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Modules over Hereditary Noetherian Prime Rings (Survey)

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1. Introduction

By a *Dedekind domain* we mean a commutative noetherian integral domain R, in which every ideal is a projective R-module. The following two theorems — essentially due to Steinitz in 1911 — can be regarded as the ultimate generalization of the Fundamental Theorem of Abelian Groups to finitely generated modules over some class of commutative noetherian domains.

Theorem 1.1 (Decomposition). *Let R be a Dedekind domain and M a finitely generated R-module. Then:*

(i) $M = P \oplus T$ *(P a projective module, T a torsion module).*

(ii) *T is a direct sum of uniserial modules of finite length, each isomorphic to $R/\mathfrak{m}^e$ for some maximal ideal $\mathfrak{m}$ and exponent e.*

(iii) *There is a decomposition*

$$P \cong H_1 \oplus \ldots \oplus H_n \qquad (\text{each } H_i \text{ an ideal} \neq 0) \tag{1.1.1}$$

In the decomposition given in (i), T is absolutely unique, since it is the *torsion submodule* of M; that is, the set of elements $m \in M$ such that $dm = 0$ for some nonzero $d \in R$. It follows immediately that P is unique up to isomorphism, namely, $P \cong M/T$. The terms in the decomposition in (ii) are unique up to isomorphism and order of occurrence. Modern readers will recognize this as a special case of the Krull Schmidt theorem for modules of finite length (see, e.g. [**Fa**, p. 43]).

Steinitz obtained the decomposition (1.1.1) by extending the familiar matrix-diagonalization argument in one of the well-known proofs of the Fundamental Theorem of Abelian Groups. The summands H_i need not be principal ideals, and thus P turns out to be projective, but not always free. Moreover, the isomorphism classes of the individual terms are not uniquely determined. However, *the number n of terms in* (1.1.1) *is uniquely determined by the isomorphism class of P;* and is usually called the *rank* of P.

The really new ingredient in Steinitz's theorem, is his analysis of the degree of uniqueness associated with the terms in decomposition (1.1.1). To state the result

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in modern terminology, we need to introduce two ingredients: (i) an abelian group $G = G(R)$, called the *ideal class group* of R; and (ii) for each finitely generated projective R-module $P \neq 0$, an element $\mathcal{S}(P) \in G$, called the *Steinitz class* of P.

THEOREM 1.2 (Uniqueness). *Let* $H_1 \ldots, H_n$ *and* $K_1, \ldots, K_n$ *be ideals* $\neq 0$ *of a Dedekind domain* R. *Then*

$$H_1 \oplus \ldots \oplus H_n \cong K_1 \oplus \ldots \oplus K_n \iff \sum_i \mathcal{S}(H_i) = \sum_i \mathcal{S}(K_i) \tag{1.2.1}$$

By taking $n = 1$ we see that we can identify $\mathcal{S}(H_i)$ with the isomorphism class of each nonzero ideal H_i of R; and this explains the term "ideal class group". The fact that P can have a lot of essentially different decompositions is the content of our next result.

EXAMPLE 1.3. Let $\aleph$ be any cardinal number. Then — for a suitable Dedekind domain R and any positive integer $n > 1$ — *there exist at least* $\aleph$ *non-isomorphic decompositions* (1.1.1) *of any finitely generated projective* R*-module* P *of rank* n.

To see this we use the fact [**Claborn, 1966**] that every abelian group G can be the ideal class group $G(R)$ of some Dedekind domain R. Fix such an R, and let P be any finitely generated projective R-module of rank n.

We construct decompositions (1.1.1) of P as follows. Arbitrarily choose $n - 1$ nonzero ideals $H_1, \ldots, H_{n-1}$ of R. Then, since G is a group, there is an ideal $H_n \neq 0$ such that $\sum_i \mathcal{S}(H_i) = \mathcal{S}(P)$. Then Theorem 1.2 shows that $P \cong \oplus_i H_i$ as desired.

As already mentioned, Theorems 1.1 and 1.2 are essentially due to [**Steinitz, 1911**], who dealt with the situation that R is the ring of all algebraic integers in an algebraic number field. Shorter proofs were subsequently given by [**Krull, 1932**] and [**Chevalley, 1936**]. A short proof, in modern form, was given by [**Kaplansky, 1952**]. He also proved that the rings that Steinitz was dealing with have the property that all ideals are projective [**Kaplansky, 1952**, Lemma 3], although the term "projective" was not yet in use.

After the appearance of Goldie's theorems in 1958, a natural problem was try to extend these theorems to an appropriate class of noncommutative notherian rings. The remainder of this survey summarizes results about this problem, and some related problems. First we fix some notation.

NOTATION 1.4. Throughout the rest of this survey, R denotes a *non-artinian* HNP (*Hereditary, Noetherian Prime*) ring. Thus, (Hereditary:) all left and all right ideals are projective R-modules, (Noetherian:) R is left and right noetherian, and (Prime:) every product of nonzero (2-sided) ideals is again nonzero.

The term "module" usually means "right module" unless the side really matters or confusion might result.

R has a 2-sided Goldie quotient ring, which we denote by R_{quo}. The significance of our assumption that R is not artinian is that, then, $R \neq R_{\text{quo}}$ and we avoid a situation in which this paper has nothing new to say.

Why prime? Prime rings are a standard noncommutative generalization of integral domains. Moreover — although it was not known at the time interest in HNP rings arose — *every hereditary noetherian ring is a direct product of prime rings and an artinian ring* [**Mc-R**, 5.4.6]. Thus, extending our study to semiprime noetherian rings is a trivial generalization. Hereditary *artinian* rings are a very interesting class of rings, and a lot of research has been done on them. But the module-theoretic results are unrelated to the Fundamental Theorem of Abelian Groups. For example, $\mathbb{Z}/n\mathbb{Z}$ is hereditary (for $n \neq 0$) if and only if the integer n is square-free; that is, $\mathbb{Z}/n\mathbb{Z}$ is a direct product of fields.

Examples 1.5. *Classical hereditary orders* are HNP rings that are finitely generated modules over a central Dedekind domain. These rings have been studied in integral representation theory. See, for example, [**R**], [**Jacobinski, 1971**], [**C-R**, §26].

The *first Weyl algebra* $A_1 = F[x, y]$, where F is a field of characteristic 0, where the elements of F commute with x and y, and with defining relation $xy - yx = 1$. This is a simple ring and an HN domain [**Mc-R**, 1.3.5 and 7.5.3]. Moreover, A_1 is not a principal ideal domain (see e.g. [**McConnell & Robson, 1973**, 5.17, 5.18]).

Since F has characteristic 0, the center of A_1 is F; and hence A_1 is not a classical hereditary order.

For additional examples of HNP rings see [**Mc-R**, 5.5.11], [**Robson, 1972**], [**Stafford & Warfield, 1984**], [**Stafford & Warfield, 1985**].

Remarks 1.6. (i) *Easy facts.* Since our rings can have zero-divisors, it is necessary to refine the idea of a "torsion element" of a module. Thus, if $xy = 0$ for elements x, y in some ring S, and m is an element of any S-module, we have either $mx = 0$ or else $(mx)y = 0$ with $mx \neq 0$. To deal with this, we define an element m of an S-module to be a *torsion element* if $md = 0$ for some *regular element* (i.e. non-zero-divisor) of S. One then easily proves [**Levy, 1963**, 1.5]:

(1.6.1) Let S be any ring. Then the set of torsion elements of every right S-module M is a submodule of M if and only if S satisfies the right Ore condition (i.e. S has a right quotient ring). (See, e.g. [**Mc-R**, Chapter 2] for basic facts about quotient rings and Goldie's theorem.)

Now return to our HNP ring R, and let M be any finitely generated R-module. It is easy to see that $M = P \oplus T$ with P projective and T the set of torsion elements of M. Moreover, P is isomorphic to the direct sum of a *unique* number of uniform right ideals of R, where *uniform* means that the intersection of any two nonzero submodules is again nonzero [**Mc-R**, 5.7.4, 5.7.5, 2.2.9].

(ii) *Not as easy.* What is the structure of T? The starting point is that, as in the commutative case, T has finite length [**Mc-R**, 5.7.4]. For more detail about T see Section 3.

What corresponds to the uniqueness theorem 1.2 in the noncommutative case? We discuss the (quite recent) answer to this in Section 4.

(iii) *Related questions.* What can be said about infinitely generated projective R-modules? See Section 5. What happens to Prüfer's theorem — the theorem that states that every abelian group whose elements have bounded order is a direct sum of cyclic groups? See Section 6.

We include a few simple proofs that illustrate how the ideas work.

2. Ext Structure of Simple Modules

Before discussing the main topics of this survey, we need to discuss one big difference between the commutative and noncommutative cases: the Ext structure of simple modules. If R (our HNP ring) is commutative and V, W are simple R-modules, then $\operatorname{Ext}(V, W) \neq 0$ if and only if $V \cong W$. This already fails for classical hereditary orders, and what replaces it is the subject of this section.

DEFINITIONS 2.1. Let V, W be simple R-modules, *with W unfaithful.* We say that W is *the successor* of V, and V is *the predecessor* of W if $\operatorname{Ext}(V, W) \neq 0$; that is, if there exists a uniserial R-module U of length 2 whose top and bottom composition factors are V and W respectively. It is important to note that *this definition only applies if W is unfaithful.*

This terminology is justified by the following result of [**Goodearl & Warfield, 1979**, §1].

LEMMA 2.2. *Every unfaithful simple R-module W has a unique predecessor V. If a simple R-module V has a successor (unfaithful!) W, then W is unique and there is no faithful X such that* $\operatorname{Ext}(V, X) \neq 0$.

What happens when we iterate the operations of successor and predecessor? The answer is given by the following result of Lenagan, Goodearl, and Warfield. (See [**Levy & Robson, 1999a**, §3] for details, in the form needed here.)

LEMMA AND DEFINITIONS 2.3 (Towers). *For every simple R-module V, exactly one of the following is true.*

(i) *V has a successor, and iterating the successor operation eventually returns to V. (We call the resulting cyclically ordered collection of simple modules a* "cycle tower"*. The trivial case of this is a single simple module that is its own successor)*

(ii) *Iterating the predecessor operation — if it can be performed — eventually leads to a faithful simple module; and iterating the successor operation — if it can be performed — eventually leads to a simple module with no successor. (We call the resulting linearly ordered collection of simple modules a* "faithful tower"*. The trivial case of this is that V is faithful and has no successor.*

In particular, the relations of predecessor and successor partition the simple R-modules into finite (cyclically or linearly) ordered sets.

Note that every cycle tower consists of unfaithful modules, while every faithful tower contains exactly one faithful module, namely its first element. We call this first element of any faithful tower its *top,* and the last element its *base.* As already mentioned, we call a 1-element tower (cycle or faithful) *trivial.*

The reason for the name "tower" is given in the next result.

LEMMA 2.4. *Let $\mathcal{T}$ be a tower of simple R-modules.*

(i) *If $\mathcal{T}$ is a cycle tower and W is any element of $\mathcal{T}$, then there is a uniserial R-module U whose composition factors — from top to base — enumerate $\mathcal{T}$, and whose base is W.*

(ii) *If $\mathcal{T}$ is a faithful tower, then there is a uniserial R-module U whose composition factors — from top to base — enumerate $\mathcal{T}$ from top to base.*

In both situations, U is unique up to isomorphism.

We call U a uniserial module *associated with $\mathcal{T}$.* Thus, if $\mathcal{T}$ is an n-element cycle tower, it has n nonisomorphic associated uniserial modules. But if $\mathcal{T}$ is a faithful tower, then it has only one associated uniserial module.

Remarks 2.5. (i) If R is a classical hereditary order, then all towers are cycle towers, and R has only finitely many nontrivial towers. (See [**Levy & Robson, 2000**, Lemma 4.2] for details.)

(ii) For $R = A_1$, the first Weyl algebra, only trivial, necessarily faithful towers occur, because all nonzero modules over any simple ring are faithful.

(iii) [**Stafford & Warfield, 1984**] and [**Stafford & Warfield, 1985**] show that it is possible for R to have towers of both types (cycle and faithful), even infinitely many nontrivial towers.

(iv) HNP rings can have uniserial modules other than those whose composition factors are segments of towers. But such uniserial modules never occur in classical hereditary orders because all simple modules belong to cycle towers; in particular, they are unfaithful.

For example, the HNP ring A_1 has uniserial modules of length 2 [**McConnell & Robson, 1973**, 5.7]. But since all nonzero modules over the simple ring A_1 are faithful, all A_1-towers are trivial.

We remark that uniserial modules whose sequence of composition factors do not enumerate segments of towers are not necessarily uniquely determined up to isomorphism by their sequence of composition factors [**McConnell & Robson, 1973**, 5.8].

In this connection, it seems worthwhile to mention a related result about finite dimensional algebras. [**B. Huisgen-Zimmermann, 1998**] gives a family of finite dimensional algebras such that each family of uniserial modules with given composition factors can be parameterized by an algebraic variety. It would be interesting to know whether such parameterizations occur for HNP rings, too.

3. Torsion Modules

Lemma 3.1. *Let V, W be simple R-modules. If V is an element of a cycle tower and W is an element of a faithful tower, then* $\mathrm{Ext}(V, W) = 0$ *and* $\mathrm{Ext}(W, V) = 0$.

Proof. Falseness of this lemma would contradict the partition of simple modules into cycle towers and faithful towers [Lemma 2.3 or 2.2]. □

It follows easily that, for every indecomposable R-module T of finite length, either all composition factors of T belong to cycle towers, or else all belong to faithful towers. Combining this with the Krull-Schmidt theorem for modules of finite length immediately yields the following result [**Kuzmanovitch, 1972**, 2.19] (or see [**Klingler & Levy, 1995**, 4.6]):

Theorem 3.2. *Every R-module T of finite length has a decomposition $T = X \oplus Y$ where all composition factorss of X belong to cycle towers and all composition factorss of Y belong to faithful towers.*

Thus we see the first surprise: there are *two* theories of torsion modules, one for the X term and one for the Y term. These theories turn out to be quite different, as we proceed to explain. The X term is quite classical in nature, and is most easily described in terms of its indecomposable summands. (See [**Klingler & Levy, 1995**, 4.6]).

Theorem 3.3 (Structure of X). *Let X be an indecomposable R-module of finite length whose composition factors all belong to cycle towers. Then X is uniserial, and its sequence of composition factors, from top to base enumerate a segment of* *a cycle tower (in the same order that they occur in the tower). Moreover, X is determined up to isomorphism by its sequence of composition factors.*

Structure of Y (All composition factors in faithful towers). This is more complicated than X. Eisenbud, McConnell, and Robson already knew the following facts in the early 1970's.

(3.3.1) (i) Every A_1-module of finite length (indecomposable or not!) is cyclic. [**Mc-R**, Lemma 5.7.3(c)]
(ii) Indecomposable A_1-modules of finite length are not necessarily uniserial. [**McConnell & Robson, 1973**, 6.2]
(iii) There exist HNP rings that have indecomposable modules of finite length requiring 2 generators. [**McConnell & Robson, 1973**, 6.3]

Thus one can ask whether Steinitz's theorem falls apart for torsion modules in noncommutative case, when R has faithful towers. The answer is "partly yes", and is contained in the following pair of results that I like to call the "Good News, Bad News" theorems.

THEOREM 3.4 (Good News). *Let Y be an R-module of finite length (indecomposable or not!) whose composition factors all belong to faithful towers. Then Y is a homomorphic image of a uniform right ideal of R.* [**Klingler & Levy, 1995**, 4.6]

Thus Y (as well as the direct sum of 1,000 copies of Y!) is always a *submodule* of a cyclic module. Therefore the following obvious consequence of Theorems 3.2–3.4 shows that the spirit of the Fundamental Theorem of Abelian Groups survives.

COROLLARY 3.5. *Every R-module of finite length is a direct sum of uniserial (hence cyclic) modules of finite length and a homomorphic image of a uniform right ideal of R.*

THEOREM 3.6 (Bad News). *The category of A_1-modules of finite length has wild representation type.* [**Klingler & Levy, 1995**, 2.13]

Wild representation type has been studied — in the context of finite dimensional algebras — for over 20 years, but is not well-known to noetherian ring theorists. Moreover, its definition is not completely standard. So a discussion of this is in order. We begin with an informal definition.

Recall that A_1 is an F-algebra. The basic idea is:

(3.6.1) A_1 has so many modules of finite length that any complete description of all A_1-modules of finite length would have to contain a description of all isomorphism classes of finite dimensional modules over *all finite dimensional F-algebras.*

Our formal definition requires notion of the *first strongly inaccessible cardinal* $\aleph_i$. We shall not define this (but see [**Klingler & Levy, 1995**, 1.1]), except to say that this cardinal is not known to exist; and — if it exists — is so large that all algebras familiar to readers of this survey are likely to be generated by $< \aleph_i$ elements. In particular, every cardinal of the form $2^{\aleph_0^{\cdot^{\cdot^{\aleph_0}}}}$ (any finite number of $\aleph_0$'s) is $< \aleph_i$.

A *full exact embedding* $\Phi\colon \mathcal{S} \to \mathcal{A}$ of module categories is an additive functor Φ such that: (i) $M \cong N$ in $\mathcal{S} \iff \Phi(M) \cong \Phi(N)$ in $\mathcal{A}$; (ii) Φ is an isomorphism on hom groups; and (iii) Φ takes exact sequences in $\mathcal{S}$ to exact sequences in $\mathcal{A}$. These conditions are easily seen to imply that an object $M \in \mathcal{S}$ is indecomposable if and only if $\Phi(M)$ is indecomposable in $\mathcal{A}$.

Caution. A_1 has no finite dimensional modules $\neq 0$, because A_1 is a simple ring and an infinite dimensional algebra.

THEOREM 3.7 (Wildness of A_1). *Let S be an F-algebra generated by $< \aleph_i$ elements. Then:*

(i) *There exists a full exact embedding Φ_S: Mod-$S \to$ Mod-A_1.*
(ii) *If S is finitely generated (as an algebra) then Φ_S carries finite dimensional S-modules to A_1-modules of finite length.*
(iii) *Every nonzero A_1-module in $\operatorname{im}(\Phi_S)$ has socle-height 2; that is, if $X \in \operatorname{im}(\Phi_S)$, then $X/\operatorname{soc}(X)$ is a semisimple A_1-module.*

Discussion. This is the strongest of several common definitions of wildness. To see how (3.6.1) manifests itself in this notion of wildness, choose your favorite finite dimensional F-algebra S. Then, by (i) and (ii) of the theorem, two finite dimensional S-modules M, N are isomorphic to each other if and only if the A_1-modules $\Phi_S(M)$ and $\Phi_S(N)$ of finite length are isomorphic. The significance of statement (iii) is that the tiny subcategory of $\operatorname{mod}(A_1)$ consisting of modules of socle-height 2 is sufficient to express this wildness.

We give several applications of this wildness theorem, and we include enough of their proofs to illustrate the use of wildness. Note that semisimple modules are, in a sense, as far as possible from being uniserial, because the composition factors of a semisimple module can be made to appear in all possible orders. We want to show that indecomposable A_1-modules of finite length can be as far as possible from uniserial. Of course, semisimple modules are never indecomposable except if they are simple. This motivates the next result.

COROLLARY 3.8. [**Klingler & Levy, 1995**, 3.1] *Let n be a positive integer. Then there is an indecomposable A_1-module X of socle-height 2 and finite length, such that both socle layers have composition length $> n$.*

Partial proof. Let $S = F[t]/(t^n)$, where is an indeterminate. Then S is a finite dimensional F-algebra and hence is generated by $< \aleph_i$ elements. (Actually, S is generated by 1 element!) Moreover, S is a uniserial S-module of length n. In particular, S is an indecomposable S-module. Therefore $X = \Phi_S(S)$ is an indecomposable A_1-module of socle-height 2.

Since Φ_S is an exact functor, it preserves monomorphisms. It follows that the length of the A_1-module X is $\geq n$, the length of the S-module S. Therefore at least one of the socle layers of X has length $\geq n/2$. This establishes the spirit of the corollary, but is not strong enough to prove the corollary as stated. For that one must use the quite explicit definition of Φ_S to see that the lengths of the socle layers of X are $3n$ and $5n$ [**Klingler & Levy, 1995**, 2.16], both of which are $> n$. ☐

The next corollary shows the enormous number of A_1-modules that exist.

COROLLARY 3.9. [**Klingler & Levy, 1995**, 2.14] *Every finite dimensional A_1-algebra is the endomorphism algebra of some A_1-module of finite length (and socle-height 2).*

PROOF. Let S be the given finite dimensional A_1-algebra. Then the endomorphism algebra of the S-module S_S is the *ring* S. Since, for any S-module M, the endomorphism ring of M is isomorphic to the endomorpihism ring of $\Phi(M)$, the A_1-module that we want is $\Phi_S(S)$. $\square$

This corollary supplements an earlier result by [**Quebbemann, 1979**], extended by [**Farkas & Snider, 1981**], which states that every division F-algebra is the endomorphism algebra of some simple A_1-module.

How far can indecomposable torsion R-modules get from being cyclic?

Corollary 3.10. [**Klingler & Levy, 1995**, 3.2] *There exists an HNP ring R such that $(\forall n)$ R has an indecomposable module of finite length and requires $> n$ generators (and has socle-height 2).*

The HNP ring in the above corollary cannot be A_1 since all A_1-modules of finite length are cyclic. The ring R actually used is the idealizer of a suitable maximal right ideal of A_1. The proof is very similar to that of Corollary 3.8.

Although the functors Φ_S that display the wildness of A_1 faithfully preserve isomorphism classes and indecomposability, and preserve exact sequences, they are quite far from Morita equivalences. For example, they do not preserve the lattice of submodules of a module. In fact, they mangle it quite badly, by converting all modules to modules of socle-height 2 — a fact we exploited in the proof of Corollary 3.8. In particular, these functors do not seem to answer the following seem to answer the following question, which I find intriguing: Let L be the lattice of submodules of some finite dimensional module over some finite dimensional F-algebra. Is L the lattice of submodules of some A_1-module of finite length? This is related to the more general question: Are there better functors that display the wildness of A_1?

4. Finitely Generated Projective Modules

Recall that every finitely generated projective R-module P is isomorphic to a direct sum

$$P \cong H_1 \oplus \dots H_n \tag{4.0.1}$$

of a unique number of uniform right ideals H_i of R. What remains to be discussed is when two such direct sums are isomorphic to each other; that is, what becomes of Steinitz's formula (1.2.1) in the noncommutative case? Recall that A_1 is an HN domain but not a principal ideal domain [Examples 1.5]. Any nonprincipal right ideal is a nonfree projective R-module. However:

Example 4.1 ([**Webber, 1970**], or see [**Mc-R**, 7.11.7]). *For every integer $n \geq 2$, the direct sum of n nonzero (necessarily uniform) right ideals of A_1 is free of rank n.*

At first sight, this seems to show that Steinitz's theorem fails completely in the noncommutative case. On the other hand, R has Krull dimension 1; and the spirit of algebraic K-theory is that one should not expect good behavior for modules that are "too small".

Definitions 4.2. Let P be a finitely generated projective R-module $\neq 0$. We define two invariants of the isomorphism class of P, its "genus" and its "Steinitz class".

The *rank* of P at an unfaithful simple R-module W — denoted by $\rho(P, W)$ — is the largest integer n such that there exists an R-linear map of P onto W^n. If P cannot be mapped onto W, we set $\rho(P, W) = 0$.

The reason that we require W to be unfaithful is that, otherwise, P can be mapped onto W^n for every n [**Mc-R**, Corollary 5.7.3].

As is customary, we often denote $\text{udim}(P)$, the *uniform dimension* of P, by $\rho(P, 0)$. In our situation (HNP rings) this equals the integer n in (4.0.1). Thus one can consider the rank function — for any given P — to be a function from the set of prime ideals of R to the nonnegative integers. (This statement makes implicit use of the fact that the only prime ideals of R are the maximal ideals and zero. Moreover, for every maximal ideal M, the ring R/M is artinian. These facts are consequences of [**Mc-R**, 5.4.5].)

In the commutative case, $\rho(P, W) = \rho(P, 0)$ for every P and every W, and is usually called the "rank" of P.

We let *genus*(P) denote the indexed family of integers $\rho(P, W)$ together with $\rho(P, 0) = \text{udim}(P)$. For projective modules over classical orders, this coincides with the notion of "genus" used in integral representation theory [**Levy, Robson, & Stafford, 1994**, 5.1].

Since every element of $K_0(R)$ is represented by a difference of projective modules, we can extend the genus function to a function from $K_0(R)$ to the integers by setting the genus of $[P] - [Q]$ to be $\text{genus}(P) - \text{genus}(Q)$. This is well-defined because the genus function is additive on direct sums.

Let $G = G(R)$ — the *ideal class group* of R — denote the set of elements of genus 0 in $K_0(R)$. Thus, for $[P]-[Q] \in G(R)$ we have $\rho(P, W) = \rho(Q, W)$ for every W, and $\text{udim}(P) = \text{udim}(Q)$. It is easy to see that, in the commutative case, this definition is equivalent to any of the usual ones.

Finally, for each P we define an element $\mathcal{S}(P) \in G(R)$ — the *Steinitz class* of P. Unfortunately the map $P \to \mathcal{S}(P)$ is not a natural map. We postpone the details of its definition to Remarks 4.6(iii). For the moment it suffices to note: *$\mathcal{S}(\dots)$, like genus, is additive on direct sums.*

We can now state the main result.

THEOREM 4.3. [**Levy & Robson, 1999b**, 1.3] *Suppose* $\text{udim}(P) \geq 2$. *Then a full set of independent invariants for the isomorphism class of P is:*

(i) *genus(P), and*
(ii) *$\mathcal{S}(P)$.*

As already mentioned, Webber's result shows that the hypothesis $\text{udim}(P) \geq 2$ cannot be deleted. We note:

COROLLARY 4.4. [**Levy & Robson, 1999b**, 3.13] *$P \oplus X \cong Q \oplus X$ with* $\text{udim}\, P \geq 2 \implies P \cong Q$. *($P, Q, X$ finitely generated projective R-modules)*

Dishonest proof. This is an immediate consequence of Theorem 4.3 since rank and Steinitz class are additive in direct sums. ☐

What is dishonest is that, although the Corollary is an immediate consequence of the theorem, the proof of the corollary is one of the main parts of the proof of the theorem given in [**Levy & Robson, 1999b**].

We note that the main difference between Corollary 4.4 and Stafford's cancellation theorem — in the context of HNP rings — is that Stafford's result also assumes that certain ranks $\rho(P, W)$ are sufficiently large. (For a version of

Stafford's cancellation theorem tailored to the needs of Krull dimension 1, see [**Levy, Robson, & Stafford, 1994**, Proposition 5.5].) However, Stafford's result is a main ingredient in the proof of Theorem 4.3 and its corollary. A theorem equivalent to 4.3 was proved by [**Jacobinski, 1971**] for classical hereditary orders whose underlying Dedekind domain is a ring of algebraic integers. The cancellation corollary was then extended by [**Odenthal, 1989**] to the situation that R is any HNP ring with "enough invertible ideals".

Remarks 4.5 (Structure of a genus). To complete the statement of the main theorem, one has to know which indexed collections of nonnegative integers can occur as the genus of some P. Let Ψ be a collection of nonnegative integers indexed by $\mathcal{W}\cup\{0\}$, where $\mathcal{W}$ is the collection of (isomorphism classes of) simple R-modules. Thus Ψ_W is a nonnegative integer for every $W \in \mathcal{W}$, and Ψ_0 is a nonnegative integer. Then:

(4.5.1) Ψ is the genus $\Psi(P)$ of some finitely generated projective R-module $P \neq 0$ if and only if Ψ satisies the two conditions "almost standard rank" and "cycle standard rank", defined below. [**Levy & Robson, 1999b**, 2.16]

We say that Ψ has *almost standard rank* if $\Psi_0 \neq 0$ and

$$\Psi_W = \frac{\Psi_0}{\operatorname{udim}(R)} \cdot \rho(R, W) \tag{4.5.2}$$

for all but possibly finitely many $W \in \mathcal{W}$.

We say that Ψ has *cycle standard rank* if, for every cycle tower $\mathcal{C}$ of simple R-modules, we have

$$\sum_{W\in\mathcal{C}} \Psi_W = \frac{\Psi_0}{\operatorname{udim}(R)} \cdot \sum_{W\in\mathcal{C}} \rho(R, W) \tag{4.5.3}$$

We note that, once almost standard rank is satisfied, it suffices to check cycle standard rank at the finite number of cycle towers at which some Ψ_W is nonstandard.

Remarks 4.6 (Dedekind closure, Steinitz class).

(i) Recall that a finitely generated projective module $P \neq 0$ is a progenerator — a generator in mod-R — if and only if P can be mapped onto every simple R-module. For commutative Dedekind domains, all nonzero projective modules are generators. But this is not true for noncommutative HNP rings. In fact, *If R is an HNP ring with at least one nontrivial tower then some essential right ideal H is not a progenerator.*

The proof of this makes a simple illustration of the ideas in the previous Remarks. Let $\mathcal{C}$ be a nontrivial tower. Construct a genus Ψ (indexed family of integers) as follows. Start with the genus of R, and alter it as follows. Choose one $W \in \mathcal{C}$ and set $\Psi_W = 0$. Since we have made only one change, Ψ has almost standard rank. If $\mathcal{C}$ is a cycle tower, then cycle standard rank no longer holds in $\mathcal{C}$. So make one further change: increase some Ψ_V, $W \neq V \in \mathcal{C}$, by $\rho(R, W)$, thus restoring cycle standard rank in $\mathcal{C}$. Then Ψ is the genus of some H. Moreover, H cannot be mapped onto W since $\rho(H, W) = \Psi_W = 0$, and therefore H is not a progenerator. Finally, since $\operatorname{udim}(H) = \operatorname{udim}(R)$, H is isomorphic to an essential right ideal of R.

Note that this construction works even if R is a classical hereditary order.

(ii) An HNP ring R is called a *Dedekind prime ring* if every nonzero right ideal is a progenerator. A basic result in integral representation theory is that every classical hereditary order R is contained in a Dedekind prime ring S whose Goldie quotient ring equals R_{quo} and such that the module S_R is finitely generated. Such an S is called a *maximal order* containing R. The reason that maximal orders are important is that their module theory is much simpler than that of general orders. A standard way of studying module questions, in integral representation theory, is to first answer the question for maximal orders, and then use that "approximation" to find the answer to the original question.

For several years it remained an open question whether an arbitrary HNP ring R is contained in a Dedekind prime ring S in R_{quo} such that S_R is finitely generated. Finally [**Stafford & Warfield, 1984**] and [**Stafford & Warfield, 1985**] showed that such a Dedekind prime ring need not exist. However, quite soon after that, [**Hodges, 1987**] — using a torsion-theoretic construction of [**Goodearl, 1974**] — showed that R is always contained in a Dedekind prime ring S in R_{quo} that is *right integral* over R in the following sense. For every finite set $s_1, \dots, s_n$ of elements of S, the ring $R[s_1, \dots, s_n]$ generated by R and the s_i is a finitely generated right R-module. He calls S a *Dedekind closure* of R. ("Right Dedekind closure" would be more precise, since right Dedekind closures need not be left Dedekind closures.) Note that Hodges does not call this *the* Dedekind closure of R, because R can have more than one such closure. See [**Levy & Robson, 1999a**] for a detailed treatment of Dedekind closures.

Hodges then proved (in our notation) that the natural map $G(R) \to G(S)$ — S any Dedekind closure of R — is a bijection. [**Jacobinski, 1971**] had proved this for hereditary orders over the ring of integers in a number field. Hodges then used this to prove that $K_0(R) \cong \ker(\Psi) \oplus \operatorname{im}(\Psi)$, where Ψ is the genus homomorphism from $K_0(R)$ to the direct product of copies of $\mathbb{Z}$. This, together with Odenthal's cancellation result for the "enough invertible ideals" case suggested to Robson and me that Theorem 4.3 is true. In part (iii) below, we give the actual definition of $\mathcal{S}(P)$.

(iii) Choose and fix a Dedekind closure S of R and a uniform right ideal U_0 of S. Then set

$$\mathcal{S}(P) = [P \otimes_R S]_S - [U_0^{(\operatorname{udim} P)}]_S \in G(S) \quad (\cong G(R)) \tag{4.6.1}$$

Here $[\dots]_S$ denotes "natural image in $K_0(S)$". The purpose of moving to S is that, over Dedekind prime rings, all ranks of projective modules are standard, hence determined by their udim. Consequently the difference in (4.6.1) is an element of $K_0(S)$ all of whose ranks are zero; that is, an element of $G(S)$, as indicated. Since $G(S) \cong G(R)$, we can think of $\mathcal{S}(P)$ as an element of $G(R)$. However, the technical details become easier if we just abandon $G(R)$ and work with $G(S)$ instead.

Note that $\mathcal{S}(\dots)$ is obviously additive on direct sums.

One interesting application of the main theorem is the analysis of the number of generators needed for right ideals of R. In the commutative case, every ideal is generated by two elements, and this remains true for Dedekind prime rings [**Mc-R**, 5.7.7]. This is no longer true for general HNP rings. If R is a classical hereditary order, it is easy to show that there is an integer n such that every right ideal is generated by n elements; and that, given any n, some R has a right ideal that requires n generators.

[**Stafford & Warfield, 1985**, comments below Example B] observe that if the cycle-tower lengths of R are unbounded, then no bound exists for the number of generators required for right ideals of R. [**Klingler & Levy, 1995**, 4.9] give an example of an HNP ring R that has no cycle towers, has exactly one nontrivial faithful tower, and again no bound exists. The full answer to the question is that lack of a bound is quite common, and it is determined by the structure of R-towers:

Theorem 4.7. *R has right ideals requiring arbitrarily many generators if and only if:*

(i) *R has at least one nontrivial faithful tower; or*
(ii) *The lengths of cycle towers are unbounded; or*
(iii) *All towers are cycle towers, and the ratios $\rho(R,V)/\rho(R,W)$ — where the tower containing V is the same as the tower containing W — are unbounded.*

Moreover, all three possibilities can occur.

Proof. All of the steps of the proof — but not the theorem itself — are contained in [**Levy & Robson, 1999b**]. So we put the pieces together here, in a proof that readers should feel free to skip!

Possibility (i) is stated in Theorem 5.3(i) of that paper, and possibility (ii) is stated in Corollary 5.5. Thus we may assume that all towers are cycle towers, and the lengths of these towers are bounded. Then possibility (iii) follows immediately from Theorem 5.3(ii) of that paper.

To see that possibility (i) can actually occur, begin with any HNP ring S that has a faithful simple module (e.g. the simple HNP ring A_1). Let N be any maximal right ideal of S such that S/N is faithful, and let R be the idealizer of N. Then R is again an HNP ring [**Mc-R**, 5.5.8]; and R has exactly one nontrivial tower, a faithful tower consisting of two simple modules. (See e.g. [**Levy & Robson, 1999a**, Lemma 5.1].)

The existence of possibility (ii) is proved in [**Stafford & Warfield, 1985**, Theorem 2.2]; and the existence of possibility (iii) is proved in [**Levy & Robson, 1999b**, Example 5.6]. □

In those situations where a bound exists, the best possible bound is determined by the structure of R-towers. See [**Levy & Robson, 1999b**, 5.3] for the actual formula.

We close this section with some questions.

Question 1. Dedekind closures of HNP rings are not uniquely determined subrings of R_{quo} [**Levy & Robson, 1999a**, 7.13(iii)]. In fact they are not even unique up to isomorphism [**Levy & Robson, 1999a**, 7.15]. However, when they are finitely generated right modules over the original HNP ring, then any two Dedekind closures are Morita equivalent. The proof (with the hypothesis not stated exactly this way) is given in [**Mc-R**, 5.2.13(ii)]. Does this Morita equivalence still hold when the Dedekind closure is not a finitely generated right module over the original HNP ring?

Question 2. Do more general noetherian prime rings of Krull dimension 1 have a “normalization” (not necessarily unique) — analogous to a Dedekind closure — that is a Dedekind prime ring? The answer is “yes” if the ring is module-finite over a central noetherian ring of Krull dimension 1; and any two such normalizations are Morita equivalent [**Levy & Odenthal, 1996**, (4.1.3),(4.1.7)]. If the general answer is “yes”, then Dedekind prime rings would play a central role in the study of noetherian prime rings of Krull dimension 1, as they do in integral representation theory and in the commutative case.

Question 3. Suppose that R has only one simple (right) module. Does it have any properties of a local ring? For example, is every finitely generated projective R-module P isomorphic to H^n for some uniform right ideal H and some n? [**Levy & Robson, 1999b**, 5.8] prove that, $G(R)$ is a cyclic group of finite order, for the rings we are discussing. If this order equals 1, then the answer to the question would be "yes" whenever $\operatorname{udim}(P) \geq 2$.

5. Infinitely Generated Projective Modules

In this section we very briefly discuss infinitely generated projective modules over our HNP ring R. *Infinitely generated* always means "not finitely generated", and *uncountably generated* means "not countably generated".

[**Kaplansky, 1952**] proved that, in the commutative case, every infinitely generated projective R-module is free. Then [**Bass, 1963**] proved the surprising generalization that, over any noncommutative noetherian ring S, every *uniformly big* projective module P is free. "Uniformly big" means that, for every maximal ideal $\mathfrak{m}$, $P/P\mathfrak{m}$ requires the same infinite number of generators that P requires. There does not seem to have been a lot of followup research to this.

Over noncommutative HNP rings R, infinitely generated projective modules are not necessarily uniformly big, as we shall see. So, the fact that finitely generated projective R-modules are now quite well-understood provided Robson and me with the motivation to ask about infinitely generated projective R-modules P that are not necessarily uniformly big. The starting point is that there is always a decomposition:

$$(5.0.1) \quad P = \oplus_{i\in I} P_i \qquad \begin{cases} \text{each } P_i \text{ isomorphic to an essential right ideal of } R; \\ \text{and } I \text{ an infinite set.} \end{cases}$$

As expected, the ideal class group becomes irrelevant to the structure of P. However, the ranks $\rho(P, W)$, which are now cardinals, obviously remain isomorphism-invariants of P, and so does $\operatorname{udim}(P) = \rho(P, 0) = |I|$. In other words, the genus of P remains an isomorphism-invariant of P. In fact, we have:

Theorem 5.1. [**Levy & Robson, 2000**, 1.3] *$P \cong Q$ (infinitely generated projective R-modules) $\iff$ $\operatorname{genus}(P) = \operatorname{genus}(Q)$.*

Example 5.2. We ilustrate the theorem by constructing an example of *an infinitely generated P that is not free,* hence (by Bass's theorem) not uniformly big.

Let R be any HNP ring that has at least one nontrivial cycle tower $\mathcal{C}$ (for example any classical hereditary order that is not a maximal order). Fix $W \in \mathcal{C}$. We claim that there is a uniform right ideal H such that $\rho(H, W) = 0$ and $\rho(H, X) \neq 0$ for every unfaithful simple $X \ncong W$.

To see this, note first that $\rho(R, X) \neq 0$ for every unfaithful simple X. Therefore it suffices, by (4.5.1), to show that we can obtain the genus of H from that of R without violating either cycle standard rank or almost standard rank. Cycle standard rank can be preserved by increasing the the rank at some $X \in \mathcal{C}$ by $\rho(R, W)$ (to compensate for decreasing the rank at W be that amount). Since this changes only two ranks, almost standard rank is preserved, and the claim is proved.

Let $P = H^\alpha$, the direct sum of some infinite number α of copies of H. Since $\rho(P, W) = 0$, P is neither free nor uniformly big, as desired.

What is perhaps surprising is that P *is a free module over some HNP ring* T *such that* $R \subset T \subset R_{\text{quo}}$ *and* T_R *is finitely generated.*

It is proved in [**Levy & Robson, 1999a**, 7.17] that, for any finite collection $\mathcal{F}$ of unfaithful simple modules containing no entire cycle tower, there is a unique overring T such that $R \subset T \subset R_{\text{quo}}$, T_R is finitely generated, and $\rho(T, Y) = 0$ if and only if $Y \in \mathcal{F}$. Let $\mathcal{F}$ consist of the single simple module W. We claim that $P \cong T^\alpha$ as an R-module.

By Theorem 5.1 it suffices to check that P and T^α have the same rank at every unfaithful simple module and the same udim. Clearly $\rho(P, W) = 0 = \rho(T^\alpha, W)$. At any other unfaithful simple module, the ranks of H and T are finite and nonzero. Therefore both $P = H^\alpha$ and T^α both have rank α at X. Also, since H and T have finite nonzero udim and α is infinite, we have $\text{udim}(P) = \alpha = \text{udim}(T^\alpha)$, completing the proof.

The fact that the nonfree P constructed above is free over some overring of R is no peculiarity, as the next two results show. Again we use the notation S^α to denote the direct sum of α copies of S.

Theorem 5.3. [**Levy & Robson, 2000**, 1.1] *Let* R *be a classical hereditary order and* P *a countably generated projective* R*-module. Then* $P \cong H \oplus S^{\aleph_0}$ *where* H *is a finitely generated projective* R*-module and* S *is a ring with* $R \subseteq S \subset R_{\text{quo}}$ *such that* S *is a finitely generated right* R*-module.*

Theorem 5.4. [**Levy & Robson, 2000**, 1.2] *Let* P *be an uncountably generated projective module over any HNP ring* R*. Then*

$$P \cong H \oplus S_1^{\alpha_1} \oplus S_2^{\alpha_2} \oplus \dots \tag{5.4.1}$$

where H *is a finitely generated projective* R*-module, the* α_n *are infinite cardinal numbers satisfying* $\alpha_1 < \alpha_2 < \dots$ *with at least one* α_i *uncountable, and* $S_1 \subset S_2 \subset \dots$ *is a finite or countably infinite sequence of rings, each a finitely generated right* R*-module with* $R \subseteq S_n \subset R_{\text{quo}}$.

Thus the spirit of Kaplansky's and Bass's results survives in an unexpected way over HNP rings.

Remarks 5.5. The following facts complete the description of infinitely generated projective modules over HNP rings.

(i) What indexed collections Ψ of cardinal numbers can occur as the genus of an infinitely generated P? The answer is simple, though somewhat technical.

> Let Ψ be a collection of cardinal numbers indexed by $\mathcal{W} \cup \{0\}$, where $\mathcal{W}$ is the set of isomorphism classes of unfaithful simple R-modules. Then Ψ is the genus of an infinitely generated projective P if and only if it satisfies the following conditions.
> (a) Ψ_0 is infinite, and $\Psi_0 \geq \Psi_W$ for all $W \in \mathcal{W}$.
> (b) (bi) For every cardinal number $\alpha < \Psi_0$, there are only finitely many $W \in \mathcal{W}$ such that $\Psi_W \leq \alpha$.
>   (bii) For every positive integer n there are only finitely many $W \in \mathcal{W}$ such that $\Psi_W \leq n \cdot \rho(R, W)$.
> (c) In each cycle tower $\mathcal{C}$ there is at least one member W such that $\Psi_W = \Psi_0$.

(ii) The previous two theorems do not describe countably (infinitely) generated projective modules P over those HNP rings that are not classical hereditary orders. We do not have a canonical form to display such P. However, part (i) of these Remarks specifies the structure of genus(P), and Theorem 5.1 states that this determines the isomorphism class of P.

Example 5.6, below, constructs a countably generated P that does not have the form displayed in the previous two theorems. It is deliberately constructed in such a way as to suggest a number of tempting variations, none of which has led Robson and me to a canonical form for P!

(iii) Let $\alpha > \beta$ be uncountable cardinal numbers, and let S be a ring such that $R \subset S \subset R_{\text{quo}}$ and S_R is finitely generated. Then the projective right R-module $P = R^\alpha \oplus S^\beta$ is not displayed in the canonical form of Theorem 5.4. However, comparing ranks easily shows that the genus of P equals the genus of R^α, and therefore $P \cong R^\alpha$.

Example 5.6. We construct a countably generated P, over a suitable R, with the following property: There is no ring S, $R \subseteq S \subset R_{\text{quo}}$, and no infinite cardinal number α, such that the R-module S^α is isomorphic to a direct summand of P.

[**Levy & Robson, 2000**, 3.5] describes an HNP ring R with an infinite sequence $W_1, W_2, \ldots$ of unfaithful simple modules and an infinite sequence of rings $R = S_1 \subset S_2 \subset \ldots \subset R_{\text{quo}}$ such that each ring S_n is a finitely generated right R-module and:

$$\rho(S_n, W_i) = 0 \iff i < n \tag{5.6.1}$$

Let $P = \oplus_n S_n$, considered as a right R-module. Finite generation of each S_n shows that every $\rho(S_n, W_i)$ is finite. It follows from this and (5.6.1) that

$$\rho(P, W_i) \quad \text{is finite for every } i. \tag{5.6.2}$$

To see that P has the claimed property, choose any S and α; and suppose that S^α is isomorphic to a direct summand of P. Since rank is additive in direct sums, we have $\rho(S^\alpha, W_i) \le \rho(P, W_i)$ for every W_i.

Case 1: S_R is finitely generated (the crux of the proof). Since R has nonzero rank at every W_i, "almost standard rank" shows that $\rho(S, W_i) = 0$ for only finitely many i, and therefore $\rho(S^\alpha, W_i)$ is infinite for almost all i, contradicting (5.6.2).

Case 2: S_R is infinitely generated. Here it suffices to prove that S_R is not projective. If it were projective, it would have a decomposition as displayed on the right-hand side of (5.0.1). Since $R \subseteq S \subseteq R_{\text{quo}}$, we see that udim($S$) equals udim($R$), which is finite. Therefore only a finite number of terms occur on the right-hand side of (5.0.1). This is a contradiction since the index set I in (5.0.1) is infinite. □

We remark that every uncountably generated P is isomorphic to a proper direct summand of itself; that is $P \cong P \oplus X$ with $X \neq 0$. (This is an immediate consequence of Theorem 5.4. But this is not true when P is countably generated. For example, we claim that the module P in Example 5.6 is not isomorphic to a proper direct summand of itself.

Proof. Suppose that $P \cong P \oplus X$ with $X \neq 0$, and let $\Psi = \text{genus}(X)$. Since $\rho(P, W_i)$ is finite for every i we see that $\rho(X, W_i) = 0$ for all of the infinitely many W_i.

Case 1: X is infinitely generated, and hence $\Psi_0 = \aleph_0$. Then the infinitely many W_i such that $\rho(X, W_i) = 0 < 1 < \aleph_0$, contradict condition (bi) of Remarks 5.5(i).

Case 2: X is finitely generated. Then the infinitely many i such that $\rho(X, W_i) = 0$ contradict almost standard rank of X. □

It is instructive to note where the preceding proof fails when P is uncountably generated (and hence udim(P) is uncountable): *If P is uncountably generated, then there are only finitely many W such that $\rho(P, W)$ is finite.* This follows from condition (bi) of Remarks 5.5(i) if we take $\alpha = \aleph_0$ and remember that $\Psi_0 =$ udim(P), which is uncountable.

6. Prüfer's Theorem

Prüfer's theorem states that every abelian group whose elements have bounded orders is a direct sum of groups of prime power orders. In particular, the Krull-Schmidt theorem holds for decompositions of such groups. We can restate the "bounded order" hypothesis in the following way, that makes sense over all HNP rings.

Hypothesis 6.1. *Let P be an infinitely generated R-module of finite socle-height.*

Theorem 6.2. [**Klingler & Levy, 1995**, 3.4] *Let R be an HNP ring with no faithful towers (e.g. any classical hereditary order). Then every P satisfying Hypothesis 6.1 is a direct sum of uniserial modules of finite length, and hence its decompositions satisfy the Krull-Schmidt theorem.*

To see how badly this can fail for other HNP rings it suffices to restrict our attention to A_1-modules of socle-height 2. These examples furnish further instances of the power of wild representation type. First we demonstrate the failure of Krull-Schmidt for these modules.

Example 6.3. *Direct-sum cancellation fails for (infinitely generated) A_1-modules of socle height 2.*

Proof. Recall that the HNP ring $R = A_1$ has a nonprincipal right ideal X [Examples 1.5] and, by Webber's result [Example 4.1],

$$X \oplus R \cong R \oplus R \tag{6.3.1}$$

Note that A_1 is generated by two elements as an F-algebra. Since all finite cardinals are less than the first strongly inaccessible cardinal, we can take $S = A_1$ in Theorem 3.7. Applying the wildness functor Φ_S to (6.3.1) converts the S-modules X and R to R-modules of socle-height 2, thus furnishing the desired example. □

A module is called *superdecomposable* if it has no indecomposable direct summands. Our final parlor trick shows how far A_1-modules can be from direct sums of uniserial modules:

Example 6.4. *There exist superdecomposable modules of socle-height 2 over A_1.*

Proof. A_1 is an F-algebra, for some field F. Moreover there exist F-algebras S that contain no primitive idempotent elements. For example, let S be the direct product of infinitely many copies of F modulo the direct sum of these copies of F.

Let $M = \Phi_S(S_S)$ where Φ_S is the wildness functor in Theorem 3.7. Then M is an R-module of socle-height 2 whose endomorphism ring is isomorphic to the S-endomorphism ring of S_S. In other words, the endomorphism ring of M_R is the *ring* S, and therefore contains no primitive idempotents. It follows that M_R has no indecomposable direct summands. □

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