INFINITE PROGENERATOR SUMS

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1. Introduction

Let A, B be modules with semilocal endomorphism rings. If $A^{(\aleph)} \cong B^{(\aleph)}$ (direct sum of $\aleph$ copies) does it follow that $A \cong B$? The answer is known to be "yes" when $\aleph$ is a finite cardinal [F, Proposition 4.8]. This note was motivated by the question of whether the answer is still "yes" when $\aleph = \aleph_0$ and (to make the question nontrivial) A and B are indecomposable. The answer to this $\aleph_0$-*root* question is "no", even if we require A and B to be progenerators [Example 3.1], but is "yes" if we require A and B to be uniserial [P].

The starting point in the construction of our counterexample is a lemma, attributed to Eilenberg, stating that for every progenerator P, the module $P^{(\aleph_0)}$ is free. This led us to the related question of whether *every* infinite direct sum of progenerators is free. The answer is "yes" if the ring is noetherian [B, Corollary 3.2], if "most" of the summands are isomorphic to each other [Corollary 2.2], or if the number of terms is "very large" [Corollary 2.3]. But the answer is "no" in general [Example 4.1].

2. General Progenerator Sums

We learned the following lemma and its proof from [C], where it is used to prove the interesting result that two rings are Morita equivalent if and only the infinite matrix rings over those rings are isomorphic.

Lemma 2.1 (Eilenberg)**.** *Let P, Q be progenerators over any ring, and $\aleph$ any infinite cardinal. Then $P^{(\aleph)} \cong Q^{(\aleph)}$ — equivalently, $P^{(\aleph)}$ is free of rank $\aleph$.*

Proof. As P is a progenerator and Q is finitely generated, there is an epimorphism $P^{(n)} \to Q$ for some n. Therefore, since Q is projective, we have $Q \oplus Q' \cong P^{(n)}$ for some Q'. Taking the direct sum of $\aleph$ copies of this isomorphism yields

$$Q^{(\aleph)} \oplus Q'^{(\aleph)} \cong P^{(\aleph)} \tag{2.1.1}$$

Substituting $Q^{(\aleph)} \oplus Q^{(\aleph)}$ for $Q^{(\aleph)}$ in (2.1.1) gives $Q^{(\aleph)} \oplus Q^{(\aleph)} \oplus Q'^{(\aleph)} \cong P^{(\aleph)}$. Therefore the isomorphism in (2.1.1) gives $Q^{(\aleph)} \oplus P^{(\aleph)} \cong P^{(\aleph)}$.

Reversing the roles of P and Q shows that $Q^{(\aleph)} \oplus P^{(\aleph)} \cong Q^{(\aleph)}$, and hence $P^{(\aleph)} \cong Q^{(\aleph)}$. □

Date: 1 July, 2004.

1991 *Mathematics Subject Classification.* 16D10, 16P40.

Key words and phrases. Progenerator, $\aleph_0$-root.

Facchini's research was partially supported by Ministero dell'Istruzione, dell'Università e della Ricerca (Italy).

Levy's research was partially supported by an NSA grant (USA).

The next Corollary follows from the so-called Eilenberg trick: if P is finitely generated projective, we have $P \oplus Q \cong R^{(n)}$ for some finite n, therefore

$$\begin{aligned} P \oplus R^{(\aleph_0)} &\cong P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \dots \\ &\cong (P \oplus Q) \oplus (P \oplus Q) \oplus \dots \cong R^{(\aleph_0)} \end{aligned}$$

and therefore $P \oplus R^{(\aleph)} \cong R^{(\aleph)}$ for every infinite cardinal $\aleph$.

Corollary 2.2. *Let $P = \oplus_i P_i$ be the direct sum of some infinite number $\aleph$ of finitely generated projective modules P_i over some ring R. Suppose that at least $\aleph$ of the summands P_i are isomorphic to some fixed progenerator Q. Then $P \cong R^{(\aleph)}$.*

Proof. Let $P = \oplus_{i\in X} P_i$, and let Y be the subset of X of all indices $i \in X$ with $P_i \cong Q$, so that X and Y have the same cardinality $\aleph$. The case $X = Y$ is Eilenberg's Lemma, so that we can suppose $X \neq Y$. By Eilenberg's Lemma, the direct sum $\oplus_{i\in Y} P_i$ is free, i.e., $\oplus_{i\in Y} P_i \cong R^{(\aleph)}$. Thus

$$P = (\oplus_{i\in Y} P_i) \oplus \left(\oplus_{i\in X\setminus Y} P_i\right) \cong R^{(\aleph)} \oplus \left(\oplus_{i\in X\setminus Y} P_i\right) \cong \oplus_{i\in X\setminus Y} \left(P_i \oplus R^{(\aleph)}\right)$$

because $X \times (X \setminus Y)$ and X have the same cardinality $\aleph$. But $P_i \oplus R^{(\aleph)} \cong R^{(\aleph)}$ by Eilenberg's trick. Thus P is free. □

Corollary 2.3. *Let $P = \oplus_i P_i$ be the direct sum of $\aleph$ progenerators P_i over some ring R, where $\aleph > \max\{|R|, \aleph_0\}$. Then $P \cong R^{(\aleph)}$*

Proof. Let $\alpha = \max\{|R|, \aleph_0\}$. Then $R^{(\aleph_0)}$ has cardinality α, so that it has at most α finitely generated submodules. Thus there are at most α finitely generated projective right R-modules up to isomorphism.

Therefore some isomorphism class must be repeated at least $\aleph$ times among the P_i. Therefore Corollary 2.2, implies $P \cong R^{(\aleph)}$. □

3. Nonunique $\aleph_0$-roots

Example 3.1. Progenerators $P \ncong Q$ that are indecomposable modules with semilocal endomorphism rings and satisfy $P^{(\aleph_0)} \cong Q^{(\aleph_0)}$.

Proof. Let e be a positive integer ≥ 2. Then there is a (noncommutative) integral domain Λ that is a module-finite algebra over a discrete valuation ring V and such that Λ has an indecomposable progenerator P such that $P^{(e)}$ maps onto Λ, but $P^{(d)}$ does not map onto Λ for $d < e$ [L, Theorem 3.1].

Since $e \geq 2$ we have $P \ncong \Lambda$; and by Eilenberg's Lemma we have $P^{(\aleph_0)} \cong \Lambda^{(\aleph_0)}$. Thus it suffices to show: (i) Λ is an indecomposable left Λ-module, and (ii) the left Λ-modules Λ and P both have semilocal endomorphism rings.

Since Λ is a domain, Λ has no non-trivial idempotents. Thus (i) follows from the fact that Λ is anti-isomorphic to the endomorphism ring of Λ as a Λ-module. Let M be any finitely generated Λ-module. Then the endomorphism ring of M is a module-finite V-algebra, and is therefore a semilocal ring. This applies, in particular, to the left Λ-modules Λ and P, and therefore proves (ii). □

4. Non-free Progenerator Sum

Example 4.1. A ring R with progenerators P_n, $n \geq 1$, such that the direct sum $G = \oplus_{n=1}^{\infty} P_n$ is not free.

Let $\mathbb{R}_{\geq 0}$ denote the nonnegative real numbers. There is a right self-injective ring R whose monoid $V(R)$ of isomorphism classes of finitely generated projective modules is isomorphic to the additive monoid $\mathbb{R}_{\geq 0}$, and such that the isomorphism $V(R) \cong \mathbb{R}_{\geq 0}$ takes the isomorphism class of R_R to the real number 1. This is a special case of [GW, Corollary 5–3.15] (which characterizes which monoids occur as $V(R)$ for regular right self-injective rings) and [GW, Proposition 3–1.11].

In more detail, there is — for every $\alpha \in \mathbb{R}_{\geq 0}$ — a finitely generated projective right R-module Q_α such that the following properties hold.

(i) For every finitely generated projective R-module Q there exists a unique $\alpha \in \mathbb{R}_{\geq 0}$ such that $Q \cong Q_\alpha$.
(ii) $Q_1 \cong R_R$.
(iii) $Q_\alpha \oplus Q_\beta \cong Q_{\alpha+\beta}$ for every α, β.
(iv) Q_α is isomorphic to a direct summand of Q_β if and only if $\alpha \leq \beta$.

We claim: *Q_α is a progenerator whenever $\alpha \neq 0$.* We have $n\alpha \geq 1$ for some positive integer n. Therefore, by (ii)–(iv), $R = Q_1$ is isomorphic to a direct summand of $(Q_\alpha)^{(n)} \cong Q_{n\alpha}$, proving the claim.

Set $G = \oplus_{n=1}^{\infty} Q_{1/2^n}$ where summation extends over all positive integers n.

We claim that there is a monomorphism of G into R because $\sum_{i=1}^{\infty} 1/2^n = 1$. In more detail, there is a decomposition $R = A_1 \oplus B_1$ with $A_1 \cong B_1 \cong Q_{1/2}$, by (ii) and (iii). Similarly, there is a decomposition $B_1 = A_2 \oplus B_2$ with $A_2 \cong B_2 \cong Q_{1/2^2}$. Thus $R = A_1 \oplus A_2 \oplus B_2$. Continuing in this way we build a submodule $A = \oplus_{n=1}^{\infty} A_n$ of R, such that every $A_n \cong Q_{1/2^n}$ and hence $A \cong G$. ($A \neq R$ because R_R is finitely generated while the infinite direct sum A is not.)

Let us prove that G is not free. Assume the contrary, that is, $G \cong R^{(\aleph)}$ for some cardinal $\aleph$. Since G is an infinite direct sum of nonzero modules, G is not cyclic. Therefore $\aleph \geq 2$. Thus there is a monomorphism of $R_R^{(2)}$ into G, and hence into R_R. Since R_R is injective, this implies that $R_R^{(2)} = Q_2$ is isomorphic to a direct summand of $R = Q_1$. This contradiction of property (iv) completes the proof.

References

[B] H. Bass, "Big projective modules are free", *Israel J. Math.* **7** (1963), pp. 24–31.

[C] V. Camillo, "Morita equivalence and infinite matrix rings", *Proc. Amer. Math. Soc.* **90** (1984) pp. 186–188.

[F] A. Facchini, *Module Theory: Endomorphism Rings and Direct Sum Decompositions in some classes of Modules,* Birkhaüser, Basel, 1998.

[GW] K. R. Goodearl and F. Wehrung, "The complete dimension theory of partially ordered systems with equivalence and orthogonality", (preprint, presently available at www.math.ucsb.edu/~goodearl/preprints.html).

[L] L. S. Levy, "Projectives of large uniform-rank, in Krull dimension 1", *Bulletin London Math. Soc.* **21** (1989), 57–64.

[P] P. Prihoda, "On uniserial modules that are not quasi-small", manuscript, 2004.

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