Γ-SEPARATED COVERS

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1. Introduction

Rings with a complex module category can often be studied by considering covers of their modules in a subcategory related to an overring with a much simpler structure. For example, by a classical result of Enochs [10], all modules over a commutative domain R have torsionfree covers, that is, covers by R-submodules of Q-vector spaces where Q is the quotient field of R. A general theory of covers was developed by Enochs' school, proving the Flat Cover Conjecture (FCC) in [4] and other interesting results, cf. [11, 19].

On the other hand, a structure theory of finitely generated modules over a class of commutative rings called "Dedekind-like" was recently introduced by Klingler and Levy [14, 15]. We postpone the somewhat technical definition of these rings [Definition 4.1], and the reason for this definition [Remark 4.2], except to say that they are commutative, reduced (no nonzero nilpotent elements), noetherian rings. Some interesting examples of these rings are (see [15, Examples 2.2]):

Naturally occurring examples of Dedekind-like rings.
(E-1) $\mathbb{Z}[\sqrt{n}\,]$ when n is squarefree.
(E-2) Integral group ring $\mathbb{Z}G_n$ (cyclic order n) when n squarefree.
(E-3) All subrings of squarefree index in $\mathbb{Z} \oplus ... \oplus \mathbb{Z}$.
(E-4) $\mathbb{R} + x\mathbb{C}[x]$ and $\mathbb{R} + x\mathbb{C}[[x]]$
(E-5) $k[x, y]/(xy)$ for any field k.

In connection with (E-1) we note that Dedekind-like rings of algebraic integers seem to be the only non-integrally-closed rings of algebraic integers whose finitely generated module category has been described since Steinitz did the integrally closed case in 1911 [18], in his description of modules over (what are now called) Dedekind domains.

The relevance of these rings to the present note is the following. The normalization Γ of an arbitrary Dedekind-like ring Λ is a direct product of Dedekind domains, and hence the structure of mod-Γ is known by Steinitz's work. Klingler and Levy call Λ-modules "Γ-separated" if they are Λ-submodules of Γ-modules. Their approach to the description of mod-Λ is to make use of what they call "Γ-separated covers" of Λ-modules [Definition 3.1 below]. These reduce the description of mod-Λ to the much simpler (and known) structure of mod-Γ. These covers are similar to — but not exactly the same as — covers in the sense of Enochs. For example, (unlike torsionfree modules over integral domains or flat modules over any ring) the class of Γ-separated Λ-modules is not closed under extensions when Λ is Dedekind-like [Theorem 4.10 and Example 4.12].

The main purpose of this note is to clarify the precise relation of these two types of covers, and use this to improve some of the Klingler-Levy results.

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Notation 1.1. Throughout this note Λ denotes a subring of a ring Γ. $\mathcal{G}$ denotes the class of all (say, right) Γ-*separated* Λ-modules; that is, all Λ-submodules of Γ-modules. $\mathcal{G}_0$ denotes the class of all finitely generated modules in $\mathcal{G}$. Mod-Λ and mod-Λ denote the classes, respectively, of all Λ-modules and finitely generated Λ-modules for any ring Λ — for definiteness: right modules unless the contrary is stated. Thus $\mathcal{G}_0 = \mathcal{G} \cap \text{mod-}\Lambda$. The notation $\mathcal{P}$ and $\mathcal{F}$ denotes the classes of projective and flat Λ-modules, respectively.

We say that two Λ-homomorphisms $f_1 \colon H_1 \to M$ and $f_2 \colon H_2 \to M$ are *isomorphic* if there is a Λ-isomorphism $\beta \colon H_1 \to H_2$ such that $f_1 = f_2\beta$. For example projective covers of a module are isomorphic.

We review the definitions of covers and covering classes in 2.1. In Theorem 2.5 we prove that $\mathcal{G}$ is a covering class. We introduce the definition of Γ-separated cover in 3.1, for an arbitrary pair $\Lambda \subseteq \Gamma$. In Theorem 3.3, we show that the (always unique) $\mathcal{G}$-cover is the largest among the (possibly non-unique) Γ-separated covers of a module.

If Λ is Dedekind-like and its normalization Γ is a finitely generated Λ-module, we show that $\mathcal{G}$-covers and Γ-separated covers of arbitrary Λ-modules coincide [Theorem 4.8(i)]. This answers, in the affirmative, Klingler-Levy's question [14, Remarks 4.8] of whether Γ-separated covers of infinitely generated Λ-modules exist in this 'classical' situation.

For arbitrary Dedekind-like rings (i.e. Γ_Λ not necessarily finitely generated), we show that $\mathcal{G}$-covers and Γ-separated covers of finitely generated Λ-modules coincide [Theorem 4.8(ii)], thus making the general theory of covers available for use here.

To deal with the fact that the class of Γ-separated Λ-modules is not closed under extensions, we make use of El Bashir's generalization [9] of FCC, providing covers in certain classes of modules not closed under extensions [Lemma 2.2]. In fact, for noetherian rings closure under extensions in the general setting is equivalent to the setting being a cotilting one [Theorem 2.5(ii)].

2. $\mathcal{G}$-covers

We begin by recalling the basics of the theory of covers of modules over an arbitrary ring Λ.

2.1. Covers. Let M be a Λ-module, $\mathcal{C}$ a class of Λ-modules, and $f : C \to M$ a Λ-homomorphism with $C \in \mathcal{C}$. Then f is a $\mathcal{C}$-*precover* of M provided that for each $C' \in \mathcal{C}$ and each Λ-homomorphism $f' : C' \to M$, f' factorizes through f (that is, there is a Λ-homomorphism $g : C' \to C$ such that $f' = fg$).

The $\mathcal{C}$-precover f is *special* if f is surjective and $\text{Ext}^1_\Lambda(\mathcal{C}, \text{Ker}(f)) = 0$. The $\mathcal{C}$-precover f is a $\mathcal{C}$-*cover* of M if f is *right minimal* (that is, each endomorphism g of C satisfying $fg = f$ is an automorphism). If $\mathcal{C}$ is closed under extensions and contains all projective modules then any $\mathcal{C}$-cover is special by the Wakamatsu Lemma [19, 2.1.1].

$\mathcal{C}$ is a *precovering (special precovering, covering)* class provided that each module $M \in \text{Mod-}\Lambda$ has a $\mathcal{C}$-precover (a special $\mathcal{C}$-precover, a $\mathcal{C}$-cover).

In general, $\mathcal{C}$-precovers need not exist, and the existence of a $\mathcal{C}$-precover of a module M does not imply existence of a $\mathcal{C}$-cover of M. However, any $\mathcal{C}$-cover is easily seen to be unique up to isomorphism of maps, as defined in Notation 1.1.

We call a class $\mathcal{C} \subseteq \text{mod-}\Lambda$ *contravariantly finite* provided that each $M \in \text{mod-}\Lambda$ has a $\mathcal{C}$-cover.

For example, the class $\mathcal{P}$ of all projective modules is a precovering class for any ring Λ. By a classical result of Bass, $\mathcal{P}$ is a covering class iff Λ is right perfect. The solution of the Flat Cover Conjecture (FCC) in [4] says that the class $\mathcal{F}$ of all flat modules is a covering class for any ring. In fact, both proofs of FCC in [4] have generalizations showing that covers are rather frequent, as the next lemma shows. For a class of modules $\mathcal{C}$, we denote by $\varinjlim \mathcal{C}$ the class of all modules that are direct limits of direct systems of modules from $\mathcal{C}$; for example, $\mathcal{F} = \varinjlim \mathcal{P}$.

Lemma 2.2. *Let $\mathcal{C}$ be a class of Λ-modules closed under finite direct sums and direct limits. Assume there is a subset $\mathcal{S} \subseteq \mathcal{C}$ such that $\mathcal{C} = \varinjlim \mathcal{S}$. Then $\mathcal{C}$ is a covering class.*

Proof. The lemma is a particular case of [9, Theorem 3.2] which proves the same result for arbitrary Grothendieck categories. □

2.3. Cotilting classes. For a module M, denote by $\operatorname{Cog}(M)$ the class of all modules cogenerated by M, that is, of all modules isomorphic to submodules of arbitrary direct products of copies of M. For a class of modules $\mathcal{C}$, put ${}^{\perp}\mathcal{C} = \operatorname{Ker} \operatorname{Ext}^1_\Lambda(-, \mathcal{C}) = \{M \in \text{Mod-}\Lambda \mid \operatorname{Ext}^1_\Lambda(M, C) = 0 \text{ for all } C \in \mathcal{C}\}$.

A Λ-module C is a *cotilting module* provided that $\operatorname{Cog}(C) = {}^{\perp}C$. Equivalently, C is cotilting iff C has injective dimension ≤ 1, $\operatorname{Ext}^1_\Lambda(C^I, C) = 0$ for any set I, and there are an injective cogenerator W for Mod-Λ and an exact sequence $0 \to C_1 \to C_0 \to W \to 0$ where C_0 and C_1 are direct summands in a (possibly infinite) direct product of copies of C. The latter definition is much longer, but shows that cotilting modules are just the category-theoretic duals of the better known (infinitely generated) tilting modules. Indeed, cotilting modules are close to being "dual": each cotilting module is pure-injective, [3, Theorem 2.8].

A class of modules $\mathcal{C}$ is a *cotilting class* provided there is a cotilting module C such that $\mathcal{C} = \operatorname{Cog}(C)$. By [8, Corollary 10], each cotilting class is a covering class in the sense of 2.1. In fact, cotilting classes are exactly the special precovering classes closed under products and submodules, [1, Theorem 2.5].

Lemma 2.4. [5] *Let Λ be a right noetherian ring. Let $\mathcal{S}$ be a class of finitely presented Λ-modules such that $\Lambda \in \mathcal{S}$, $\mathcal{S}$ is closed under finite direct sums, submodules, and extensions. Let $\mathcal{C} = \varinjlim \mathcal{S}$. Then $\mathcal{C}$ is a cotilting class.*

Proof. By [7, Lemma 4.4], $\mathcal{C}$ is a torsion-free class in Mod-Λ. By [2, Lemma 2.1(iii) and Theorem 2.3], $\mathcal{C} = {}^{\perp}\mathcal{I}$ for a class of pure-injective modules $\mathcal{I}$, so $\mathcal{C}$ is a covering class by [8, Corollary 10]. By the Wakamatsu lemma and [1, Theorem 2.5], $\mathcal{C}$ is a cotilting class. □

A module M is *cotorsion* if $\operatorname{Ext}^1_\Lambda(\mathcal{F}, M) = 0$. For example, any pure-injective module is cotorsion.

Theorem 2.5. *Let Λ be a ring. Then (notation as in 1.1):*

(i) *$\mathcal{G} = \varinjlim \mathcal{G}_0$, and $\mathcal{G}$ is a covering class containing $\mathcal{F}$ and closed under $\varinjlim$. Each $\mathcal{G}$-cover is a Λ-epimorphism.*

(ii) *Assume Λ is right noetherian. Then $\mathcal{G}$ is a cotilting class if and only if $\mathcal{G}_0$ is closed under extensions. In this case, for any Λ-module M, the (unique) $\mathcal{G}$-cover of M, $g : G \to M$, is special, and $\operatorname{Ker}(g)$ is a cotorsion Λ-module of injective dimension ≤ 1.*

Proof. (i) Clearly $\mathcal{G}$ is closed under submodules and products. Since $\Lambda \in \mathcal{G}$, we have $\mathcal{P} \subseteq \mathcal{G}$. *Caution:* But $\mathcal{G}$ need not be closed under extensions, as we show in Theorem 4.10 and Example 4.12.

Since $\mathcal{G}$ is closed under submodules, we have $\mathcal{G} \subseteq \varinjlim \mathcal{G}_0$.

Let $M \in \mathcal{G}$. Then there is $N \in \text{Mod-}\Gamma$ such that $M \subseteq N$. Consider the canonical maps $\nu_M : M \to M \otimes_\Lambda \Gamma$ and $\eta_M : M \otimes_\Lambda \Gamma \to M{\cdot}\Gamma\ (\subseteq N)$. Since $\eta_M \nu_M$ equals the identity on M, ν_M is monic.

Let $M = \varinjlim_{i\in I} G_i$ where $(I, \leq)$ is an upper directed set and $(G_i, g_{ij} \mid i \leq j \in I)$ a direct system of elements of $\mathcal{G}$; in particular, for each $i \in I$, the map ν_{G_i} is monic. The induced system $(G_i \otimes_\Lambda \Gamma, g_{ij} \otimes_\Lambda 1_\Gamma \mid i \leq j \in I)$ of Γ-modules is also direct, and for all $i \leq j \in I$, there is a commutative diagram

$$\begin{array}{ccc} G_i & \xrightarrow{\nu_{G_i}} & G_i \otimes_\Lambda \Gamma \\ {\scriptstyle g_{ij}}\downarrow & & \downarrow{\scriptstyle g_{ij}\otimes_\Lambda 1_\Gamma} \\ G_j & \xrightarrow{\nu_{G_j}} & G_j \otimes_\Lambda \Gamma \end{array}$$

Since $\varinjlim$ is a left exact functor, we infer that the canonical Λ-homomorphism $M \to \varinjlim_{i\in I}(G_i \otimes_\Lambda \Gamma)$ is monic. Since the functor $- \otimes_\Lambda \Gamma$ commutes with direct limits, we also have the canonical Γ-isomorphism $\varinjlim_{i\in I}(G_i \otimes_\Lambda \Gamma) \cong M \otimes_\Lambda \Gamma$. It follows that $M \in \mathcal{G}$, so $\varinjlim \mathcal{G}_0 \subseteq \varinjlim \mathcal{G} \subseteq \mathcal{G}$, and hence $\mathcal{G} = \varinjlim \mathcal{G}_0$ is closed under direct limits.

By Lemma 2.2, $\mathcal{G}$ is a covering class of right Λ-modules. Since $\mathcal{P} \in \mathcal{G}$, each $\mathcal{G}$-cover is a Λ-epimorphism.

(ii) If $\mathcal{G}$ is a cotilting class then $\mathcal{G}$, and hence also $\mathcal{G}_0$, is closed under extensions. Conversely, since $\mathcal{G}_0$ consists of finitely presented modules, and $\mathcal{G} = \varinjlim \mathcal{G}_0$ by part (i), $\mathcal{G}$ is a cotilting class by Lemma 2.4.

Finally, since $\mathcal{P} \subseteq \mathcal{G}$ and $\mathcal{G}$ is closed under extensions, $\mathcal{G}$-covers are special by the Wakamatsu lemma. In particular, if K is the kernel of a $\mathcal{G}$-cover then $\mathrm{Ext}^1_\Lambda(\mathcal{F}, K) = 0$ by part (i), that is, K is a cotorsion module. Since $\mathcal{G} = {}^\perp\{C\}$ where C has injective dimension ≤ 1, the condition $\mathrm{Ext}^1_R(\mathcal{G}, K) = 0$ implies that K has injective dimension ≤ 1 by the Baer Test of Injectivity and dimension shifting. □

Remark 2.6. Though $\mathcal{G}$ is a covering class closed under submodules and products, it is not cotilting in general: $\mathcal{G}_0$ is not closed under extensions for any Dedekind-like ring $\Lambda \neq \Gamma$ [Theorem 4.10 and Example 4.12]. In that case, $\mathcal{G}$ is not special precovering; in fact, if W is an injective cogenerator for Mod-Λ then the $\mathcal{G}$-cover of W is not special by [1, Theorem 2.5].

Example 2.7. Let Λ be a commutative domain, $\Gamma = E(\Lambda)$ its quotient field, and $K = \Gamma/\Lambda$. To avoid trivialities, we assume $K \neq 0$. We show:

(i) $\mathcal{G}$ (= the class of all torsionfree modules) is a cotilting class.

(ii) Assume that Λ is noetherian and is not a complete local ring. Then the $\mathcal{G}$-cover (= torsionfree cover) of every nonzero Λ-module of finite length is infinitely generated.

Proof. (i) $\mathcal{G}$ is a covering class by [10]. So by the Wakamatsu lemma and [1, Theorem 2.5], $\mathcal{G}$ is a cotilting class. In fact, it is easy to construct a cotilting module C cogenerating the class $\mathcal{G}$ as follows.

We have $\mathcal{G} = \mathrm{Cog}_\Lambda(\Gamma)$. By [6, VII.2.2], $\mathcal{G} = \{M \mid \mathrm{Tor}^R_1(M, K) = 0\}$, and the Ext-Tor relations [6, VI.5.1] yield that $\mathcal{G} = {}^\perp\{K^*\}$ where $K^* = \mathrm{Hom}_\mathbb{Z}(K, \mathbb{Q}/\mathbb{Z})$. Moreover, K^* is a torsion-free Λ-module by [6, VII.1.5]. So $C = \Gamma \oplus K^*$ is a cotilting module such that $\mathcal{G} = \mathrm{Cog}(C) = {}^\perp\{C\}$.

(ii) Let $0 \neq M \in$ mod-Λ have finite length, and assume that its torsionfree cover $f \colon F \to M$ is finitely generated. Since M has finite length and is nonzero, it has a simple submodule, necessarily isomorphic to $k(\mathfrak{m}) = \Lambda/\mathfrak{m} = \Lambda_\mathfrak{m}/\mathfrak{m}_\mathfrak{m} = \hat{\Lambda}_\mathfrak{m}/\hat{\mathfrak{m}}_\mathfrak{m}$ where $\mathfrak{m}$ is a maximal ideal of Λ, and $\hat{\Lambda}_\mathfrak{m}$ the $\mathfrak{m}$-adic completion of $\Lambda_\mathfrak{m}$. Therefore there exists a nonzero Λ-homomorphism $g \colon \Lambda_\mathfrak{m} \to M$, and g factors through the torsionfree cover of M, say $g \colon \Lambda_\mathfrak{m} \xrightarrow{h} F \xrightarrow{f} M$. We consider two cases.

Case 1: Λ is not a local ring. We claim that $h(1) \neq 0$. Otherwise $h(\Lambda) = \Lambda h(1) = 0$. Then, for any $x/d \in \Lambda_{\mathfrak{m}}$ we have $d{\cdot}h(x/d) = h(x) = 0$. Since F is torsionfree, we have $h(x/d) = 0$; that is, $h(\Lambda_{\mathfrak{m}}) = 0$, a contradiction.

Next we claim that h is monic. Suppose not. Then $h(x/d) = 0$ for some $x, d \in \Lambda$ with $x \neq 0$ and $d \notin \mathfrak{m}$. Then $h(x) = d{\cdot}h(x/d) = 0$, and hence $x{\cdot}h(1) = 0$. Since F is torsionfree, this yields the contradiction $h(1) = 0$. Thus h is monic.

Since F is finitely generated and Λ is noetherian, the image of the monic map h is finitely generated; and hence $\Lambda_{\mathfrak{m}}$ is a finitely generated Λ-module. Let $0 \neq d \in R$ be a common denominator for some finite set of generators. Then $d\Lambda_{\mathfrak{m}} \subseteq \Lambda$.

Since Λ is not local, it has a maximal ideal $\mathfrak{n} \neq \mathfrak{m}$. Choose an element $x \in \mathfrak{n} - \mathfrak{m}$. Then x is a unit in $\Lambda_{\mathfrak{m}}$, and hence $dx^{-i} \in \Lambda$ for every positive integer i. But then $d \in \cap_{i=1}^{\infty} \mathfrak{n}^i$. Since Λ is a noetherian domain this intersection equals zero, by the Krull intersection theorem. Thus we have the contradiction that $d = 0$.

Case 2: Λ is local with maximal ideal $\mathfrak{m}$, and $\Lambda \neq \hat{\Lambda}_{\mathfrak{m}}$. First we prove a simple lemma, for which we do not know a reference: *If $\hat{\Lambda}_{\mathfrak{m}}$ is a finitely generated Λ-module, then $\Lambda = \hat{\Lambda}_{\mathfrak{m}}$.* We want to show that the natural map $\nu\colon \Lambda \to \hat{\Lambda}$ is an isomorphism. Since $\hat{\Lambda}_{\mathfrak{m}}$ is a faithfully flat Λ-module, it suffices to show that the induced map $\hat{\nu}\colon \hat{\Lambda}_{\mathfrak{m}} \otimes_{\Lambda} \Lambda \to \hat{\Lambda}_{\mathfrak{m}} \otimes_{\Lambda} \hat{\Lambda}_{\mathfrak{m}}$ is an isomorphism. Since, moreover, both Λ and $\hat{\Lambda}_{\mathfrak{m}}$ are finitely generated Λ-modules, their $\mathfrak{m}$-adic completions are given by tensoring with $\hat{\Lambda}_{\mathfrak{m}}$. Therefore $\hat{\nu}$ can be identified with the identity map on $\hat{\Lambda}_{\mathfrak{m}}$. In particular, it is an isomorphism, completing the proof of the lemma.

As in the paragraph before Case 1, there is a nonzero map $g'\colon \hat{\Lambda}_{\mathfrak{m}} \to M$, and g' factors through the torsionfree cover of M, say $g'\colon \hat{\Lambda}_{\mathfrak{m}} \xrightarrow{h'} F \xrightarrow{f} M$. We claim that the restriction $h = h' \restriction \Lambda$ is nonzero.

Suppose that $h = 0$, and choose any $\hat{x} \in \hat{\Lambda}$. Say $\hat{x} = \lim_{n=1}^{\infty} x_n$ with each $x_n \in \Lambda$. By passing to a subsequence, we may assume that $\hat{x} - x_n \in \hat{\mathfrak{m}}^n = \mathfrak{m}^n \hat{\Lambda}$ for all $\mathfrak{n}$. Then $h'(\hat{x}) \in \cap_n \mathfrak{m}^n F$ which equals zero by the Krull intersection theorem since F is finitely generated. Thus we have the contradiction that $h' = 0$, proving the claim.

Next note that $h\colon \Lambda \to F$ is monic because Λ/B is a torsion module for every nonzero ideal B and F is torsionfree. Therefore we may assume that $\Lambda \subseteq F$ and h equals the identity on Λ. We claim that h' is monic.

Take any $\hat{x} \in \mathrm{Ker}(h')$ and, as above, write $\hat{x} = \lim_{n=1}^{\infty} x_n$, the limit of a Cauchy sequence in Λ with $\hat{x} - x_n \in \mathfrak{m}^n \hat{\Lambda}_{\mathfrak{m}}$ for every n. Since h' equals the identity on Λ, applying h' to the previous "$\in$" statement yields $x_n \in \mathfrak{m}^n F$ for all n. Therefore the sequence $x_1, x_2, \ldots$ is a Cauchy sequence in F converging to 0. Since Λ is a submodule of the finitely generated Λ-module F, the $\mathfrak{m}$-adic topology on Λ coincides with the topology induced by the $\mathfrak{m}$-adic topology on F [17, Theorem 8.6]. Therefore the sequence $x_1, x_2, \ldots$ is also a Cauchy sequence in Λ converging to 0. Therefore the limit $\hat{x}$ of this sequence equals 0, completing the proof of the claim.

Since F_{Λ} is finitely generated and h' is monic, we see that $\hat{\Lambda}_{\mathfrak{m}}$ is a finitely generated Λ-module. Therefore the lemma at the beginning of Case 2 of this proof yields the contradiction $\Lambda = \hat{\Lambda}$. □

3. Γ-separated covers

In this section we define Γ-separated covers, and compare them with $\mathcal{G}$-covers and $\mathcal{G}_0$-covers. We do this in the context of arbitrary rings — a much more general context than that considered by Klingler and Levy. In this generality, Γ-separated covers are easily seen not to be unique [Example 3.5]. But they always exist [Theorem 3.2].

Definition 3.1. Let Λ and Γ be given (as in Notation 1.1), which determines $\mathcal{G}$ and $\mathcal{G}_0$.

We call a homomorphism $g : G \to M$ of Λ-modules a Γ-*separated cover* of M provided that:

(i) g is surjective;
(ii) $G \in \mathcal{G}$; and
(iii) In every factorization $G \xrightarrow{h} G' \xrightarrow{g'} M$ of g, with h surjective and $G' \in \mathcal{G}$, h must be an isomorphism. (Intuitively: G' is no closer to M than G is.)

Notice that g is close to being right minimal: If $h : G \to G$ is a Λ-homomorphism such that $g = gh$ then h is a monomorphism. However, h need not be an isomorphism in the present generality: let $\Lambda = \mathbb{Z}$, $\Gamma = \mathbb{Q}$, and $g : \mathbb{Z} \to \mathbb{Z}_2$ be the Γ-separated cover of $\mathbb{Z}_2$ given by the projection modulo 2. Then $g = gh$ where (the non-surjective map) $h : \mathbb{Z} \to \mathbb{Z}$ maps 1 to 3. However (in less generality) see Theorem 4.8.

If $g : G \to M$ is a Γ-separated cover of a finitely generated module M such that $G \in \mathcal{G}_0$ then g is called a *finitely generated* Γ-*separated cover* of M.

For Λ right noetherian and M_Λ finitely generated, Klingler and Levy observed that (for any Γ) it is trivial that M has a Γ-separated cover [14, Proposition 4.7]. In [14, 4.8] they cited several instances in which the finite generation hypothesis can be dropped, and asked whethether it is always unnecessary. Our first result shows that this is indeed the case.

Theorem 3.2. *Every right Λ-module has a Γ-separated cover. In more detail:*

(i) *Let $f\colon H \to M$ be a Λ-epimorphism with $H \in \mathcal{G}$. Then there exists a factorization $H \xrightarrow{h} G \xrightarrow{g} M$ of f such that h is surjective and g is a Γ-separated cover of M.*
(ii) *If M is a κ-generated Λ-module (κ any finite or infinite cardinal) then M has a Γ-separated cover $g\colon G \to M$ where G is κ-generated.*

Proof. (i) Let $\mathcal{K}$ be the set of all submodules $K \subseteq \operatorname{Ker}(f)$ such that $H/K \in \mathcal{G}$. It suffices to show that $\mathcal{K}$ has a maximal submodule K_0 (with respect to $\subseteq$). For then the factorization $H \xrightarrow{h} G = H/K_0 \xrightarrow{g} M$ of f satisfies the desired conditions. Therefore, by a simple appliction of Zorn's Lemma, it suffices to show that the union U of every totally ordered subset $\mathcal{T}$ of $\mathcal{K}$ is again in $\mathcal{K}$; that is, $H/U \in \mathcal{G}$.

Every $T' \subseteq T'' \in \mathcal{T}$ induces a natural map $H/T' \to H/T''$, and these maps make the set of modules $\{H/T \mid T \in \mathcal{T}\}$ into a direct (in fact, totally ordered) system whose direct limit is H/U (because the maps in the system are surjective). Since $\mathcal{G}$ is closed under direct limits [Theorem 2.5(i)], we see that $H/U \in \mathcal{G}$, completing the proof.

(ii) If M is κ-generated then applying part (i) to an epimorphism $f\colon H = \Lambda^{(\kappa)} \to M$, we get a Γ-separated cover $g\colon G \to M$ with $f = gh$ for an epimorphism $h : \Lambda^{(\kappa)} \to G$, so G is κ-generated. □

Although Γ-separated covers are not always unique, in the generality considered in this section, there is a unique largest such cover of any M, namely the $\mathcal{G}$-cover f of M; and all other Γ-separated covers of M are isomorphic to restrictions of f, as described in the next theorem.

Theorem 3.3. *Let Λ be a ring, M a Λ-module, $f\colon H \to M$ the $\mathcal{G}$-cover of M, and $g\colon G \to M$ a Γ-separated cover of M. Then:*

(i) *f is a Γ-separated cover of M.*
(ii) *There is a submodule $H' \subseteq H$ such that the restriction $f \restriction H'$ is a Γ-separated cover of M isomorphic to g.*

(iii) *If g is a $\mathcal{G}$-precover of M then g is the $\mathcal{G}$-cover of M (necessarily isomorphic to f).*

Proof. (ii) Since f is a $\mathcal{G}$-precover, there is a factorization $g\colon G \xrightarrow{h} H \xrightarrow{f} M$. Put $H' = \mathrm{Im}(h)$. Since $H' \in \mathcal{G}$ and g is a Γ-separated cover of M, we have $\mathrm{Ker}(h) = 0$, as desired.

(i) Since $\mathcal{G}$-covers are surjective [Theorem 2.5], Theorem 3.2 yields a factorization $f\colon H \xrightarrow{h} G' \xrightarrow{g'} M$, with h and g' surjective, such that g' is a Γ-separated cover of M. Part (ii) yields a submodule $H' \subseteq H$ such that $f \restriction H'$ is isomorphic to g'. Thus there is an isomorphism $\theta\colon G' \cong H'$ such that $g' = f\theta$.

We also have $f = g'h$, and therefore $f = f(\theta h)$. Since f is right minimal, we have $\mathrm{Ker}(h) = 0$ and $H' = H$; that is, f and g' are isomorphic Γ-separated covers of M.

(iii) Since g is a precover there is a factorization $f = g\alpha$ for some $\alpha\colon H \to G$. Since f is a $\mathcal{G}$-cover, there is a factorization $g = f\beta$ for some $\beta\colon G \to H$. Therefore $f = f(\beta\alpha)$. Right minimality of f implies that $\beta\alpha$ is an automorphism of H, and hence β is surjective. Since g is a Γ-separated cover, β is an isomorphism $G \cong H$; and this shows that g is isomorphic to the $\mathcal{G}$-cover f, and hence is itself a $\mathcal{G}$-cover. □

There is a similar result for finitely generated modules:

Theorem 3.4. *Let Λ be a ring, M a finitely generated Λ-module, $f\colon H \to M$ the $\mathcal{G}$-cover of M. Assume there exists a $\mathcal{G}_0$-cover $f_0\colon H_0 \to M$. Then:*

(i) *f_0 is a finitely generated Γ-separated cover of M.*
(ii) *Every finitely generated Γ-separated cover of M is isomorphic to a restriction of f_0.*
(iii) *Let g be a finitely generated Γ-separated cover of M. If g is a $\mathcal{G}_0$-precover of M, then g is a $\mathcal{G}_0$-cover of M (necessarily isomorphic to f_0).*
(iv) *There is a finitely generated pure submodule $H' \subseteq H$ such that $f \restriction H'$ is a $\mathcal{G}_0$-cover of M isomorphic to f_0.*

Proof. (i) We claim that f_0 is surjective. There is a surjective map $\phi\colon F \to M$ with F free of finite rank. The claim holds since ϕ factors through f_0.

Now choose a factorization $f_0 = \beta\alpha$ with both factors surjective, $\alpha\colon H_0 \to K_0$, and $K_0 \in \mathcal{G}_0$. We need to show tht α is monic. Since f_0 is a $\mathcal{G}_0$-cover, there is a factorization $\beta = f_0\gamma$. Then right minimality of $f_0 = f_0(\gamma\alpha)$ shows that $\gamma\alpha$ is an automorphism of H_0, and hence α is monic.

(ii) and (iii) We omit the details which are the same as in the proof of Theorem 3.3, (ii) and (iii), with $\mathcal{G}_0$ replacing $\mathcal{G}$.

(iv) By Theorem 3.3(ii) and by part (i), there is a finitely generated submodule $H' \subseteq H$ such that $f \restriction H'$ is a Γ-separated cover isomorphic to f_0. Since $f \restriction H'$ is isomorphic to the $\mathcal{G}_0$-cover f_0, we see that $f \restriction H'$ is a $\mathcal{G}_0$-cover of M, as desired.

It now suffices to prove that H' is pure in H. H is the directed union of all finitely generated Λ-modules L such that $H' \subseteq L \subseteq H$. Therefore if we can show that H' is a direct summand of every such L — and hence pure in L — we will know that H' is pure in the directed union H of these submodules L. Fix such an L and, for brevity, write $f_L = f \restriction L$ and $f_H = f \restriction H$.

Since $f(H') = M$ and $H' \subseteq L$ we also have $f(L) = M$, and therefore we have $H' + \mathrm{Ker}(f_L) = L$. It therefore suffices to show that $H' \cap \mathrm{Ker}(f_L) = 0$.

Since $f_{H'}$ is a $\mathcal{G}_0$-cover of M there is a factorization $f\colon L \xrightarrow{\pi} H' \xrightarrow{f} M$. Thus $f = f\pi$ on L and hence on H'. Then right minimality of $f_{H'}$ shows that π is an automorphism on H'. In particular, $H' \cap \mathrm{Ker}(f_L) = 0$, as desired. □

Example 3.5 (Non-uniqueness of Γ-separated covers)**.** Let Λ be a commutative domain, Γ the quotient field of Λ, and assume that $\Lambda \neq \Gamma$. Thus $\mathcal{G}$ is the class of all torsion-free Λ-modules. We show:

(i) If Λ has a nonprincipal finitely generated ideal, then some simple Λ-module k has nonisomorphic finitely generated Γ-separated covers.

(ii) If Λ is noetherian but not local and complete, then k also has an infinitely generated Γ-separated cover.

Proof. (i) Let $A \neq 0$ be a finitely generated ideal of Λ. Then A has a maximal Λ-submodule, and hence A maps onto a simple Λ-module k.

We claim that every epimorphism $g\colon A \to k$ is a Γ-separated cover. We need to show that there is no surjective factorization $g\colon A \to A/B \to k$ with A/B torsionfree and $B \neq 0$. But since any nonzero element of B annihilates A/B, this is obvious.

Thus, choosing A to be finitely generated and nonprincipal we get one Γ-separated cover $A \to k$. A second such cover, not isomorphic to the first, is (the case $A = \Lambda$:) any surjection $\Lambda \to k$.

(ii) Let $g\colon G \to k$ be the $\mathcal{G}$ (= torsionfree) cover of k, and hence a Γ-separated cover of k [Theorem 3.3(i)]. Since k_Λ has finite length, Example 2.7(ii) shows that G_Λ is not finitely generated. □

4. The Dedekind-like case

In this section we define Dedekind-like rings, and give the reason for this rather techincal definition. Then we compare Γ-separated covers with $\mathcal{G}$-covers and $\mathcal{G}_0$-covers in the context of these rings.

Definition 4.1. Let Λ be a reduced (no nonzero nilpotent elements) commutative noetherian ring with normalization Γ. Following [15, 10.1], we call Λ *Dedekind-like* provided that the following conditions hold:

(i) Γ is a direct sum of Dedekind domains.
(ii) $(\Gamma/\Lambda)_\mathfrak{m}$ is either a simple $\Lambda_\mathfrak{m}$-module or 0 for all maximal ideals $\mathfrak{m}$ of Λ.
(iii) $\mathfrak{m}_\mathfrak{m} = \mathrm{rad}(\Gamma_\mathfrak{m})$ in $\Gamma_\mathfrak{m}$ (the Jacobson radical) for all maximal ideals $\mathfrak{m}$ of Λ.

We do not consider fields to be Dedekind domains. Therefore Dedekind-like rings have Krull dimension 1.

We call a Dedekind-like ring *classical* if Γ is a finitely generated Λ-module. All of examples (E-1)–(E-5) in the Introduction to this note are classical. An example of a nonclassical Dedekind-like ring is constructed in [12].

Remark 4.2 (Reason for the name "Dedekind-like")**.** Let Ω be a commutative noetherian ring. For the purpose of discussing Ω-modules we assume, without loss of generality, that the ring Ω is indecomposable.

In [15, Theorem 14.5] it is proved that if the category of Ω-modules of finite length does not have wild representation type, then Ω is either a homomorphic image of a Dedekind-like ring or else is an artinian local ring of (composition) length 4, called a "Klein ring".

Then [15] describes the detailed structure of finitely generated Λ-modules when Λ is Dedekind-like, extending Steinitz's well-known results for Dedekind domains [18]. There is a possible slight exception to this new structure theory, involving characteristic 2 [15, Additional Hypothesis 10.2]. But this possible exception does not apply to the results in the present paper.

We note tbat the structure of finitely generated modules over Klein rings can also be described [14, §11].

There is a formal relation between the classical and nonclassical cases that we need:

Lemma 4.3. *Let* $\mathfrak{m}$ *be a maximal ideal of a Dedekind-like ring* Λ *with normalization* Γ*. Then the local ring* $\Lambda_{\mathfrak{m}}$ *is a classical Dedekind-like ring with normalization* $\Gamma_{\mathfrak{m}}$*. Moreover, if* $g\colon G \to M_\Lambda$ *is a finitely generated* Γ*-separated cover, then* $g_{\mathfrak{m}}\colon G_{\mathfrak{m}} \to M_{\mathfrak{m}}$ *is a finitely generated* $\Gamma_{\mathfrak{m}}$*-separated cover.*

Proof. First note that all local Dedekind-like rings are classical. In fact, by Definition 4.1(ii), Γ_Λ is generated by 2 elements. For the statement about localizing Λ and Γ see [14, Proposition 10.6 and Remarks 5.3(i)]. For the statement about separated covers see [15, Theorem 18.13]. □

Our results about classical Dedekind-like rings are more complete than those about non-classical ones. Also, our results relating $\mathcal{G}$-covers to Γ-separated covers apply to a class of (commutative and noncommutative) rings much wider than classical Dedekind-like rings. The next lemma identifies these rings.

Lemma 4.4. *Let* $\rho\colon \Gamma \to \bar{\Gamma}$ *be a surjective ring homomorphism, where* Γ *is right noetherian and* $\bar{\Gamma}$ *is semisimple artinian. Let* $\bar{\Lambda}$ *be a subring of* $\bar{\Gamma}$ *such that* $\bar{\Gamma}$ *is a finitely generated* $\bar{\Lambda}$*-module, and let*

$$\Lambda = \{x \in \Gamma \mid \rho(x) \in \bar{\Lambda}\}$$

Then Λ *is a right noetherian ring, and every classical Dedekind-like ring with normalization* Γ *is of this form.*

For the proof that Λ is right noetherian, see [14, Lemma 4.2]. The proof that all classical Dedekind-like rings have this form is the case $\Omega = \Gamma$ of [15, Proposition 18.3(ii)] (because Γ_Λ is finitely generated in the classical case).

The main property of Γ-separated covers proved in [14, 15] is:

Theorem 4.5 (Almost functorial property)**.** *Let* Λ *and* Γ *be as in Lemma 4.4 (e.g. any classical Dedekind-like ring with normalization* Γ*). Let* $f\colon M_1 \to M_2$ *be a* Λ*-homomorphism, and let* $\phi_i\colon G_i \to M_i$ $(i = 1, 2)$ *be* Γ*-separated covers. Then:*

(i) *f can be lifted to a* Λ*-homomorphism* $\theta\colon G_1 \to G_2$ *such that* $f\phi_1 = \phi_2\theta$*.*
(ii) *If* f *is monic or epic, so is any such* θ*.*

If Λ *is a nonclassical Dedekind-like ring and* M_1, M_2 *are finitely generated, the same conclusions hold.*

See [15, Theorem 18.10] for the case of Dedekind-like rings (classical or not), and [14, Remarks 4.8(ii) and Theorem 4.12] for the situation in Lemma 4.4.

An immediate consequence of the almost functorial property is:

Corollary 4.6. *Let* M *be a right* Λ*-module.*

(i) *If* Λ *and* Γ *are as in Lemma 4.4, then* M *has a unique* Γ*-separated cover* $g\colon G \to M$ *(up to isomorphism of maps), and if* M *is finitely generated, so is* G*.*
(ii) *If* Λ *is a nonclassical Dedekind-like ring with normalization* Γ*, and* M *is finitely generated, then* M *has a finitely generated* Γ*-separated cover* g*, and every* Γ*-separated cover of* M *is isomorphic to* g*.*

Proof. By Theorem 3.2, M has a Γ-separated cover, which can be chosen to be finitely generated if M is. To complete the proof, apply the almost functorial property with $M_1 = M_2 = M$ and f the identity map on M. □

We note the following property of Γ-separated covers:

Theorem 4.7. *Let* $g\colon G \to M$ *be a* Γ*-separated cover. If either of the following conditions holds, then* g *is a "minimal epimorphism" (no submodule properly smaller than* G *is mapped by* g *onto* M*).*

(i) Λ *and* Γ *are as in Lemma 4.4; or*

(ii) Λ *is a nonclassical Dedekind-like ring with normalization* Γ*, and* M *(hence* G*) is finitely generated.*

Proof. In situation (i) this is proved in [14, Lemma 4.10 and Remarks 4.8(ii)]. For Dedekind-like rings (classical or not) see [15, Theorem 18.15]. Note that, in part (ii), finite generation in G results from the uniqueness statement in Corollary 4.6(ii). □

Theorem 4.8. *Let* Λ *be a ring and* M *a right* Λ*-module. (Thus* M *has at least one* Γ*-separated cover, say* $g\colon G \to M$ *[Theorem 3.2].)*

(i) *If* Λ *and* Γ *are as in Lemma 4.4, then* g *is the* $\mathcal{G}$*-cover of* M*. If, in addition,* M *is finitely generated, then* g *is also the* $\mathcal{G}_0$*-cover of* M*.*

(ii) *If* Λ *is a nonclassical Dedekind-like ring with normalization* Γ*, and* M *is finitely generated, then* g *is the* $\mathcal{G}$*-cover and the* $\mathcal{G}_0$*-cover of* M*.*

Thus, in either situation, $\mathcal{G}_0$ *is contravariantly finite.*

Proof. Let f be the $\mathcal{G}$-cover of M (which exists by Theorem 2.5). Then, by Theorem 3.3(i), f is also a Γ-separated cover of M. Parts (i) and (ii) of Corollary 4.6] give the uniqueness of Γ-separated covers, hence an isomorphism of f to g in the cases (i) and (ii), respectively.

If M is finitely generated then, by Corollary 4.6, G is also finitely generated, and hence the $\mathcal{G}$-cover g is also the $\mathcal{G}_0$-cover of M. □

We now begin working toward the proof that $\mathcal{G}_0$ (and hence $\mathcal{G}$) is far from being closed under extensions, when Λ is Dedekind-like.

Lemma 4.9. *Let* $\phi\colon S \to M$ *be a* Γ*-separated cover of a finitely generated* Λ*-module, where* Λ *is Dedekind-like with normalization* Γ*, let* $K = \mathrm{Ker}(\phi)$*, and let* X *be a* Γ*-module containing* S*. Then:*

(i) $\Gamma K \subseteq S$ *(where* ΓK *denotes the* Γ*-submodule of* X *generated by* K*).*

(ii) ΓK *is semisimple as a* Γ*-module and as a* Λ*-module.*

(iii) *Every semisimple* Λ*-module is* Γ*-separated.*

Proof. (i) It suffices to show that $\Gamma_\mathfrak{m} K_\mathfrak{m} \subseteq S_\mathfrak{m}$ (in $X_\mathfrak{m}$) for every maximal ideal $\mathfrak{m}$ of Λ. Therefore, after a change of notation, we may assume that Λ is a local ring with maximal ideal $\mathfrak{m}$. Moreover, after this change of notation, Λ remains Dedekind-like with normalization Γ, and ϕ remains a Γ-separated cover with kernel K [Lemma 4.3].

What is gained by this reduction to the local case is that $\mathfrak{m}$ is now an ideal of Γ [Definition 4.1(iii)] and $K \subseteq \mathfrak{m}S$ [14, Lemma 4.9]. But then $\Gamma K \subseteq \Gamma \mathfrak{m} S = \mathfrak{m} S \subseteq S$, as desired.

(ii) Γ-semisimplicity of ΓK is proved in [15, Corollary 18.9]. Thus it suffices to prove that every simple Γ-module Y is Λ-semisimple. In fact, Y is the direct sum of at most two simple Λ-modules, by [15, Theorem and Definition 11.3] together with the "lying over" theorem for integral extensions of commutative rings.

(iii) It suffices to show that every *simple* Λ-module N is Γ-separated. Recall that over any commutative noetherian ring, every module of finite length is the direct sum of its (finitely many) nonzero localizations at maximal ideals. (See e.g. [15, Lemma 6.3].) Since N is simple, this implies that there is a maximal ideal $\mathfrak{m}$ of Λ such that $N = N_\mathfrak{m}$. Thus N is isomorphic to the unique simple $\Lambda_\mathfrak{m}$-module $\Lambda_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}$. But, by the definition of "Dedekind-like ring", $\mathfrak{m}_\mathfrak{m}$ is an ideal of $\Gamma_\mathfrak{m}$. Therefore the inclusion $\Lambda_\mathfrak{m}/\mathfrak{m}_\mathfrak{m} \subseteq \Gamma_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}$ shows that $\Lambda_\mathfrak{m}$ is $\Gamma_\mathfrak{m}$-separated, and hence Γ-separated. □

Theorem 4.10. *Let* Λ *be a Dedekind-like ring with normalization* Γ*. Then every finitely generated* Λ*-module is an extension of a* Γ*-separated module by a* Γ*-separated module.*

Proof. Let M be a finitely generated Λ-module and $\phi\colon S \to M$ a Γ-separated cover. Let $K = \mathrm{Ker}(\phi)$ so that we may assume that ϕ is the natural homomorphism $S \to S/K = M$. Since S is Γ-separated, there is a Γ-module X such that $S \subseteq \Gamma S = X$.

We have $K \subseteq \Gamma K \subseteq S$ by Lemma 4.9(i). Hence we have the following short exact sequence of Λ-modules.

$$0 \to (\Gamma K)/K \to S/K \to S/(\Gamma K) \to 0$$

It therefore suffices to prove that the Λ-modules $(\Gamma K)/K$ and $S/(\Gamma K)$ are Γ-separated. This holds for $S/(\Gamma K)$ since $S/(\Gamma K) \subseteq (\Gamma S)/(\Gamma K)$, a Γ-module.

ΓK is semisimple as a Λ-module by Lemma 4.9(ii), and hence so is its homomorphic image $(\Gamma K)/K$. Therefore, by Lemma 4.9(iii), $(\Gamma K)/K$ is a Γ-separated Λ-module. □

Lemma 4.11. *Let Λ be a local Dedekind-like ring with maximal ideal $\mathfrak{m}$ and normalization Γ, and let S be a Λ-module.*

Suppose that S is Γ-separated, $S \neq \mathfrak{m}S$, and $\mathfrak{m}S$ has a simple Λ-submodule A that is not a Γ-submodule of $\mathfrak{m}S$. Then the Λ-module S/A is not Γ-separated, and the natural map $S \to S/A$ is a Γ-separated cover.

Proof Since Λ is local, its maximal ideal $\mathfrak{m}$ is an ideal of Γ [Definition 4.1]. Since S is Γ-separated, this implies that $\mathfrak{m}S$ is a Γ-submodule of S. (*Caution:* If S is not Γ-separated, $\mathfrak{m}S$ can fail to be a Γ-module. The difficulty is that the left-hand side of the relation $\gamma(ms) = (\gamma m)s$ does not make sense if S is not Γ-separated.)

Note that we can consider S to be a Λ-submodule of $X = \Gamma \otimes_\Lambda S$. For, since S is Γ-separated, the composite map $S \to \Gamma \otimes_\Lambda S \to \Gamma S$ makes sense and equals 1_S. When this is done, we have $X = \Gamma S$.

We claim that S/A is not Γ-separated. It suffices to show that the canonical map $\tau'\colon S/A \to \Gamma \otimes_\Lambda (S/A)$ is not an injection. By right-exactness of $\otimes$, applied to the short exact sequence $A \to S \to S/A$ we obtain the identification (i.e. Γ-isomorphism)

$$\Gamma \otimes_\Lambda S/A = X/\Gamma A \quad \text{via} \quad \gamma \otimes (s + A) \to \gamma s + \Gamma A$$

In terms of this identification we can identify the map τ' with the map $\nu\colon S/A \to X/\Gamma A$ given by $\nu(s + A) = s + \Gamma A$. Since the Λ-submodule A of $\mathfrak{m}S$ is not a Γ-submodule of $\mathfrak{m}S$, there exist $\gamma \in \Gamma$ and $a \in A$ such that $\gamma a \notin A$ (but $\gamma a \in \mathfrak{m}S \subseteq S$), and hence $0 \neq \gamma a + A \in \mathrm{Ker}(\nu)$, proving the claim.

Since A is a simple Λ-module, the natural surjection $S \to S/A$ has no proper surjective factorizations, where "proper" means that neither factor has nonzero kenel. Since S is Γ-separated and (by the above claim) S/A is not, we conclude that $S \to S/A$ is a Γ-separated cover. □

To complete the proof that $\mathcal{G}_0$ is not closed under extensions we need to know that non-Γ-separated modules actually exist over some Dedekind-like ring Λ.

Example 4.12. Suppose $\Lambda \neq \Gamma$. We show that there exists a cyclic non-Γ-separated Λ-module M of finite length, and display its Γ-separated cover $\phi\colon S \to M$.

Proof. Suppose first that Λ is local with maximal ideal $\mathfrak{m}$ and residue field $k = \Lambda/\mathfrak{m}$. Then Γ_Λ is finitely generated and $\mathfrak{m} = \mathrm{rad}(\Gamma)$ [Definition 4.1].

We claim that, in this situation, Γ is a direct product of semilocal principal ideal domains. By the previous paragraph, the ring $\Gamma/\mathfrak{m}$ is a finite dimensional algebra over the field $k = \Lambda/\mathfrak{m}$, and therefore has only finitely many maximal ideals. Since $\mathfrak{m} = \mathrm{rad}(\Gamma)$, every maximal ideal of Γ contains $\mathfrak{m}$, and hence Γ has only finitely many maximal ideals. By the definition of "Dedekind-like", Γ is a direct product of

Dedekind domains; and since Γ is semilocal, so are all of these Dedekind domains. Thus the claim follows from the well-known (and easily proved) fact that every semilocal Dedekind domain is a principal ideal domain.

Next we claim that the Γ-module $\mathfrak{m}/\mathfrak{m}^2$ has a simple Λ-submodule A that is not a Γ-submodule.

Since Γ is a direct product of semilocal principal ideal domains, $\mathfrak{m} = \mathrm{rad}(\Gamma)$ is a principal ideal of Γ (but not of Λ), say $\mathfrak{m} = \Gamma p$ where p is a non-zero-divisor of Γ. Therefore $\mathfrak{m}/\mathfrak{m}^2 \cong \Gamma/\mathfrak{m}$ as Γ-modules. Therefore it suffices to show that $\Gamma/\mathfrak{m}$ has a simple Λ-submodule that is not a Γ-submodule. The simple Λ-submodule $\Lambda/\mathfrak{m}$ of $\Gamma/\mathfrak{m}$ satisfies the required conditions since $\Lambda \neq \Gamma$ and $\Gamma\Lambda = \Gamma$.

Let $S = \Lambda/\mathfrak{m}^2$. By the previous claim, there is a Λ-submodule A of $\mathfrak{m}S = \mathfrak{m}/\mathfrak{m}^2$ that is not a Γ-submodule of $\mathfrak{m}S$. Then the natural map ϕ: $S \to M = S/A$ is a Γ-separated cover of the non-Γ-separated cyclic Λ-module M [Lemma 4.11]. Moreover, S_Λ and M_Λ have finite length because $\Lambda/\mathfrak{m} \cong k$ and $\mathfrak{m}/\mathfrak{m}^2 \cong \Gamma/\mathfrak{m}$ which (as shown above) is a finite dimensional k-algebra.

Now consider a general (non-local) Λ. Since $\Lambda \neq \Gamma$ there is a maximal ideal $\mathfrak{m}$ of Λ such that $\Lambda_\mathfrak{m} \neq \Gamma_\mathfrak{m}$ (in $\Gamma_\mathfrak{m}$). Recall that $\Lambda_\mathfrak{m}$ is again Dedekind-like with normalization $\Gamma_\mathfrak{m}$ [Lemma 4.3], and let ϕ: $S \to M$ be the $\Gamma_\mathfrak{m}$-separated cover of the non-separated $\Lambda_\mathfrak{m}$-module M obtained above.

To complete the proof it suffices to note that every $\Lambda_\mathfrak{m}$-module of finite length is a Λ-module whose Λ-submodules are all $\Lambda_\mathfrak{m}$-submodules [15, Lemma 6.2]. □

5. Open Problems

1. Let M be a module over a nonclassical Dedekind-like ring. If M is not finitely generated, then M has Γ-separated covers [Theorem 3.2], but we do not know whether these covers satisify the almost functorial property [Theorem 4.5], are unique [Corollary 4.6], or have the minimal epimorphism property [Theorem 4.7].

2. In the general setting where Λ is an arbitrary right noetherian ring, and Γ is arbitrary, does contravariant finiteness of $\mathcal{G}_0$ imply that finitely generated Γ-separated covers of all finitely generated modules M are isomorphic? (hence isomorphic to the $\mathcal{G}_0$-cover of M.)

We have affirmative answers for the rings in Theorem 4.8, in particular, for all Dedekind-like rings. (Moreover, by Theorem 4.8, $\mathcal{G}_0$-covers coincide with the $\mathcal{G}$-covers in this case).

Also, the answer is affirmative if Λ is a DVR with the quotient field Γ. Then $\mathcal{G}_0$ ($\mathcal{G}$) is the class of all finitely generated projective modules (flat modules), so any finitely generated Γ-separated cover g of M is isomorphic to a (surjective) restriction of the projective cover f of M, hence g is isomorphic to f. (However, if Λ is not complete and M is a nonzero module of finite length, then f is not isomorphic to the $\mathcal{G}$-cover of M by Example 2.7(ii).)

3. Can the semisimplicity condition in Lemma 4.4 be weakened in any reasonable way that allows the theorems about these rings in Section 4 — especially the almost functorial property, uniqueness of separated covers, and minimal epimorphism properties — to remain true?

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