Spectra of computable models of strongly minimal disintegrated theories

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(Joint work with Uri Andrews)

October 12, 2016
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The following theorem will allow us to define spectra:

**Theorem (Baldwin/Lachlan 1971)**

The countable models of any $\aleph_1$-categorical but not totally categorical theory $T$ in any countable language form an elementary chain

$$M_0 \prec M_1 \prec \ldots \prec M_\omega$$

where $M_0$ is the prime model and $M_\omega$ is the countable saturated model of $T$. 

**Definition**

The spectrum of computable models of an $\aleph_1$-categorical but not totally categorical theory $T$ in any computable language is

$$SCM(T) = \{\alpha \leq \omega | M_\alpha \text{ is computable}\}.$$
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Warning: $\mathcal{M}_\alpha$ may have dimension $k + \alpha$ for fixed $k > 0$. 

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**Corollary**

For any strongly minimal disintegrated theory $T$, the spectrum of $T$ is a $\Sigma^0_5$-set.
Theorem

The following are all previously known spectra of computable models of strongly minimal (indeed, all $\aleph_1$-categorical) theories:

- $\emptyset$ and $[0, \omega]$ (trivial)
- $\{0\}$ (Goncharov 1978) and $[0, n]$ ($n \in \omega$, Kudaibergenov 1980)
- $\{\omega\}$ (Hirschfeldt/Khoussainov/Semukhin 2006)
- $\{0, \omega\}$ (Andrews 2011, the first known non-interval!)
- All spectra except for the last are for a strongly minimal disintegrated theory; the last is by a Hrushovski construction.
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**Theorem (Andrews/Medvedev 2014)**

If $T$ is a strongly minimal disintegrated theory in a *finite* language $\mathcal{L}$, then the possible spectra of computable models are exactly $\emptyset$, $[0, \omega]$, and $\{0\}$. 

This shows that the Herwig/Lempp/Ziegler model was "essentially" the only way to construct a nontrivial spectrum for a strongly minimal disintegrated theory in a finite language.

In addition to disintegrated theories, the result of Andrews/Medvedev also extends to locally modular expansions of a group and, by Poizat (1988), to field-like theories, i.e., to "most" trichotomous theories.
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For infinite languages, the situation is more difficult.

**Theorem (Andrews/Lempp)**

If $T$ is a strongly minimal disintegrated theory in a (possibly infinite) *binary relational* language $\mathcal{L}$, then the possible spectra of computable models are exactly the following seven sets: $\emptyset$, $[0, \omega]$, $\{0\}$, $\{1\}$, $\{0, 1\}$, $\{\omega\}$, and $[1, \omega]$. 
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Our recent work has been motivated by the following sweeping

**Conjecture**

If $T$ is a strongly minimal disintegrated theory in a (possibly infinite) relational language $\mathcal{L}$ of arity at most $n$, then there are only finitely many possible spectra of computable models.
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**Conjecture**

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The following constitutes progress toward, and related to, this conjecture.
In a strongly minimal model $\mathcal{M}$, a relation $R \subseteq M^n$

- *has (Morley) rank* 0 if $R$ is finite (and nonempty);
- *has (Morley) rank* at most 1 if for any $\bar{a} \in M^n$ with $\mathcal{M} \models R(\bar{a})$, $\dim(\text{acl}(\bar{a}))$ is at most 1, i.e., $\bar{a}$ does not contain two mutually generic elements.
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**Theorem (Andrews/Lempp)**

If $T$ is a strongly minimal disintegrated theory in a relational language $\mathcal{L}$ of bounded arity such that in each model $\mathcal{M}$ of $T$, any relation $R^\mathcal{M}$ has rank at most 1, then the possible spectra of computable models are exactly the following nine or ten sets: $\emptyset$, $[0, \omega]$, $\{0\}$, $\{1\}$, $\{0, 1\}$, $\{\omega\}$, $[1, \omega]$, $\{0, \omega\}$, and $\{0, 1, \omega\}$, and possibly $\{1, \omega\}$. 
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**Theorem (Andrews/Lempp)**

If $T$ is a strongly minimal disintegrated theory in a relational language $L$ of bounded arity such that in each model $\mathcal{M}$ of $T$, any relation $R^\mathcal{M}$ has rank at most 1, then the possible spectra of computable models are exactly the following nine or ten sets: $\emptyset$, $[0, \omega]$, $\{0\}$, $\{1\}$, $\{0, 1\}$, $\{\omega\}$, $[1, \omega]$, $\{0, \omega\}$, and $\{0, 1, \omega\}$, and possibly $\{1, \omega\}$.

Among the two additional spectra, $\{0, \omega\}$ was not known before to be the spectrum of a disintegrated theory; and $\{0, 1, \omega\}$ was not even known to be a spectrum at all.
The assumption of bounded arity in the previous theorem was crucial since we also have:

**Theorem (Andrews/Lempp)**

If $T$ is a strongly minimal disintegrated theory in a relational language $\mathcal{L}$ (of any arity) such that in each model $\mathcal{M}$ of $T$, any relation $R^\mathcal{M}$ has rank at most 1, then the possible spectra of computable models are exactly the nine or ten spectra from the previous theorem as well as the sets $[0, \alpha)$ and $[0, \alpha) \cup \{\omega\}$ for all $\alpha \leq \omega$. 
With a trick, we can “almost” reduce the ternary case to the rank-1 case and obtain the following

**Theorem (Andrews/Lempp)**

If $T$ is a strongly minimal disintegrated theory in a ternary relational language $\mathcal{L}$, then there are at least nine and at most eighteen possible spectra of computable models:

For any spectrum $S$, $[3, \omega) \cap S \neq \emptyset$ implies $[1, \omega] \subseteq S$. 
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Binary $\mathcal{L}$: If $\mathcal{M}_\alpha$ for some $\alpha \geq 2$ is computable, then fix two mutually generic $a, b \in M_\alpha$. Now $R^{\mathcal{M}_\alpha}$ has rank 2 iff $\mathcal{M}_\alpha \models R(a, b)$, so in that case we replace $R$ by $\neg R$ (which is at most rank 1).
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**Ternary \( \mathcal{L} \):** If \( M_\alpha \) for some \( \alpha \geq 3 \) is computable, then fix three mutually generic \( a, b, c \in M_\alpha \).

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Ternary $\mathcal{L}$: If $\mathcal{M}_\alpha$ for some $\alpha \geq 3$ is computable, then fix three mutually generic $a, b, c \in \mathcal{M}_\alpha$. First reduce to rank at most 2 as in the binary case. Then $\mathcal{M}_\alpha \models \exists^\infty w R(w, y, z)$ iff at least two of $\mathcal{M}_\alpha \models R(a, y, z)$, $\mathcal{M}_\alpha \models R(b, y, z)$, and $\mathcal{M}_\alpha \models R(c, y, z)$ hold,
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Now all of $\exists^\infty w R(w, y, z)$, $\exists^\infty w R(x, w, z)$, $\exists^\infty w R(x, y, w)$, $R(x, y, z) \setminus [\exists^\infty w R(w, y, z) \lor \exists^\infty w R(x, w, z) \lor \exists^\infty w R(x, y, w)]$, $[\exists^\infty w R(w, y, z) \lor \exists^\infty w R(x, w, z) \lor \exists^\infty w R(x, y, w)] \setminus R(x, y, z)$ have rank at most 1 and are effectively interdefinable with $R(x, y, z)$.
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For a basis $B$ of a strongly minimal disintegrated model $\mathcal{M}_\alpha$, we have

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Suppose

- $\mathcal{M}_\beta \subset \mathcal{M}_\alpha$ for $\beta < \alpha \leq \omega$,
- $\mathcal{M}_\alpha$ is a computable model,
- $M_\beta$ is a $\Delta^0_2$-subset of $M_\alpha$, and
- $M_\beta$ contains an infinite $\Sigma^0_1$-subset $S$.

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Then $M_\beta$ has a computable copy:
Let $\dim(M_\beta) = k + \beta$, fix $k + \beta$ many mutually generics $\bar{a}$ in $M_\alpha$ and construct $acl(\bar{a})$, “discarding mistakes” into $S$. 
Step 3: Complexity of $\text{acl}(\emptyset)$ and $\text{iacl}(a)$:

If all relations in $M_\alpha$ are at most rank 1, then both $\text{acl}(\emptyset)$ and $\text{iacl}(a)$ (for every generic $a \in M_\alpha$) are $\Sigma_0^2$-subsets of $M_\alpha$ (nonuniformly in $a$); so they are $\Delta_0^2$-subsets if $\alpha < \omega$.

Proof: Define the $n$-neighborhood $Nbh_n(a)$ of $a \in M_\alpha$ by recursion:

$Nbh_0(a) = \{a\}$

$Nbh_{n+1}(a) = \{b \in M_\alpha | \exists c \in Nbh_n(a) [c, b \text{ directly connected}]\}$

where $c$ and $b$ are "directly connected" if the binary projection of an $m$-ary relation $R \in L$ holds (or fails) between $c$ and $b$ but not between $c$ and cofinitely many elements of $M_\alpha$, nor between $b$ and cofinitely many elements of $M_\alpha$.

Then $0'$ can compute canonical indices for $Nbh_n(a)$ (uniformly in $n$ but nonuniformly in $a$).
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Then $0'$ can compute canonical indices for $\text{Nbh}_n(a)$ (uniformly in $n$ but nonuniformly in $a$).
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Case I: $B$ is finite: Then for any generic $a \in M_k$, $\text{iacl}(a)$ is a $\Sigma^0_1$-subset of $M_k$ (finite or infinite).
**Step 4:** “Down”: If all relations in $\mathcal{M}_\alpha \models T$ are at most rank 1 and $k \in \text{SCM}(T) \cap [2, \omega)$, then $k - 1 \in \text{SCM}(T)$:

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In either case, we can apply the previous steps to see that $M_{k-1}$ is computable.
Step 5: “Up”: If all relations in $M_\alpha \models T$ are at most rank 1 and of bounded arity, and if $k \in SCM(T) \cap [2, \omega)$, then $k + 1 \in SCM(T)$ (uniformly in $k$; so $\omega \in SCM(T)$ as well):
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Again, assume $\mathcal{L}$ is “closed under permutation of variables”.

Case I: For generic $a \in M_k$, there are infinitely many disjoint tuples $\bar{b}$ in $M_k$ such that

$$\mathcal{M}_k \models \exists i \left( R_i(a, \bar{b}) \land \exists^{< \infty} x R_i(x, \bar{b}) \right)$$
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Then we can generate a $\Sigma^0_1$-set of such disjoint tuples and then construct $\mathcal{M}_{k+1}$ as $\mathcal{M}_k \sqcup \text{iacl}(g)$ for a new generic element $g$. 

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Step 5: “Up”: If all relations in $\mathcal{M}_\alpha \models T$ are at most rank 1 and of bounded arity, and if $k \in \text{SCM}(T) \cap [2, \omega)$, then $k + 1 \in \text{SCM}(T)$ (uniformly in $k$; so $\omega \in \text{SCM}(T)$ as well):

Again, assume $\mathcal{L}$ is “closed under permutation of variables”.

Case I: For generic $a \in M_k$, there are infinitely many disjoint tuples $\bar{b}$ in $M_k$ such that

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Then we can generate a $\Sigma_1^0$-set of such disjoint tuples and then construct $\mathcal{M}_{k+1}$ as $\mathcal{M}_k \sqcup \text{iacl}(g)$ for a new generic element $g$.

Case II: Otherwise there is a finite set $\{h_0, \ldots, h_n\}$ of elements involved in all $R_i$: We can then generate a new language $\mathcal{L}'$ of lower arity consisting of all $R_i$ with fixed $h_j$, and iterate Case I vs. Case II for $\mathcal{L}'$, etc., until we reach Case I or a binary language.
Binary $\mathcal{L}$: We also need to show

$$\{0, 1\} \cap \text{SCM}(T) \neq \emptyset \text{ and } \omega \in \text{SCM}(T) \implies 2 \in \text{SCM}(T)$$
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Finally: Several priority arguments to establish new spectra.
Thanks!