_{j,t} \geq r_{i,t}$, there is an infinite, nondense $\mathcal{K}_{r_{i,t}}$-interval of $\mathcal{K}_t$. Since $i \notin W_t$, we can conclude that $i \in \Phi(\mathrm{Succ}(\mathcal{K}))[t]$, as otherwise R_i would require attention at stage t. Let $u < t$ be the stage at which the computation $i \in \Phi(\mathrm{Succ}(\mathcal{K}))[t]$ was defined by R_i.

Since $u \in I_{i,t}$, we have $r_{j,t} > u$. Hence, the use of the computation asserting that $j \in \Phi(\mathrm{Succ}(\mathcal{K}))[s]$ contains every successor pair in $\mathcal{K}_{u+1}$, and so every successor pair in the use of the computation $j \in \Phi(\mathrm{Succ}(\mathcal{K}))[t]$. At stage t, we let F_{t+1} extend F_t. This shows that the persistence of the computation $j \in \Phi(\mathrm{Succ}(\mathcal{K}))[t+1]$ to s also shows the persistence of the computation $i \in \Phi(\mathrm{Succ}(\mathcal{K}))[t]$ to s. □

Claim 4.4.2. For all $j, s < \omega_1$ with $j < s$, the map F_s extends the map $F_{r_{j,s}}$.

Proof. We prove this by induction on s. As we take limits at limit stages, we need only consider a successor stage s. Suppose that $r_{j,s} < s$. By induction, the map F_{s-1} extends the map F_r. If F_s extends F_{s-1} then we are done. Suppose otherwise. Some requirement R_i receives attention at stage $s-1$, and extracts i from $\Phi(\mathrm{Succ}(\mathcal{K}))[s]$. Since $r_{j,s} < s$, we must have $i \geq j$. Let $t < s-1$ be the stage at which the computation $i \in \Phi(\mathrm{Succ}(\mathcal{K}))[s-1]$ was defined. Note that we have $F_s(x) = F_{s-1}(x)$ for all $x \notin A_{s-1}$.

Since $i \geq j$, we have $r_{i,s-1} \geq r_{j,s-1} = r_{j,s}$. The persistence of the computation $i \in \Phi(\mathrm{Succ}(\mathcal{K}))[s-1]$ from stage t to stage $s-1$, and Claim 4.4.1, show that no requirement R_k stronger than R_i requires attention at any stage in $[t, s-1]$. Hence $r_{i,s-1} = r_{i,t}$. Since A_t is a $\mathcal{K}_{r_{i,t}}$-interval, it follows that A_s is an $\mathcal{K}_{r_{j,s}}$-interval. Hence A_s and $\mathcal{K}_{r_{j,s}}$ are disjoint; so F_s and F_{s-1} agree on $\mathcal{K}_{r_{j,s}}$ as required. □

The rest of the verification is straightforward. The argument defining $F_{<s}$ for a limit stage s shows that $F := F_{<\omega_1} := \lim_{s<\omega_1} F_s$ is well-defined, and is an isomorphism from $\mathcal{K} := \mathcal{K}_{<\omega_1}$ to $\mathcal{L}$.

For all $j < \omega_1$, $r_{j,\omega_1} < \omega_1$, and requirement R_j is met. This is proved by induction on j. If $r_{j,\omega_1} < \omega_1$, then we show that I_{j,ω_1} is bounded. If $j \in W$, then R_j requires attention at most once after a stage s at which $j \in W_s$; and $j \notin \Phi(\mathrm{Succ}(\mathcal{K}))$. Suppose that $j \notin W$. Let $S := \mathcal{K}_{r_{j,\omega_1}}$. Since $\mathcal{L}$ is not weakly separable, so is $\mathcal{K}$. Hence there is some S-interval of $\mathcal{K}$ which is infinite and nondense. Since S is countable, there is a stage $t \geq r_{j,\omega_1}$ after which if $(a,b) \in \mathrm{Succ}(\mathcal{K}_s)$ and $a, b \in S$ then $(a,b) \in \mathrm{Succ}(\mathcal{K})$. Let (S_1, S_2) be the $<_{\omega_1}$-least cut of S such that $(S_1,S_2)_{\mathcal{K}}$ is infinite and nondense. Then (S_1, S_2) witnesses that eventually, the requirement R_j requires attention and eventually successfully defines a computation $j \in \Phi(\mathrm{Succ}(\mathcal{K}))$. After that successful definition, the requirement R_j never requires attention.

We claim that F is computable from W. Let $x \in \mathcal{K}$. To compute $F(x)$ with oracle W, find a stage $s < \omega_1$ and an index j such that $x \in \mathcal{K}_{r_{j,s}}$, $W \restriction j = W_s \restriction j$, and $\Phi(\mathrm{Succ}(\mathcal{K}))(i) \neq W(i)\,[s]$ for all $i < j$. We claim that $F(x) = F_s(x) = F_{r_{j,s}}(x)$. This is because no requirement R_i, for $i < j$, will cause a redefinition of F_t after stage s, and so for all $t > s$, the map $F_{r_{j,t}}$ extends the map $F_{r_{j,s}}$, and so F_t extends $F_{r_{j,s}}$ (Claim 4.4.2).

Since W computes both F and $\mathrm{Succ}(\mathcal{L})$, it computes $\mathrm{Succ}(\mathcal{K})$. This completes the proof. □

In light of Example 4.2 and Theorem 4.4, it is natural to ask how badly upward closure can fail in weakly separable linear orders. We require the following definition.

Definition 4.5. Let A be an uncountable ω_1-c.e. set. If $f : \omega_1 \to A$ and $g : \omega_1 \to A$ are injective computable enumerations of A, then for all $B \subseteq A$, the sets $f^{-1}B$ and $g^{-1}B$ are Turing equivalent (indeed they are 1-1 equivalent). We thus define, for all $B \subseteq A$, $\deg_{\mathrm{T}}(B|A)$ to be the Turing degree of $f^{-1}B$, where f is any injective computable enumeration of A.

The point is that passing from B to $f^{-1}B$ erases the complexity of A. Certainly $\deg_{\mathrm{T}}(A|A) = \mathbf{0}$. For all $B \subseteq A$, $\deg_{\mathrm{T}}(B|A) \leq \deg_{\mathrm{T}}(B)$. If A is computable, then for all $B \subseteq A$, $\deg_{\mathrm{T}}(B|A) = \deg_{\mathrm{T}}(B)$.

The degree $\deg_{\mathrm{T}}(B|A)$ is the amount of information coded in B once we know that it is a subset of A. This intuition is explained as follows. For all C, $\deg_{\mathrm{T}} C \leq \deg_{\mathrm{T}}(B|A)$ if and only if there is a reduction of C to B which only queries the oracle on elements of A. Similarly, $\deg_{\mathrm{T}}(B|A) \leq \deg_{\mathrm{T}}(C)$ if and only if there is a partial reduction Φ such that for all $x \in A$, $B(x) = \Phi(C, x)$; the reduction $\Phi(C)$ may not halt on inputs outside A. This is why we informally write, for example, $C \leq_{\mathrm{T}} (B|A)$, even though there is no fixed *set* $B|A$.

This definition also works for strong reducibilities. We say that $B \leq_{\mathrm{wtt}} C$ (where B and C are subsets of ω_1) if there is a Turing functional Φ and a computable function φ such that $\Phi(C) = B$ and such that for all $x < \omega_1$, $\Phi(C \restriction \varphi(x)) \supseteq B \restriction x$. In other words, the use of the computation is bounded by φ. For $B \subseteq A$, we write $\deg_{\mathrm{wtt}}(B|A)$ for $\deg_{\mathrm{wtt}}(f^{-1}B)$, where f is any injective computable enumeration of A. Similarly, we write $\deg_{\mathrm{m}}(B|A)$ for $\deg_{\mathrm{m}}(f^{-1}B)$ for any such B. We note that neither of these depend on the choice of computable function f.

Definition 4.6. Let $\mathcal{L}$ be a computable, weakly separable linear order, witnessed by a countable subset Q of $\mathcal{L}$.

For a set $\mathfrak{C}$ of cardinals, we let $I^Q_{\mathfrak{C}}(\mathcal{L})$ be the set of cuts (Q_1, Q_2) of Q such that the size of $(Q_1, Q_2)_{\mathcal{L}}$ is in $\mathfrak{C}$.

We use obvious abbreviations: For example, we write $I^Q_\kappa(\mathcal{L})$ for $I^Q_{\{\kappa\}}(\mathcal{L})$, $I^Q_{>\kappa}(\mathcal{L})$ for $I^Q_{(\kappa,\aleph_1]}(\mathcal{L})$, and $I^Q_\infty(\mathcal{L})$ for $I^Q_{\{\aleph_0,\aleph_1\}}(\mathcal{L})$.

Lemma 4.7. *Let λ be a computable, weakly separable order-type. Let $\mathfrak{C}$ be a set of finite cardinals. Then $\deg_{\mathrm{m}}(I^Q_{\mathfrak{C}}(\mathcal{L}))$ does not depend on the choice of the computable presentation $\mathcal{L}$ of λ and the countable subset Q of $\mathcal{L}$ witnessing that $\mathcal{L}$ is weakly separable.*

Proof. If $\mathcal{L}$ and $\mathcal{K}$ are computable presentations of λ, $F\colon \mathcal{L} \to \mathcal{K}$ is an isomorphism, and Q witnesses that $\mathcal{L}$ is weakly separable, then $I^Q_{\mathfrak{C}}(\mathcal{L})$ and $I^{F[Q]}_{\mathfrak{C}}(\mathcal{K})$ are 1-1 equivalent. Here F need not be computable, since we only use the countable parameter $F \upharpoonright Q$.

Hence it suffices to fix a computable presentation $\mathcal{L}$ of λ and show that if S and Q both witness that $\mathcal{L}$ is weakly separable, then $I^Q_{\mathfrak{C}}(\mathcal{L}) \leq_{\mathrm{m}} I^S_{\mathfrak{C}}(\mathcal{L})$.

Fix such Q and S. If $I^Q_{\mathfrak{C}}(\mathcal{L})$ is computable, then there is nothing to show. Thus we may assume that $\mathcal{L}$ contains uncountably many maximal finite blocks. Since Q and S are countable, they intersect only countably many maximal (finite) blocks of $\mathcal{L}$. Since infinite S- and Q-intervals of $\mathcal{L}$ are dense, it follows that for all but countably many cuts (Q_1, Q_2) of Q, if the interval $(Q_1, Q_2)_{\mathcal{L}}$ is finite, then it is a maximal block of $\mathcal{L}$, and must be an S-interval as well.

So outside a countable set of cuts, given a cut (Q_1, Q_2) of Q, we search for either a cut (S_1, S_2) such that $(Q_1, Q_2)_{\mathcal{L}} = (S_1, S_2)_{\mathcal{L}}$, or a stage at which we see that $(Q_1, Q_2)_{\mathcal{L}}$ is infinite. In the former case, we, of course, know that $(Q_1, Q_2) \in I^Q_{\mathfrak{C}}(\mathcal{L})$ if and only if $(S_1, S_2) \in I^S_{\mathfrak{C}}(\mathcal{L})$. In the latter case, we know without consulting the oracle that $(Q_1, Q_2) \notin I^Q_{\mathfrak{C}}(\mathcal{L})$. □

Definition 4.8. Let λ be a computable weakly separable order-type with uncountably many adjacencies. For any computable presentation $\mathcal{L}$ of λ and any set $Q \subseteq \mathcal{L}$ witnessing that $\mathcal{L}$ is weakly separable, the set $I^Q_{>1}(\mathcal{L})$ must be uncountable. Certainly $I^Q_{>1}(\mathcal{L})$ is c.e. By Lemma 4.7, the degrees

$$\deg_{\mathrm{T}}\left(I^Q_\infty(\mathcal{L}) \,\middle|\, I^Q_{>1}(\mathcal{L})\right)$$

and

$$\bigvee_{n\geq 2} \deg_{\mathrm{T}}\left(I^Q_n(\mathcal{L}) \,\middle|\, I^Q_{>1}(\mathcal{L})\right)$$

do not depend on the choice of $\mathcal{L}$ and Q, and so we respectively name them $\mathbf{min}(\lambda)$ and $\mathbf{max}(\lambda)$.

The set $I^Q_\infty(\mathcal{L})$ is c.e., and so $\mathbf{min}(\lambda)$ is a c.e. degree. It is not immediately clear, but we will see that $\mathbf{max}(\lambda)$ is also a c.e. degree.

We also let

$$\mathbf{min}_{\mathrm{wtt}}(\lambda) := \deg_{\mathrm{wtt}}\left(I^Q_\infty(\mathcal{L}) \,\middle|\, I^Q_{>1}(\mathcal{L})\right)$$

and

$$\mathbf{max}_{\mathrm{wtt}}(\lambda) := \bigvee_{n\geq 2} \deg_{\mathrm{wtt}}\left(I^Q_n(\mathcal{L}) \,\middle|\, I^Q_{>1}(\mathcal{L})\right);$$

again, Lemma 4.7 shows this does not depend on the choice of $\mathcal{L}$ and Q.

As we shall immediately see, the degrees $\mathbf{min}(\lambda)$ and $\mathbf{max}(\lambda)$ constrain the degree spectrum $\mathrm{DegSpec}_{\mathrm{Succ}}(\lambda)$. This explains why they are both defined inside $I^Q_{>1}(\mathcal{L})$: In measuring the complexity of $\mathrm{Succ}(\mathcal{L})$, we need to avoid the false complexity that can be added by the set of intervals containing fewer than two points. Of course, such intervals cannot add complexity to the successor relation.

Theorem 4.9. *Let λ be a computable, weakly separable order-type with uncountably many adjacencies. Then* $\mathrm{DegSpec}_{\mathrm{Succ}}(\lambda)$ *is contained in the interval of degrees* $[\mathbf{min}(\lambda), \mathbf{max}(\lambda)]$.

In fact, for every computable presentation $\mathcal{L}$ of λ, $\mathrm{Succ}(\mathcal{L}) \leq_{\mathrm{wtt}} \mathbf{max}_{\mathrm{wtt}}(\lambda)$.

Proof. Let $\mathcal{L}$ be a computable presentation of λ; let Q witness that $\mathcal{L}$ is weakly separable. We need to show that $\mathrm{Succ}(\mathcal{L}) \geq_{\mathrm{T}} \left(I^Q_\infty(\mathcal{L}) \,|\, I^Q_{>1}(\mathcal{L})\right)$ and that $\mathrm{Succ}(\mathcal{L}) \leq_{\mathrm{wtt}} \bigoplus_{n\geq 2} \left(I^Q_n(\mathcal{L}) \,|\, I^Q_{>1}(\mathcal{L})\right)$.

Since $I^Q_\infty(\mathcal{L})$ is c.e., to compute it from $\mathrm{Succ}(\mathcal{L})$ inside $I^Q_{>1}(\mathcal{L})$ it is sufficient to enumerate its complement inside $I^Q_{>1}(\mathcal{L})$, i.e., to enumerate the set $\bigcup_{n\geq 2} I^Q_n(\mathcal{L})$ with oracle $\mathrm{Succ}(\mathcal{L})$. To do so, given some cut (Q_1, Q_2) such that the interval $(Q_1, Q_2)_{\mathcal{L}}$ contains at least two points, we enumerate (Q_1, Q_2) if we find some pair (a, b) in $\mathrm{Succ}(\mathcal{L})$ with $a, b \in (Q_1, Q_2)_{\mathcal{L}}$; the point, of course, is that the interval is infinite if and only if it dense. Note that the use of this enumeration may not be bounded by a computable function, as the $<_{\omega_1}$-least successor pair in a finite interval $(Q_1, Q_2)_{\mathcal{L}}$ may appear much later than the cut (Q_1, Q_2).

For the second reduction, we first note that

$$\left(I^Q_\infty(\mathcal{L}) | I^Q_{>1}(\mathcal{L})\right) \leq_{\mathrm{wtt}} \bigoplus_{n\geq 2} \left(I^Q_n(\mathcal{K}) | I^Q_{>1}(\mathcal{L})\right),$$

which is, of course, necessary for the theorem. This is because inside $I^Q_{>1}(\mathcal{L})$, $I^Q_\infty(\mathcal{L})$ and $\bigcup_{n\geq 2} I^Q_n(\mathcal{L})$ are complements. In other words, given $(Q_1, Q_2) \in I^Q_{>1}(\mathcal{L})$, we need only countably many queries to $\bigoplus_{n\geq 2} I^Q_n(\mathcal{L})$ to decide whether $(Q_1, Q_2)_{\mathcal{L}}$ is infinite, and these queries are bounded computably in (Q_1, Q_2) as the pairing function is computable.

We compute $\mathrm{Succ}(\mathcal{L})$ from $\bigoplus_{n\geq 2} \left(I^Q_n(\mathcal{L}) \,|\, I^Q_{>1}(\mathcal{L})\right)$. Let $a <_{\mathcal{L}} b$ be elements of $\mathcal{L}$; we want to decide if $(a, b) \in \mathrm{Succ}(\mathcal{L})$. We may assume that $a, b \notin Q$. This is because $\mathrm{Succ}(\mathcal{L}) \cap ((Q \times \mathcal{L}) \cup (\mathcal{L} \times Q))$ is countable, as Q is countable and every element of Q has at most one successor and one predecessor.

We first decide if a and b are in the same Q-interval; if not, then $(a, b) \notin \mathrm{Succ}(\mathcal{L})$. If so, let (Q_1, Q_2) be the cut of Q such that $a, b \in (Q_1, Q_2)_{\mathcal{K}}$. Then $(Q_1, Q_2) \in I^Q_{>1}(\mathcal{L})$. We may therefore ask the oracle if the interval $(Q_1, Q_2)_{\mathcal{L}}$ is finite. If not, then it is dense, and so $(a, b) \notin \mathrm{Succ}(\mathcal{L})$. If so, the oracle gives us the size n of $(Q_1, Q_2)_{\mathcal{K}}$. We wait for a stage s such that $(Q_1, Q_2)_{\mathcal{K}\restriction s}$ already contains n points; then $(a, b) \in \mathrm{Succ}(\mathcal{L})$ if and only if a and b are adjacent in $\mathcal{L} \restriction s$.

The use of this computation is bounded by a computable function because the cut (Q_1, Q_2) is obtained effectively from a and b. □

Having shown that the complexity of the successor relation is bounded within an interval, we turn to seeing which degrees in this interval belong to the spectrum of the successor relation. We first show that both endpoints always belong to the spectrum.

Theorem 4.10. *Let λ be a computable, weakly separable order-type with uncountably many adjacencies. Then* $\mathbf{max}(\lambda) \in \mathrm{DegSpec}_{\mathrm{Succ}}(\lambda)$.

In particular, the degree $\mathbf{max}(\lambda)$ is c.e.

Proof. Let $\mathcal{L}$ be a computable presentation of λ, and let Q witness that $\mathcal{L}$ is weakly separable. We build a computable copy $\mathcal{K}$ of $\mathcal{L}$ and an isomorphism $F\colon \mathcal{K} \to \mathcal{L}$ such that $F^{-1}Q = Q$, and such that for all $n \geq 2$, $(I^Q_n(\mathcal{K}) \,|\, I^Q_{>1}(\mathcal{K})) \leq_{\mathrm{T}} \mathrm{Succ}(\mathcal{K})$. By Theorem 4.9 this is sufficient. Note that uniformity in n is free in ω_1-computability, but is anyway obvious from the proof.

By regularity of ω_1, let $\langle \mathcal{L}_s \rangle$ be a continuous, computable and increasing sequence of countable linear orderings such that $\mathcal{L} = \bigcup_s \mathcal{L}_s$, such that $\mathcal{L}_0 = Q$, and such that for all s, every Q-interval of $\mathcal{L}_s$ is either finite or dense. We define $\mathcal{K}$ as the union of a computable and increasing sequence $\langle \mathcal{K}_s \rangle$; for all s, we define an isomorphism $F_s\colon \mathcal{K}_s \to \mathcal{L}_s$. We start with $\mathcal{K}_0 = \mathcal{L}_0 = Q$ and $F_0 = \mathrm{id}_Q$. For all s, F_s will extend F_0, so to define $\mathcal{K}_s$ and F_s, it is sufficient, given a nonempty Q-interval $B_s = (Q_1, Q_2)_{\mathcal{L}_s}$ of $\mathcal{L}_s$, to define $A_s = (Q_1, Q_2)_{\mathcal{K}_s}$ and the isomorphism $F_s \restriction A_s$ from A_s to B_s.

The idea for coding $I^Q_n(\mathcal{L})$ for each $n \geq 2$ into $\mathrm{Succ}(\mathcal{K})$ is by copying $\mathcal{L}$, but whenever we extend a finite Q-interval A_s to a larger A_{s+1}, we insert new points so that we destroy at least one adjacency in A_s. This way, $\mathrm{Succ}(\mathcal{K})$ can keep track of the size of $(Q_1, Q_2)_{\mathcal{L}}$.

So the instructions are simple. At stage s, given $\mathcal{K}_s$ and F_s, fix a cut (Q_1, Q_2) of Q such that $B_{s+1} = (Q_1, Q_2)_{\mathcal{L}_{s+1}}$ is nonempty. Suppose that $B_{s+1} \neq B_s$ (where, of course, $B_s = (Q_1, Q_2)_{\mathcal{L}_s}$), that B_{s+1} is finite and that $A_s = (Q_1, Q_2)_{\mathcal{K}_s}$ contains at least two points. We then define A_{s+1} extending A_s which has the same size as B_{s+1}, but such that some $a, b \in A_s$ which are adjacent in A_s are no longer adjacent in A_{s+1}. We then let $F_{s+1} \restriction A_{s+1}$ be the unique isomorphism from A_{s+1} to B_{s+1}.

In all other cases (if $B_{s+1} = B_s$, or $|A_s| \leq 1$, or B_{s+1} is infinite), we let $F_{s+1} \restriction A_{s+1}$ be an extension of $F_s \restriction A_s$ to an isomorphism from A_{s+1} to B_{s+1}, and, of course, define A_{s+1} accordingly.

At a limit stage s, let $\mathcal{K}_{<s} = \bigcup_{t<s} \mathcal{K}_t$. Let $B_s = (Q_1, Q_2)_{\mathcal{L}_s}$ be a nonempty Q-interval of $\mathcal{L}_s$; let $A_{<s} = (Q_1, Q_2)_{\mathcal{K}_{<s}} = \bigcup_{t<s} A_t$, where, of course, $A_t = (Q_1, Q_2)_{\mathcal{K}_t}$. We define an embedding $F_{<s} \restriction A_{<s}$ from $A_{<s}$ to B_s, and then extend it to an isomorphism $F_s \restriction A_s$ from A_s to B_s by adding points to $A_{<s}$. If $\langle F_t \restriction A_t \rangle_{t<s}$ is increasing on some final segment of s, then we let $F_{<s} \restriction A_{<s}$ be the limit of these maps. Otherwise, since $F_t \restriction A_t$ only changes when $B_{t+1} \neq B_t$, we see that B_s is infinite, and so dense, so we let $F_{<s}$ be any embedding of $A_{<s}$ into B_s.

This defines $\mathcal{K}$. We argue that $F = \lim_s F_s$ is an isomorphism from $\mathcal{K}$ to $\mathcal{L}$. This is because for every Q-interval A_{ω_1} of $\mathcal{K}$, the sequence $\langle F_s \restriction A_s \rangle$ is eventually increasing. For either A_{ω_1} is finite, in which case eventually the sequence stabilizes; or eventually A_s is infinite, after which the sequence is increasing.

Now let $n \geq 2$; we see how to compute $I^Q_n(\mathcal{K}) | I^Q_{>1}(\mathcal{K})$ from $\mathrm{Succ}(\mathcal{K})$. Let (Q_1, Q_2) be a cut of Q, and suppose that $A_{\omega_1} = (Q_1, Q_2)_{\mathcal{K}}$ contains at least two points. With oracle $\mathrm{Succ}(\mathcal{K})$ we can find a stage s such that either $A_s = (Q_1, Q_2)_{\mathcal{K}_s}$ is infinite, or A_s is finite, contains at least two points, and every adjacency in A_s is an adjacency in $\mathcal{K}$. The construction ensures that if A_s is finite then $A_s = A_{\omega_1}$, so we can compute the size of A_{ω_1}.

Again we emphasize the need to work within $I^Q_{>1}(\mathcal{L})$. The procedure above will not halt if we start with a cut (Q_1, Q_2) such that $(Q_1, Q_2)_{\mathcal{L}}$ contains at most one point. This is why $\deg_{\mathrm{T}}(\mathrm{Succ}(\mathcal{K}))$ lies above each $\deg_{\mathrm{T}}(I_n^Q(\mathcal{K}) \,|\, I_{>1}^Q(\mathcal{K}))$, and not necessarily above $\deg_{\mathrm{T}}(I_n^Q(\mathcal{K}))$. □

Note that the use of the reduction of $(I_n^Q(\mathcal{K}) \,|\, I_{>1}^Q(\mathcal{K}))$ to $\mathrm{Succ}(\mathcal{K})$ is not necessarily computably bounded. We do not know if there is always a computable presentation $\mathcal{L}$ of λ such that $\mathrm{Succ}(\mathcal{L}) \in \mathbf{max}_{\mathrm{wtt}}(\lambda)$.

Theorem 4.11. *Let λ be a computable, weakly separable order-type with uncountably many adjacencies. Then $\mathbf{min}(\lambda) \in \mathrm{DegSpec}_{\mathrm{Succ}}(\lambda)$.*

In fact, we can build a computable presentation $\mathcal{L}$ of λ such that $\mathrm{Succ}(\mathcal{L}) \in \mathbf{min}_{\mathrm{wtt}}(\lambda)$.

Proof. The construction is the opposite of that of Theorem 4.10. We fix $\langle \mathcal{L}_s \rangle$ and build $\langle \mathcal{K}_s \rangle$ and $\langle F_s \rangle$ as before, but in this construction we preserve adjacencies in finite Q-intervals. So the construction is identical to that of the previous proposition, but when extending A_s to A_{s+1} in the case that A_s contains at least two points and B_{s+1} is finite, we make sure to define A_{s+1} so that every adjacency in A_s is still an adjacency in A_{s+1} (by say enumerating all new points in A_{s+1} to the right of A_s). This too may require changing the value of F on A_s, as some adjacencies in B_s may no longer be adjacencies in B_{s+1}.

Given $a <_{\mathcal{K}} b$, we want to decide, with oracle $I_\infty^Q(\mathcal{K}) | I_{>1}^Q(\mathcal{K})$, whether $(a, b) \in \mathrm{Succ}(\mathcal{K})$. As in the proof of Theorem 4.9, we may assume that $a, b \notin Q$, and that a and b lie in the same Q-interval $(Q_1, Q_2)_{\mathcal{K}}$. We know that this interval contains at least two points, so we can ask the oracle if this interval is infinite or not. If it is infinite, then it is dense, so $(a, b) \notin \mathrm{Succ}(\mathcal{K})$. If it is finite, then $(a, b) \in \mathrm{Succ}(\mathcal{K})$ if and only if $(a, b) \in \mathrm{Succ}(\mathcal{K}_s)$ where s is any stage such that $a, b \in \mathcal{K}_s$. This has bounded use since Q_1 and Q_2 can be effectively determined from a and b.

For the other direction, we modify slightly the algorithm given in the proof of Theorem 4.9. Given some cut (Q_1, Q_2), we wait until the first stage s such that $|(Q_1, Q_2)_{\mathcal{K}}| > 1$. Then if for some $a, b \in (Q_1, Q_2)_{\mathcal{K}_s}$, $(a, b) \in \mathrm{Succ}(\mathcal{K})$, we know $(Q_1, Q_2) \notin I_\infty^Q(\mathcal{K})$. Otherwise, we know $(Q_1, Q_2) \in I_\infty^Q(\mathcal{K})$.

Note that for any $(Q_1, Q_2) \in I_{>1}^Q(\mathcal{K})$, this algorithm will halt. Furthermore, the use is given by the (partial) function $(Q_1, Q_2) \mapsto ((Q_1, Q_2)_{\mathcal{K}_s})^2$, where s is least such that $(Q_1, Q_2)_{\mathcal{K}_s}$ is non-empty and size greater than one. Note that this function is computable, and converges on all $(Q_1, Q_2) \in I_{>1}^Q(\mathcal{K})$, which is all that is required to establish $(I_\infty^Q(\mathcal{K}) | I_{>1}^Q(\mathcal{K})) \leq_{\mathrm{wtt}} \mathrm{Succ}(\mathcal{K})$. □

We note that for $\lambda = 2 \cdot \rho$ (Example 4.2), $\mathbf{max}(\lambda) = \mathbf{min}(\lambda) = \mathbf{0}$, which explains why $\mathrm{DegSpec}_{\mathrm{Succ}}(\lambda) = \{\mathbf{0}\}$. We generalize this example.

Proposition 4.12. *If $\mathbf{a}$, $\mathbf{b}$ are c.e. degrees and $\mathbf{a} \leq \mathbf{b}$, then there is a computable weakly separable order-type λ with uncountably many adjacencies such that $\mathbf{min}(\lambda) = \mathbf{a}$ and $\mathbf{max}(\lambda) = \mathbf{b}$.*

Proof. Let $A \in \mathbf{a}$ and $B \in \mathbf{b}$ be c.e., disjoint subsets of the collection of cuts of the rationals $\mathbb{Q}$. Define a computable linear order $\mathcal{L}$ by starting with $\mathbb{Q}$, and defining $(Q_1, Q_2)_{\mathcal{L}}$ for every cut (Q_1, Q_2) of $\mathbb{Q}$:

$$(Q_1, Q_2)_{\mathcal{L}} \cong \begin{cases} \mathbb{Q}, & \text{if } (Q_1, Q_2) \in A; \\ 3, & \text{if } (Q_1, Q_2) \in B; \textit{and} \\ 2, & \text{if } (Q_1, Q_2) \notin A \cup B. \end{cases}$$

Then $I^{\mathbb{Q}}_{>1}(\mathcal{L})$ is computable, $I^{\mathbb{Q}}_{\infty}(\mathcal{L}) = A$, and

$$\bigoplus_{n\geq 2} I^{\mathbb{Q}}_n(\mathcal{L}) \equiv_{\mathrm{T}} B \oplus (\omega_1 \setminus (A \cup B)) \equiv_{\mathrm{T}} B. \qquad \square$$

Corollary 4.13. *For every c.e. degree* $\mathbf{d}$ *there is a computable order-type such that* $\mathrm{DegSpec}_{\mathrm{Succ}}(\lambda) = \{\mathbf{d}\}$.

We note that Corollary 4.13 fails for ω-computability: By the Downey-Lempp-Wu theorem, if λ is an ω-computable order-type and $\mathrm{DegSpec}_{\mathrm{Succ}}(\lambda)$ is a singleton, then it must be $\{\mathbf{0}'\}$. Downey and Moses [7] constructed an ω-computable order-type such that $\mathrm{DegSpec}_{\mathrm{Succ}}(\lambda) = \{\mathbf{0}'\}$ (a computable linear ordering with an *intrinsically complete* successor relation). Their construction is much more difficult than ours.

We turn to investigate how many of the intermediate degrees in the interval $[\mathbf{min}(\lambda), \mathbf{max}(\lambda)]$ must be contained in $\mathrm{DegSpec}_{\mathrm{Succ}}(\lambda)$.

Theorem 4.14. *There is a computable, weakly separable order-type* λ *with uncountably many adjacencies such that* $\mathrm{DegSpec}_{\mathrm{Succ}}(\lambda) \neq [\mathbf{min}(\lambda), \mathbf{max}(\lambda)]$.

Furthermore, the example we give is not an immediate corollary of the degree-theoretic properties of the interval $[\mathbf{min}(\lambda), \mathbf{max}(\lambda)]$. Let λ be a computable, weakly separable order-type with uncountably many adjacencies. If $\mathbf{m}$ in the interval $[\mathbf{min}(\lambda), \mathbf{max}(\lambda)]$ is a degree such that no c.e. element M of $\mathbf{m}$ is weak truth-table reducible to $\mathbf{max}_{\mathrm{wtt}}(\lambda)$, then by Theorem 4.9, $\mathbf{m} \notin \mathrm{DegSpec}_{\mathrm{Succ}}(\mathcal{L})$. In the proof below, we construct λ and $\mathbf{m} \in [\mathbf{min}(\lambda), \mathbf{max}(\lambda)] \setminus \mathrm{DegSpec}_{\mathrm{Succ}}(\mathcal{L})$ such that there is a c.e. set $M \in \mathbf{m}$ which is weak truth-table reducible to $\mathbf{max}_{\mathrm{wtt}}(\lambda)$.

Proof. We build a computable linear ordering $\mathcal{L}$ by starting with $\mathbb{Q}$ and inserting either two or three points into every cut of $\mathbb{Q}$. This means that every cut of $\mathbb{Q}$ is in $I^{\mathbb{Q}}_{>1}(\mathcal{L})$, so $I^{\mathbb{Q}}_{>1}(\mathcal{L})$ is computable. Also $I^{\mathbb{Q}}_{\infty}(\mathcal{L})$ is empty. So $\mathbf{min}(\lambda) = \mathbf{0}$, and $\mathbf{max}_{\mathrm{wtt}}(\lambda) = \deg_{\mathrm{wtt}}(I^{\mathbb{Q}}_3(\mathcal{L}))$.

Hence, it is sufficient to build $\mathcal{L}$ and a c.e. set M such that $M \leq_{\mathrm{wtt}} I^{\mathbb{Q}}_3(\mathcal{L})$, but $\deg_{\mathrm{T}}(M) \notin \mathrm{DegSpec}_{\mathrm{Succ}}(\mathcal{L})$. We build $\mathcal{L}$ by enumerating $I^{\mathbb{Q}}_3(\mathcal{L})$. That is, we enumerate a c.e. set P of cuts of $\mathbb{Q}$. The order $\mathcal{L}$ is then defined by letting, for any cut (Q_1, Q_2) of $\mathbb{Q}$, $(Q_1, Q_2)_{\mathcal{L}}$ contain two points if $(Q_1, Q_2) \notin P$, and three points if $(Q_1, Q_2) \in P$; whence $P = I^{\mathbb{Q}}_3(\mathcal{L})$.

We can effectively list all "partial" computable orderings, that is, computable linear orders of c.e. domains. We use this to get a list $\langle \mathcal{A}_i, \Phi_i, \Psi_i, \pi_i \rangle$ of all quadruples consisting of a partial computable linear order, two Turing functionals, and an injective countable function π whose domain is $\mathbb{Q}$. The intended oracle of Ψ_i is $\mathrm{Succ}(\mathcal{A}_i)$; we require that any query Ψ_i makes to the oracle does not mention pairs involving elements in the range of π_i.

For all $i < \omega_1$, the requirement R_i states that one of three outcomes must happen:

(a) There is no isomorphism from $\mathcal{L}$ to $\mathcal{A}_i$ extending π_i.
(b) $\Phi_i(M) \neq \mathrm{Succ}(\mathcal{A}_i)$.
(c) $M \neq \Psi_i(\mathrm{Succ}(\mathcal{A}_i))$.

If every requirement R_i is met, then $\deg_{\mathrm{T}}(M) \notin \mathrm{DegSpec}_{\mathrm{Succ}}(\mathcal{L})$. For suppose that $\mathcal{A}$ is a computable copy of $\mathcal{L}$, and that $\mathrm{Succ}(\mathcal{A}) \equiv_{\mathrm{T}} M$. Let $F\colon \mathcal{L} \to \mathcal{A}$ be an isomorphism. The point is that there is a reduction of M to $\mathrm{Succ}(\mathcal{A})$ which does not query any pairs containing elements of $F \restriction \mathbb{Q}$, as there are only countably many such pairs. This shows that there is some i for which R_i fails.

The construction is a priority argument. A requirement R_i may be assigned a witness – a cut $(Q_1(i), Q_2(i))$ of $\mathbb{Q}$ – to work with. If a requirement R_i receives attention at stage s, then the witnesses $(Q_1(j), Q_2(j))$ for $j > i$ are all canceled, and will need to be later redefined (with large value). In this way, the requirement R_i imposes restraint on weaker requirements R_j. If not reset by stronger requirements, the witness persists to the next stage and across limit stages. A requirement R_i may also appoint a *follower* $m(i)$, targeted for M; the same rules apply.

A requirement R_i only requires attention at stage s if range $\pi_i \subseteq \mathcal{A}_{i,s}$, π_i is an embedding of $\mathbb{Q}$ into $\mathcal{A}_i$, and for all cuts $(Q_1, Q_2) <_{\omega_1} s$ of $\mathbb{Q}$, $(\pi_i[Q_1], \pi_i[Q_2])_{\mathcal{A}_{i,s}}$ contains two points if $(Q_1, Q_2) \notin P_s$, and three points if $(Q_1, Q_2) \in P_s$. We say that $\mathcal{A}_i$ *appears correct* at stage s. The point, of course, is that if F is an isomorphism from $\mathcal{L}$ to $\mathcal{A}_i$ which extends π_i, then for all cuts (Q_1, Q_2) of $\mathbb{Q}$, $F(Q_1, Q_2)_{\mathcal{L}} = (\pi_i[Q_1], \pi_i[Q_2])_{\mathcal{A}_i}$, and so the latter contains two points if $(Q_1, Q_2) \notin P$, and three otherwise.

If the witness $(Q_1(i), Q_2(i))$ is defined at stage s, and $\mathcal{A}_i$ appears correct at stage s, then the interval $(\pi_i[Q_1(i)], \pi_i[Q_2(i)])_{\mathcal{A}_i}$ contains at least two points; we let $a(i)$ and $b(i)$ be the two points which are first enumerated in this interval.

A requirement R_i *requires attention* at stage s if $\mathcal{A}_i$ appears correct at stage s, and one of the following hold:

(1) A witness $(Q_1(i), Q_2(i))$ is not defined at stage s.
(2) A witness $(Q_1(i), Q_2(i))$ is defined, and s is the least stage at which we observe a computation $\Phi_i(M, (a(i), b(i)))\!\downarrow = 1\,[s]$.
(3) A follower $m(i)$ is defined, and s is the least stage at which we observe a computation $\Psi_i(\mathrm{Succ}(\mathcal{A}_i), m(i))\!\downarrow = 0\,[s]$.

At stage s give attention to the strongest requirement asking for it. Say R_i receives attention at stage s. In case (1), we define a new witness $(Q_1(i), Q_2(i))$ with large value. In case (2), we appoint a new follower $m(i)$ with large value. In case (3), we enumerate $m(i)$ into M_{s+1} so as not to break the existing adjacency, and $(Q_1(i), Q_2(i))$ into P_{s+1}. This construction defines M and P, and so defines $\mathcal{L}$.

We first show that $M \leq_{\mathrm{wtt}} I_3^{\mathbb{Q}}(\mathcal{L})$. For $x \in M$ only if x is chosen as a follower for some requirement by stage x. If x is a follower for R_i at stage x, then $x \in M$ if and only if the interval $(Q_1(i), Q_2(i))_{\mathcal{L}}$ contains three points. The cut $(Q_1(i), Q_2(i))$ is obtained effectively from x, and so the use of this reduction is computably bounded. We note that this reduction is the only driver for making intervals of size 3; the requirements R_i would be easily met if every $\mathbb{Q}$-interval has two elements, making $\mathrm{Succ}(\mathcal{L})$ intrinsically computable and making M noncomputable.

Finally, we see that every requirement is met. An inductive "countably injury" argument shows that for every $i < \omega_1$, R_i is only reset countably many times. For if R_i is never injured after stage s, then R_i receives attention at most three times after stage s, possibly once at step (1), maybe later at step (2), and then maybe later at step (3).

Fix $i < \omega_1$; we show that the requirement R_i is met. Let r^* be a stage after which R_i is never reset. Suppose that there is an isomorphism F from $\mathcal{L}$ to $\mathcal{A}_i$ extending π_i, so R_i is not satisfied by clause (a) above. The regularity of ω_1 shows that the set of stages at which $\mathcal{A}_i$ looks correct is closed and unbounded in ω_1.

This means that there is some stage $s > r^*$ at which R_i receives attention by step (1), appointing a witness cut $(Q_1(i), Q_2(i))$. This witness is never canceled. Since F extends π_i, the interval $(\pi_i[Q_1(i)], \pi_i[Q_2(i)])_{\mathcal{A}_i}$ has the same number of points as the interval $(Q_1, Q_2)_{\mathcal{L}}$, namely three if R_i ever reaches step (3) after stage r^*, and two otherwise. Let $a(i)$ and $b(i)$ be the two points which are enumerated earliest into $(\pi_i[Q_1(i)], \pi_i[Q_2(i)])_{\mathcal{A}_i}$.

If R_i never reaches step (2) after stage r^*, then $(a(i), b(i)) \in \mathrm{Succ}(\mathcal{A}_i)$, but it is not the case that $\Phi_i(M, (a(i), b(i))) = 1$. In this case, R_i is satisfied by clause (b) above. Suppose that R_i reaches step (2) at some stage $s' > r^*$. The resetting of weaker requirements at stage s', the fact that $s' > r^*$, and the fact that at step (2), $m(i)$ is chosen to be large, show that $\Phi_i(M, (a(i), b(i))) = 1$.

At step (2), R_i appoints a follower $m(i)$ which is never canceled. If R_i never reaches step (3) after that, then $m(i) \notin M$ but $\Psi_i(\mathrm{Succ}(\mathcal{A}_i), m(i)) \neq 0$, so R_i is satisfied by clause (c) above. Suppose that R_i reaches step (3) at some stage $t > r^*$. Then $m(i) \in M$; we argue that $\Psi_i(\mathrm{Succ}(\mathcal{A}_i), m(i)) = 0$, which would mean that R_i is satisfied by clause (c).

At stage t, $\Psi_i(\mathrm{Succ}(\mathcal{A}_i), m(i)) = 0$. We show that $\mathrm{Succ}(\mathcal{A}_i)$ and $\mathrm{Succ}(\mathcal{A}_{i,t})$ agree on the use of the computation at stage t. Since Ψ_i does not query pairs involving elements of $\pi_i[\mathbb{Q}]$, and since pairs of elements from distinct $\pi_i[\mathbb{Q}]$-intervals of $\mathcal{A}_i$ are not successor pairs in either $\mathcal{A}_i$ or $\mathcal{A}_{i,t}$, it suffices to show that for all $\mathbb{Q}$-cuts $(Q_1, Q_2) <_{\omega_1} t$, for all $a <_{\mathcal{A}_i} b$ in $(\pi_i[Q_1], \pi_i[Q_2])_{\mathcal{A}_{i,t}}$, $(a, b) \in \mathrm{Succ}(\mathcal{A}_{i,t})$ if and only if $(a, b) \in \mathrm{Succ}(\mathcal{A}_i)$.

Let $(Q_1, Q_2) <_{\omega_1} t$ be a cut of $\mathbb{Q}$, and let $a <_{\mathcal{A}_i} b$ be elements of the interval $(\pi_i[Q_1], \pi_i[Q_2])_{\mathcal{A}_{i,t}}$. There are two cases. If $(Q_1, Q_2) \neq (Q_1(i), Q_2(i))$, then the fact that $t > r^*$, and the fact that R_i resets weaker requirements at stage s (and later these requirements choose large witnesses) means that $(Q_1, Q_2) \in P$ if and only if $(Q_1, Q_2) \in P_t$. At stage t, $\mathcal{A}_i$ appears correct, so $(\pi_i[Q_1], \pi_i[Q_2])_{\mathcal{A}_{i,t}}$ contains three points if and only if $(Q_1, Q_2) \in P_t$; and since F extends π_i, the interval $(\pi_i[Q_1], \pi_i[Q_2])_{\mathcal{A}_i}$ contains three points if and only if $(Q_1, Q_2) \in P$. It follows that $(\pi_i[Q_1], \pi_i[Q_2])_{\mathcal{A}_i,t} = (\pi_i[Q_1], \pi_i[Q_2])_{\mathcal{A}_i}$, so $\mathrm{Succ}(\mathcal{A}_i)$ cannot change on (a, b) after stage t.

If $(Q_1, Q_2) = (Q_1(i), Q_2(i))$, then as $(Q_1, Q_2) \notin P_t$, we must have $a = a(i)$ and $b = b(i)$. We have $(a(i), b(i)) \in \mathrm{Succ}(\mathcal{A}_{i,t})$, and by assumption, $(a(i), b(i)) \in \mathrm{Succ}(\mathcal{A}_i)$. In other words, the third point enumerated into $(\pi_i[Q_1], \pi_i[Q_2])_{\mathcal{A}_i}$ after stage t does not break the adjacency $(a(i), b(i))$, or otherwise R_i is already satisfied by clause (b) as explained above. □

At the opposite extreme, there is a computable linear order $\mathcal{L}$ such that the degree spectrum of its successor relation contains every ω_1-c.e. degree. This follows from Theorem 4.4, applying it to any computable linear ordering $\mathcal{L}$ which is not weakly separable but such that $\mathrm{Succ}(\mathcal{L})$ is computable; an example for such an ordering is $(\omega_1, <)$. Here, we show that the example can be weakly separable.

Theorem 4.15. *There is a computable, weakly separable order-type λ such that the degree spectrum $\mathrm{DegSpec}_{\mathrm{Succ}}(\lambda)$ contains every c.e. degree.*

In fact, we construct λ so that every c.e. set is weak truth-table equivalent to $\mathrm{Succ}(\mathcal{L})$ for some computable presentation of $\mathcal{L}$.

Proof. The idea is to effectively encode the set W_α into $\mathcal{L}$ by replacing the $(\alpha, \beta)^{\text{th}}$ irrational with the order-type 2 or 3 depending on whether $\beta \in W_\alpha$. Fix an effective list $\langle r_{\alpha,\beta} \rangle_{\alpha,\beta<\omega_1}$ of all the irrational numbers.

The order $\mathcal{L}$ is obtained from $\mathbb{R}$ by replacing $r_{\alpha,\beta}$ by two points if $\beta \notin W_\alpha$ and by three points if $\beta \in W_\alpha$. Then $\mathcal{L}$ is computable, and $\mathbb{Q} \subseteq \mathcal{L}$ witnesses that $\mathcal{L}$ is weakly separable.

For any $\gamma < \omega_1$, we construct a computable $\mathcal{A} \cong \mathcal{L}$ such that $\text{Succ}(\mathcal{A}) \equiv_{\text{wtt}} W_\gamma$. We start with $\mathbb{Q}$; for any irrational number r, let C_r be the $\mathbb{Q}$-interval of $\mathcal{A}$ replacing r. We start by enumerating two points into each C_r. If β enters W_α, and $\alpha \neq \gamma$, we enumerate a third point into $C_{r_{\alpha,\beta}}$ to the right of the existing two points. If β enters W_γ, then we enumerate a third point into $C_{r_{\gamma,\beta}}$ between the existing two points.

To compute $\text{Succ}(\mathcal{A})$ from W_γ, we take $a <_{\mathcal{A}} b$; again, we may assume that $a, b \notin \mathbb{Q}$, and that they lie in the same $\mathbb{Q}$-interval $C_{r_{\alpha,\beta}}$; α and β are effectively obtained from a and b. If $\alpha \neq \gamma$, then $(a, b) \in \text{Succ}(\mathcal{A})$ if and only if $(a, b) \in \text{Succ}(\mathcal{A}_s)$ for any stage s at which $a, b \in \mathcal{A}_s$. If $\alpha = \gamma$, then W_γ tells us the size of $C_{r_{\gamma,\beta}}$, and so a stage s at which $C_{r_{\gamma,\beta},s} = C_{r_{\gamma,\beta}}$; then, of course, $(a, b) \in \text{Succ}(\mathcal{A})$ if and only if $(a, b) \in \text{Succ}(\mathcal{A}_s)$.

To compute W_γ from $\text{Succ}(\mathcal{A})$, for $\beta < \omega_1$, we let $a <_{\mathcal{A}} b$ be the first two points enumerated into $C_{r_{\gamma,\beta}}$; these are obtained effectively from β. Then $\beta \in W_\gamma$ if and only if $(a, b) \in \text{Succ}(\mathcal{A})$. □

5. Open Questions

We close with natural questions that are related to, but not addressed by, the work in this paper. We begin with questions not answered by Section 2.

Question 5.1. Is there an order-type of size $\aleph_1$ whose degree spectrum consists of the nonzero ω_1-degrees?

Though it is unclear what definition should be taken for $A^{(\alpha)}$ for $A \subseteq \omega_1$ and $\alpha \geq \omega$, we ask the following question anyway.

Question 5.2. Is there, for each ordinal $\alpha < \omega_1$, an order-type with proper α^{th} jump degree $\mathbf{a}^{(\alpha)}$?

If every infinite linear order had proper self-embeddings, it would seem possible to generalize the characterization of computable categoricity to higher cardinals. As this is not the case for linear orders of size $\aleph_1$ (see [8]), we ask for a characterization of computable categoricity at the next cardinal.

Question 5.3. Which ω_2-computable linear orders are ω_2-computably categorical?

Question 5.4. Is there a computable weakly separable linear order such that $\mathbf{min}(\lambda) < \mathbf{max}(\lambda)$ but $\text{DegSpec}_{\text{Succ}}(\lambda) = \{\mathbf{min}(\lambda), \mathbf{max}(\lambda)\}$? In general, what are the possible relations between $\text{DegSpec}_{\text{Succ}}(\lambda)$ and the interval $[\mathbf{min}(\lambda), \mathbf{max}(\lambda)]$?

Question 5.5. What can be said about the complexity of the block relation "$a <_{\mathcal{L}} b$ and $(a, b)_{\mathcal{L}}$ is finite" and the relation "$a <_{\mathcal{L}} b$ and $(a, b)_{\mathcal{L}}$ is countable" in uncountable computable linear orderings?

We also pose a methodological question:

Question 5.6. What effects do combinatorial principles such as $\Diamond$ have on the effectiveness properties of uncountable linear orders? We note that Jensen's original proof of $\Diamond$ shows the existence of an ω_1-computable $\Diamond$-sequence.

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Department of Mathematics, Victoria University of Wellington, Wellington, New Zealand
E-mail address: Noam.Greenberg@msor.vuw.ac.nz
URL: http://homepages.mcs.vuw.ac.nz/~greenberg/

Department of Mathematics, University of Connecticut-Storrs, Storrs, CT 06269-3009, USA
E-mail address: asher.kach@uconn.edu
URL: http://www.math.uconn.edu/~kach/

Department of Mathematics, University of Wisconsin, Madison, WI 53706-1388, USA
E-mail address: lempp@math.wisc.edu
URL: http://www.math.wisc.edu/~lempp/

Department of Mathematics, Victoria University of Wellington, Wellington, New Zealand
E-mail address: dan.turetsky@msor.vuw.ac.nz
URL: http://msor.victoria.ac.nz/Main/DanTuretsky