REPRESENTATION TYPE OF COMMUTATIVE NOETHERIAN RINGS (INTRODUCTION)

LEE KLINGLER AND LAWRENCE S. LEVY

Abstract. We introduce a series of papers [KL1, KL2, KL3] that describe the isomorphism classes of finitely generated modules *and their direct sum relations* over all commutative noetherian rings that do not have wild representation type [see Definition 2.1]. (There is a possible slight exception to our structure results, involving characteristic 2 [see Remarks 2.4].)

1. Background

Let Ω be a commutative noetherian ring whose finitely generated module category fingen(Ω) we wish to describe. For example, the case $\Omega = \mathbb{Z}$ is given by the structure theorem for finitely generated abelian groups. A well-known slight extension of the case $\Omega = \mathbb{Z}$ is given by the situation that Ω is any principal ideal domain. This was further extended to the case that Ω is any Dedekind domain by Steinitz in 1911-1912 [S]. Steinitz's motivation was to describe fingen(Ω) where Ω is the ring of integers in any algebraic number field.

In modern terminology, a *Dedekind domain* is any integral domain (necessarily noetherian) in which every ideal is a projective Ω-module. Over Dedekind domains, fingen(Ω) is not a *Krull-Schmidt category;* that is, modules in fingen(Ω) are not necessarily *unique* direct sums of indecomposables (up to isomorphism). One of the most interesting ingredients in Steinitz's papers is that he was able to describe the direct-sum relations in fingen(Ω).

As far as we know, Dedekind domains remained the only class of noetherian domains for which fingen(Ω) could be described until 1985, when Levy described a very restricted class of such rings called "Dedekind-like" [L2], a proper subset of the rings called Dedekind-like in the present paper. There were, however, an assortment of results for commutative local noetherian rings Ω, some of them non-artinian (Nazarova-Roiter [NR, 1969], corrected in [NRSB, 1975], Ringel [R, 1975], Drozd [D2, 1991]).

The obstruction to getting a very general theorem is "wild representation type", which for more than 30 years has been a well-known obstruction to obtaining a structure theorem for fingen(Ω) when Ω is a finite dimensional (possibly noncommutative) algebra over an algebraically closed field. Furthermore, the vast majority of finite dimensional algebras are known to have wild representation type.

Date: June, 2004, correction July, 2005. Based on a talk presented at University of Lisbon, Portugal, on July 17, 2003 by L. S. Levy.

1991 *Mathematics Subject Classification.* 13E05, 16G60.

Key words and phrases. Tame representation type, wild representation type, commutative noetherian ring, Dedekind-like ring.

Levy's research was partially supported by an NSA grant.

Thus the present project overlaps two research areas that do not normally have much interaction: (i) modules over commutative noetherian rings (and the associated local versus global, and direct-sum relations, which have no counterpart in finite dimensional representation theory); and (ii) tameness versus wildness (and the associated matrix problems), a familiar subject in finite dimensional representation theory, but not very familiar over commutative noetherian rings. Since the two communities — commutative noetherian ring theorists and finite dimensional representation theorists — are sometimes unfamiliar with ideas that are well-known in the other community, we ask the forbearance of readers from each community as we define terms that "everybody knows" in their community but not the other.

2. Introduction

Throughout this paper, *ring* means "commutative ring" unless otherwise specified, and *local ring* means "noetherian local ring". For a maximal ideal $\mathfrak{m}$ of a noetherian ring Ω, the notation $\Omega_{\mathfrak{m}}$ and $\hat{\Omega}_{\mathfrak{m}}$ denotes $\mathfrak{m}$-localization and $\mathfrak{m}$-adic completion, respectively.

Definitions 2.1 (tame, wild)**.** Let Ω be a noetherian ring, not necessarily an algebra over a field. We say that Ω fingen-*tame* — or more completely, that fingen(Ω) has *tame representation type* — if we can describe all isomorphism classes of finitely generated modules *and their direct-sum relations,* and give some information about the homomorphisms in this category.

We supplement this informal definition of "tameness" by the following formal definition of wildness.

Let $k = \Omega/\mathfrak{m}$ be some residue field of our noetherian ring Ω. We say that Ω is finlen-*wild* (wrt $\mathfrak{m}$) — or more completely, that finlen(Ω), the category of Ω-modules of finite length, has *wild representation type (with respect to* $\mathfrak{m}$*)* — if the following condition holds for every finite dimensional (possibly noncommutative!) k-algebra A. There exist $\mathcal{W} = \mathcal{W}_A$, a full subcategory of finlen(Ω) and an additive functor: $\Phi = \Phi_A\colon \mathcal{W} \to$ finlen(A) (say, left A-modules) such that Φ is a *representation equivalence.* In other words:

(2.1.1) (i) Φ maps $\mathcal{W}$ onto all isomorphism classes in finlen(A);
(ii) For $M, N \in \mathcal{W}$: $\quad M \cong N \iff \Phi(M) \cong \Phi(N)$ in finlen(A); and
(iii) Φ is a surjection on Hom groups.

It follows easily that ${}_{\Omega}M$ is indecomposable $\iff$ $\Phi(M)$ is indecomposable.

If we say that Ω is finlen-wild without mentioning any $\mathfrak{m}$, we mean that Ω is finlen-wild (wrt some $\mathfrak{m}$).

The definition of "wild" is not completely standard. Often one requires each Φ to be an isomorphism on Hom groups in (2.1.1)(iii). Then each functor Φ becomes a category equivalence of $\mathcal{W}$ with finlen(A); and we call Ω *strictly wild (wrt* $\mathfrak{m}$*).*

This strict wildness is not what occurs in our main results. But it does occur in our discussion of them.

Remark 2.2 (Introduction to wildness)**.** For readers not familiar with the notion of wildness, the following comments may be helpful. Suppose that we know that Ω is finlen-wild (wrt $\mathfrak{m}$), and that we actually know all of the associated functors Φ_A. Let $k = \Omega/\mathfrak{m}$, let A be any finite dimensional k-algebra, and choose any two modules $X, Y \in$ finlen(A). Are they isomorphic?

Since $\Phi = \Phi_A$ is onto all isomorphism classes, there exist $M, N \in \operatorname{finlen}(\Omega)$ such that $\Phi(M) \cong X$ and $\Phi(N) \cong Y$. Moreover, by property (2.1.1)(ii) of wildness we have $X \cong Y \iff M \cong N$ in $\operatorname{mod}(\Omega)$. Thus we have transferred the original isomorphism question for A-modules — for an arbitrary finite dimensional k-algebra — to an isomorphism question about modules over the single ring Ω.

In less formal language, $\operatorname{finlen}(\Omega)$ has such a rich system of modules of finite length that any classification of their isomorphism classes would result in a corresponding A-isomorphism classification in $\operatorname{finlen}(A)$ for every finite dimensional k-algebra A. The seeming hopelessness of ever describing such a classification is responsible for the terminology "wild representation type". Indeed no Ω of wild representation type has ever had its finite-length module category explicitly described. [But we know of no theorem from mathematical logic saying that such a classification is impossible.]

Our starting point, in this survey, is the following theorem.

Theorem 2.3. *Let Ω be a noetherian ring, and suppose that there is no maximal ideal $\mathfrak{m}$ of Ω such that Ω is* finlen-*wild (wrt $\mathfrak{m}$). Then one of the following two situations holds.*

(i) *Ω is* fingen-*tame — we can explicitly describe the isomorphism classes and direct-sum relations in* $\operatorname{fingen}(\Lambda)$*; or*

(ii) *Ω is one of the "exceptional" rings described in Definition 4.3 (and we do not know whether Ω is tame, wild, or neither).*

Remarks 2.4 (on the "exception")**.** This exceptional class of rings is very small. For example, it contains no rings whose residue fields are all finite (e.g. rings of algebraic integers). Nor does it contain any rings that are algebras over a field of characteristic $\neq 2$, or over a perfect (e.g. algebraically closed) field of characteristic 2.

Another "loose end" is that we cannot prove that none of our tame rings is also wild, except in special cases [KL3, §37 Problem 2].

In order to make Theorem 2.3 useful, we need to say which rings are wild, which are tame, and — in the tame case — give some of the basics of the module-classification in $\operatorname{fingen}(\Lambda)$. We begin with the wild case.

3. Artinian Triads and Drozd Rings

The notation $\mu_\Omega(M)$ denotes the minimum number of elements needed to generate a module $M \in \operatorname{fingen}(\Omega)$.

Definitions 3.1 ([KL1, 2.4])**.** We call an artinian local ring $(\Omega, \mathfrak{m}, k)$ (maximal ideal $\mathfrak{m}$, residue field k) an *artinian triad* if $\mu_\Omega(\mathfrak{m}) = 3$ and $\mathfrak{m}^2 = 0$.

The simplest example is the k-algebra $\Omega = k[X, Y, Z]/(X, Y, Z)^2$, where k is any field and X, Y, and Z are indeterminates.

We call an artinian local ring $(\Omega, \mathfrak{m}, k)$ a *Drozd ring* if $\mu_\Omega(\mathfrak{m}) = \mu_\Omega(\mathfrak{m}^2) = 2$, $\mathfrak{m}^3 = 0$, and there is an element $x \in \mathfrak{m} - \mathfrak{m}^2$ such that $x^2 = 0$.

Remarks 3.2 (Drozd Rings)**.** The above definition of Drozd rings has the advantage of brevity, but leaves one wondering what such rings really look like.

(i) The simplest example is the well-known 5-dimensional *Drozd algebra* over any field k. This is the 5-dimensional k-algebra with k-basis $1, x, y, xy, y^2$ and all other monomials equal to zero.

(ii) Every Drozd ring $(\Omega, \mathfrak{m}, k)$ has the following easily-proved property [KL1, Lemma 4.2] that makes it look very much like the Drozd algebra. *For each element $c \in \mathfrak{m}$, there is an expression $c = u_1x + u_2y + u_3xy + u_4y^2$ with each u_i a unit or* 0. (We do not claim uniqueness of the coefficients u_i.)

(iii) An example of a Drozd ring that is not an algebra over a field (probably the simplest such example) is easily seen to be $A_p = \mathbb{Z}[X]/(X^2, p^3, p^2X)$, where p denotes any prime number and X is an indeterminate [KL1, 6.1].

Definition 3.3 ([KL1, 2.8])**.** We call a (necessarily artinian) local ring $(\Omega, \mathfrak{m}, k)$ a *Klein ring* if $\mu_\Omega(\mathfrak{m}) = 2$, $\mu_\Omega(\mathfrak{m}^2) = 1$, $\mathfrak{m}^3 = 0$, and $x^2 = 0$ $(\forall x \in \mathfrak{m})$.

We will not dwell on Klein rings in this survey. They have characteristic 2 or 4 [KL1, Lemma 2.9] and are quasi-Frobenius rings. For a few more details and references, see subsection 7.7. An example is the group ring of the Klein 4-group over any field of characteristic 2. For an example that has characteristic 4 (and hence is not an algebra over a field), see [KL1, Example 5.4].

Theorem 3.4. *Artinian triads and Drozd rings are* finlen-*wild (wrt their maximal ideal). Klein rings are* fingen-*tame (=* finlen-*tame since Klein rings are artinian).*

Artinian triads are wild by a theorem of Warfield [GLW, Lemma 3].

Remarks 3.5 ($\mathcal{S}(1\frac{1}{2})$, $\mathcal{S}(2)$, $\mathcal{S}(m)$)**.** In order to discuss our proof of wildness of Drozd rings, we need to review some related categories that are familiar to everyone working in representations of finite dimensional algebras. Let k be a field.

The classical unsolved "wildness problem" is that of classifying ordered pairs (A, B) of square matirices over k, up to simultaneous similarity. Define the category $\mathcal{S}(2)$ to be the category whose objects are ordered pairs (A, B) of $n \times n$ matrices over k, where n ranges over all positive integers. Morphisms are matrices τ (of appropriate sizes) such that the following diagrams commute, where $k^{(n)}$ denotes columns of length n.

(3.5.1) $$\mathcal{S}(2):\quad \begin{array}{ccccc} k^{(n)} & \xrightarrow{A} & k^{(n)} & \xrightarrow{B} & k^{(n)} \\ \tau\downarrow & & \tau\downarrow & & \tau\downarrow \\ k^{(n')} & \xrightarrow{A'} & k^{(n')} & \xrightarrow{B'} & k^{(n')} \end{array}$$

More generally, the category $\mathcal{S}(m)$ is the analogously formed category of m-tuples of matrices over k.

The connection of this with wildness, in the modern sense, is made by considering the noncommutative free algebra F on m indeterminates. Every element $(A_1, \ldots, A_m) \in \mathcal{S}(m)$, where the matrices have size (say) $n \times n$, makes the k-vector space $k^{(n)}$ into a left F-module if we define multiplication by the i^{th} indeterminate to be left multiplication by A_i. A straightforward computation shows that the category $\mathcal{S}(m)$ is equivalent to finlen(F) — and therefore contains finlen(C) for every m-generated finite dimensional noncommutative k-algebra C. A simple, but ingenious observation of S. Brenner [B, Theorem 3] is that $\mathcal{S}(2)$ contains a full subcategory equivalent to $\mathcal{S}(m)$ for every m. Thus $\mathcal{S}(2)$ is strictly wild in the sense defined in the present paper.

Define the category $\mathcal{S}(1\frac{1}{2})$ ("one and one-half similarity") over k, to consist of ordered triples (m, n, ϕ) where m, n, are positive integers with $m \leq n$ and ϕ is an

$n \times n$ matrix over k. Morphisms in $\mathcal{S}(1\frac{1}{2})$ are pairs of (σ, τ) of matrices such that the following diagram commutes (where I is an identity matrix).

$$\mathcal{S}(1\frac{1}{2}): \quad \begin{array}{ccccc} k^{(m)} & \xrightarrow{i\,=\,[I\;\;0]} & k^{(n)} & \xrightarrow{\phi} & k^{(n)} \\ \sigma\downarrow & & \tau\downarrow & & \tau\downarrow \\ k^{(m')} & \xrightarrow{i'\,=\,[I'\;\;0]} & k^{(n')} & \xrightarrow{\phi'} & k^{(n')} \end{array} \tag{3.5.2}$$

A theorem of Nazarova [Nz, Lemma 1] is that $\mathcal{S}(1\frac{1}{2})$ contains a subcategory equivalent to $\mathcal{S}(2)$, and hence is strictly wild.

3.6. **Wildness of Drozd rings.** This wildness is proved in [KL1, §4]. The proof follows Ringel's suggested simplification [R] of Drozd's original proof [D1] for k-algebras. Let $(\Omega, \mathfrak{m}, k)$ be a Drozd ring. For each triple (m, n, ϕ) where $m \leq n$ are positive integers and ϕ is an $n \times n$ matrix over k, Ringel defines a module $M(m, n, \phi)$ over the Drozd k-algebra that makes sense over an arbitrary Drozd ring Ω. However, morphisms in fingen(Ω) are no longer k-linear maps when Ω is not a k-algebra.

Let $\mathcal{W}$ be the full subcategory of finlen(Ω) consisting of all $M(m, n, \phi)$. Our basic idea is that modules over artinian rings have projective covers. Let $f\colon M \to N$ be a homomorphism in finlen(Ω). Then f can be lifted to a homomorphism F of the projective covers of M and N respectively. Since projective modules over commutative local rings are free, F can be represented by a matrix — which we again call F — over Ω (rather than over its residue field k). Reduction of the entries of F modulo the maximal ideal $\mathfrak{m}$ of Ω yields a matrix $\bar{F}$ over k. If the given modules M, N are in $\mathcal{W}$, extraction of certain diagonal blocks of $\bar{F}$ yield a morphism $\mathcal{E}(f) \in \mathcal{S}(1\frac{1}{2})$. A rather complicated matrix computation then shows that the pair of correspondences $M(m, n, \phi) \to \phi$ and $f \to \mathcal{E}(f)$ forms the desired representation equivalence $\mathcal{W} \to \mathcal{S}(1\frac{1}{2})$ showing that Ω is finlen-wild.

An interesting aspect of the proof is that our matrices are substantially smaller than Ringel's. However, this is deceptive because an element of Ω holds more information than an element of k holds, roughly five times as much since the Drozd algebra has dimension 5. Extraction of the needed k-information from Ω-information is less convenient than one might expect, and is responsible for the complications in our proof.

Tameness of Klein rings is proved in [KL2, §11, especially Theorem.11.3]. See subsection 7.7 of the present survey for a few more details.

The next step, in our discussion of which rings are tame and which are wild, is the following bit of pure commutative algebra. Neither its statement nor its proof involves the concept of tameness or wildness. One of the types of rings that it mentions — Dedekind-like rings — will be defined in the next section.

Theorem 3.7 (Ring-theoretic Dichotomy)**.** *Let Ω be an indecomposable noetherian ring. Then exactly one of the following two possibilities occurs.*

(i) *Ω has an artinian triad or a Drozd ring as a homomorphic image.*
(ii) *Ω is a Klein ring or a homomorphic image of a Dedekind-like ring.*

This is proved for complete local rings in [KL1, Theorem 3.1], and extended to the nonlocal case in [KL3, Theorem 14.3].

Any ring that maps onto a finlen-wild ring is obviously itself finlen-wild. Therefore our tame-wild problem is now reduced to deciding whether Dedekind-like rings and their homomorphic images are tame or wild or neither. It is here that we encounter the possible exception mentioned earlier.

3.8. **Historical Remark.** The idea of fingen-tame versus finlen-wild is older than one might suspect. In [NR, 1969] Nazarova and Roiter described the isomorphism classes in fingen(Λ), where Λ is what we call a strictly split local Dedekind-like ring in subsection 7.1.

Subsequently, Drozd [D1, 1972] considered complete local noetherian rings Ω with residue field k, assuming that k is algebraically closed and Ω is a finitely generated k-algebra. He proved that the only the following possibilities occur.

(i) Ω is a homomorphic image of $k[[xy]]/(xy)$.
(ii) k has characteristic 2 and Ω is the group algebra of the Klein 4-group.
(iii) Ω maps onto $k[[x, y, z]]/(x, y, z)^2$.
(iv) Ω maps onto the Drozd algebra [defined in our Remarks 3.2(ii)].

The rings in (i) were known to be fingen-tame by [NR], and Drozd showed that the ring (ii) is isomorphic to a ring known to be fingen-tame (= finlen-tame in this case). Drozd proved that the rings in (iii) and (iv) are finlen-wild.

In the subsequent explosion of work on finite-dimensional representations of noncommutative algebras, this fingen-tame/finlen-wild phenomenon seems to have been overlooked except by the present authors and one later paper of Drozd [D2, 1991], which is mainly about the noncommutative case. (New results proved there do not go beyond [D1] in the commutative case.)

4. Dedekind-like Rings

Notation 4.1. If a ring Ω is *reduced* (no nonzero nilpotent elements) it has a *normalization* (integral closure in its total quotient ring). We denote the set of maximal ideals of Ω by maxspec(Ω).

Definition 4.2. We call a noetherian ring Λ *Dedekind-like* if Λ is reduced and its normalization Γ has the following properties.

(i) Γ is a direct sum of Dedekind domains;
(ii) $(\Gamma/\Lambda)_{\mathfrak{m}}$ is either a simple $\Lambda_{\mathfrak{m}}$-module or 0 $\big(\forall \mathfrak{m} \in \text{maxspec}(\Lambda)\big)$;
(iii) $\mathfrak{m}_{\mathfrak{m}} = J(\Gamma_{\mathfrak{m}})$, the Jacobson radical $\big(\forall \mathfrak{m} \in \text{maxspec}(\Lambda)\big)$; and
(iv) (nontriviality:) $\Lambda_{\mathfrak{m}}$ is never a field $\big(\forall \mathfrak{m} \in \text{maxspec}(\Lambda)\big)$.

This form of the definition is taken from [KL3, Definition 10.1]. Dedekind-like rings Λ have the following property [KL3, Proposition 10.9]:

(4.2.1) Every ideal of Λ is generated by 2 elements.

An immediate consequence of statement (ii) is that $\big(\forall \mathfrak{m} \in \text{maxspec}(\Lambda)\big)$ the normalization $\Gamma_{\mathfrak{m}}$ of $\Lambda_{\mathfrak{m}}$ is a finitely generated $\Lambda_{\mathfrak{m}}$-module. However, ${}_{\Lambda}\Gamma$ *is not necessarily finitely generated.* Examples should eventually appear in [HL].

We always consider Γ and fingen(Γ) to be "approximations" to Λ and fingen(Λ) respectively, in the sense that almost every time we describe a property of Λ we do so by relating it to a property of Γ. This already occurred in the definition of "Dedekind-like" given above.

Our reason for calling this generalization of Dedekind domains "Dedekind-like" is given in the Epilog on Terminology, Section 9.

As with Drozd rings and Klein rings, our definition of Dedekind-like rings leaves one wondering what these rings really look like. We display the structure of complete local Dedekind-like rings quite explicitly in subsections 7.1–7.3. These explicit descriptions are related to the general (i.e. nonlocal) situation by the fact that a noetherian ring is Dedekind-like if and only all of its completions at maximal ideals are Dedekind-like [Proposition 5.1].

Definition 4.3 (Exceptional Dedekind-like rings)**.** Let Λ be a Dedekind-like ring with normalization Γ. We say that Λ is *exceptional* if Λ has a maximal ideal $\mathfrak{m}$ such that the ring $\Gamma/\Gamma\mathfrak{m}$ is a 2-dimensional inseparable field-extension of the field $\Lambda/\mathfrak{m}$ (and hence both fields have characteristic 2). (The terminology "exceptional" is not used in [KL1]–[KL3]. Instead, the equivalent "Additional Hypothesis" [KL3, 10.2] and [KL2, (1.1.3)] are implicitly invoked whenever a tameness result is stated.)

For an example of an exceptional Dedekind-like ring, see subsection 7.2.

Theorem 4.4. *Every non-exceptional Dedekind-like ring is* fingen*-tame. (We do not know whether exceptional Dedekind-like rings are tame or wild or neither.)*

Remarks 4.5 (Module Classification, part 1)**.** The proof of Theorem 4.4, in the complete local case, occupies almost all of [KL2]. The extension to the non-local situation occupies almost all of [KL3]. Tamemess is more complicated than wildness for the two reasons given in items (i) and (ii) below.

(i) The form of wildness that we use is finlen-wild. Moreover, for every maximal ideal $\mathfrak{m}$ of Λ the category finlen$(\hat{\Lambda}_{\mathfrak{m}})$ — $\hat{\Lambda}_{\mathfrak{m}}$ the $\mathfrak{m}$-adic completion of Λ — is a full subcategory of finlen(Λ). Also, every indecomposable module in finlen(Λ) is an indecomposable module in finlen$(\hat{\Lambda}_{\mathfrak{m}})$ for some maximal ideal $\mathfrak{m}$, and finlen(Λ) is a Krull-Schmidt category. This reduces the description of module structure in finlen(Λ) to the the description of indecomposable modules in the complete local case, described in [KL2].

In other words, finlen-wildness is determined in the simpler, complete local situation. We postpone further discussion of the complete local case to Section 7, where we also discuss the role of "1-parameter families" of modules. (These are always families of modules of finite length.)

(ii) The relationship between fingen$(\hat{\Lambda}_{\mathfrak{m}})$ and fingen(Λ) is much more complicated, because fingen$(\hat{\Lambda}_{\mathfrak{m}})$ is not a subcategory of fingen(Λ). Moreover, fingen(Λ) is not a Krull-Schmidt category. Therefore, unlike the situation in finitely dimensional algebras, we can no longer focus exclusively on indecomposable Λ-modules. In fact, they play a rather minor role in our module-description, even though indecomposable modules in fingen(Λ) play a major role in the complete local case, and their properties are needed in the nonlocal case.

(iii) Let fingen$_\infty(\Lambda)$ denote the category of finitely generated Λ-modules with no direct summands of finite length. Since Λ is noetherian, it is easy to see that every $M \in$ fingen(Λ) has a unique decomposition (up to isomorphism) $M = M_\infty \oplus M_0$ where $M_\infty \in$ fingen$_\infty(\Lambda)$ and $M_0 \in$ finlen(Λ). Moreover, if $N = N_\infty \oplus N_0$, then $M_\infty \oplus N_\infty \in$ fingen$_\infty(\Lambda)$ and $M_0 \oplus N_0 \in$ finlen(Λ). This separates the study of fingen(Λ) into two disjoint parts (except for the zero module, which belongs to both categories). We comment briefly on each part separately, in items (iv) and (v) below. For proofs and more details of this reduction, which applies to rings much more general than Dedekind-like rings, see [KL3, §7].

(iv) It therefore remains to discuss the category $\mathrm{fingen}_\infty(\Lambda)$. It turns out that a module $M \in \mathrm{fingen}(\Lambda)$ belongs to $\mathrm{fingen}_\infty(\Lambda)$ if and only if every $\mathfrak{m}$-adic completion $\hat{M}_\mathfrak{m} \in \mathrm{fingen}_\infty(\hat{\Lambda}_\mathfrak{m})$. And, as mentioned in (iii), $\mathrm{fingen}_\infty(\Lambda)$ is closed under (finite) direct sums. Thus we have reduced the study of modules and their direct sums in $\mathrm{fingen}(\Lambda)$ to the analogous problem in $\mathrm{fingen}_\infty(\Lambda)$. This is subject of Section 5.

We continue this discussion in Remarks 5.2. The following theorem summarizes what we know so far, except for a clear description of what the tameness in part (iii)(a) really means. That occupies most of the rest of this survey.

Theorem 4.6. *Let Ω be an indecomposable noetherian ring. Then exactly one of the following occurs.*

(i) *Ω maps onto an artinian triad or a Drozd ring. Here Ω is* finlen-*wild.*
(ii) *Ω is a Klein ring. Here Ω is* fingen-*tame [see subsection 7.7].*
(iii) *Some Dedekind-like ring Λ maps onto Ω.*
   (a) *If Λ is non-exceptional, then Ω is* fingen-*tame.*
   (b) *If every Λ that maps onto Ω is exceptional [Definition 4.3], then we do not know whether Ω is tame or wild or neither.*

Since every homomorphic image of a fingen-tame ring is obviously finlen-tame, we limit our discussion of the tamenees in Theorem 4.6(iii)(a) to modules over Dedekind-like rings. This discussion begins in the next section.

Corollary 4.7. *Every noetherian ring Ω with Krull dimension ≥ 2 is* finlen-*wild.*

Proof. Dedekind domains have Krull dimension 1. Thus the normalization Γ of every Dedekind-like ring Λ has Krull dimension 1. It follows easily that Dedekind-like rings — exceptional or not — have Krull dimension 1; in fact, all maximal ideals of Λ have height 1 [KL3, Proposition 10.6]. Therefore no Dedekind-like ring can map onto Ω. Since rings of Krull dimension ≥ 2 are not artinian, it follows from our ring-theoretic dichotomy [Theorem 3.7] that Ω maps onto an artinian triad or Drozd ring, and hence is finlen-wild [Theorem 4.6]. □

Corollary 4.8. *Let Ω be a noetherian ring with Krull dimension ≤ 1.*

(i) *If some ideal of Ω requires 3 or more generators, then Ω is* finlen-*wild.*
(ii) *If every ideal of Ω is principal, then Ω is* fingen-*tame.*

Proof. (i) As observed in (4.2.1), all ideals of Dedekind-like rings are 2-generated. Therefore no Dedekind-like ring maps onto Ω. As in the proof of the previous corollary, we conclude that Ω is finlen-wild.

(ii) The well-known fact that all principal ideal rings are fingen-tame, in the sense in which we are using the term, is a much older result than the definition of "tame". □

By the previous two corollaries, "tame noetherian" is a very small class of rings of Krull dimension ≤ 1. Does the class contain any interesting non-artinian rings (other than direct sums of Dedekind domains and their homomorphic images)?

Examples 4.9 (Natural Examples of Dedekind-like rings)**.**

(i) $\mathbb{Z}[\sqrt{n}\,]$ when n is squarefree.
(ii) $\mathbb{Z}G_n$ (integral group ring of a cyclic group of order n) when n is squarefree.

(iii) All subrings of squarefree index in $\mathbb{Z} \oplus ... \oplus \mathbb{Z}$.
(iv) (Over any algebraically closed field:) The coordinate ring of any affine curve whose singularities are all simple nodes.
(v) $k[x,y]/(xy)$ and $k[[x,y]]/(xy)$ for any field k.
(vi) $\mathbb{R} + x\mathbb{C}[x]$ and $\mathbb{R} + x\mathbb{C}[[x]]$ (polynomials and formal power series rings over the complex numbers, with real constant term).

For proofs, see the following. (i): [KL3, Example 36.3]. (ii): See [L3]. But beware of the differently-phrased definition of "Dedekind-like" in this earlier paper, a narrower class of rings than the Dedekind-like rings in the present paper. (iii): Combine Proposition 5.1 with [KL2, 12.5]. (v) and (vi): The power series rings are Dedekind-like by [KL1, 2.17]. For the polynomial version, use this and Proposition 5.1, remembering that discrete valuation rings are Dedekind-like. (iv) A purely algebraic statement of a property, from which (iv) follows, is: *Suppose that the completion of a noetherian ring Ω at each maximal ideal is either isomorphic to $k[[x,y]]/(xy)$ or $k[[x]]$, for some field k. Then Ω is Dedekind-like.* The proof is the same as that of statement (v), with the polynomial ring replaced by Ω.

We included rings (i) because rings of algebraic integers are the rings that interested Steinitz; and Dedekind-like rings of algebraic integers may be the only non-integrally closed rings of algebraic integers whose finitely generated module category has been described.

We included examples (vi) because they seem to be the simplest known fingen-tame rings that are infinite dimensional algebras over a non-algebraically closed field and have no counterpart over algebraically closed fields. We discuss its modules later [Section 8].

Example 4.10 (Super-wild ring)**.** The ring $\Omega = \mathbb{Z}[x]/(x^2)$ is finlen-wild (and might be the most easily described nonlocal finlen-wild ring of Krull dimension 1).

This ring has an amusing property that does not occur for finite dimensional algebras, and that we call *superwild:* For each prime number p there is a maximal ideal $\mathfrak{m}$ of Λ such that $\Lambda/\mathfrak{m}$ has characteristic p and Λ is finlen-wild (wrt $\mathfrak{m}$).

Proof. For every p, Ω maps onto the ring $\mathbb{Z}[X]/(X^2, p^3, p^2X)$, which we have observed to be a Drozd ring [Remarks 3.2]. Hence Ω is finlen-wild (wrt the appropriate $\mathfrak{m}$) [Theorem 3.4] □

5. Local-global and direct-sum relations

Throughout this section Λ denotes a non-exceptional Dedekind-like ring with normalization Γ. We focus on Dedekind-like rings for the rest of this survey, except for the short subsection 7.7 on Klein rings, because all known indecomposable fingen-tame noetherian rings except Klein rings are homomorphic images of Dedekind-like rings. Two important stability properties of the class of Dedekind-like rings are the following.

Proposition 5.1. *Let Ω be a noetherian ring.*

(i) *Ω is Dedekind-like if and only if all localizations $\Omega_{\mathfrak{m}}$ at maximal ideals are Dedekind-like.*
(ii) *If Ω is local with maximal ideal $\mathfrak{m}$, then it is Dedekind-like if and only its $\mathfrak{m}$-adic completion $\hat{\Omega}_{\mathfrak{m}}$ is Dedekind-like.*

When the conditions hold, the normalizations of $\Omega_{\mathfrak{m}}$ and $\hat{\Omega}_{\mathfrak{m}}$ are $\Gamma_{\mathfrak{m}}$ and $\hat{\Gamma}_{\mathfrak{m}}$ respectively, where Γ is the normalization of Ω.

Proof. (i) is an almost immediate from the definition of Dedekind-like. For details see [KL3, Corollary 10.7]. For the well-known fact that $\Gamma_{\mathfrak{m}}$ is the normalization of $\Omega_{\mathfrak{m}}$ see, for example, [KL3, Remarks 5.3].

For (ii) and its supplementary statement about the normalization in this case, see [KL3, Lemma 11.8]. □

Remarks 5.2 (Module Classification, part 2)**.** If we wish to describe the isomorphism classes and direct-sum relations in fingen(Λ), it now suffices, for the reasons given in Remarks 4.5, to work with the subcategory $\text{fingen}_\infty(\Lambda)$. We make this assumption whenever convenient. We now outline what must be done.

Complete local case. The details of this appeared in [KL2], and we give its flavor in Sections 7 and 8.

Nonlocal case. Let $M \in \text{fingen}(\Lambda)$ be given. Then we know the structure of every $\mathfrak{m}$-adic completion $\hat{M}_{\mathfrak{m}}$ by Proposition 5.1 and the complete local case. The rest of this section deals with the following three questions.

(i) The *package deal question:* For which families $\{M(\mathfrak{m})\}$, where each $M(\mathfrak{m}) \in \text{fingen}(\hat{\Lambda}_{\mathfrak{m}})$, does there exist $M \in \text{fingen}(\Lambda)$ such that $\hat{M}_{\mathfrak{m}} \cong M(\mathfrak{m})$ for all $\mathfrak{m}$?

(ii) What *additional information* is needed, to determine the isomorphism class of M? Here it is usually more convenient to work in $\text{fingen}_\infty(\Lambda)$.

(iii) What are the *direct-sum relations* in fingen(Λ)? Again, working in $\text{fingen}_\infty(\Lambda)$ is more convenient.

Question (i): package deals. The answer is straightforward. For almost all $\mathfrak{m} \in \text{maxspec}(\Lambda)$, $\hat{M}_{\mathfrak{m}}$ must be $\hat{\Lambda}_{\mathfrak{m}}$-free; and the torsionfree ranks of the $M(\mathfrak{m})$ must satisfy a simple, obviously necessary consistency condition: $M(\mathfrak{m})_{\mathfrak{p}} \cong M(\mathfrak{m}')_{\mathfrak{p}}$ whenever the maximal ideals $\mathfrak{m}, \mathfrak{m}'$ contain a common minimal prime ideal $\mathfrak{p}$. We refer to these isomorphisms as "equality of torsionfree ranks" because $\hat{\Lambda}_{\mathfrak{p}}$ is always a field, and hence its modules are determined up to isomorphism by their vector-space dimension.

These consistency conditions remain necessary and sufficient for the existence of M in much more generality than Dedekind-like rings. They have long been well-known for torsionfree modules over arbitrary reduced noetherian rings Ω of Krull dimension 1, when the normalization is a finitely generated Ω-module. For the more general result needed here — dropping the "torsionfree" and "finite normalization" hypotheses, see [LO, §2]. In slightly more detail: Unlike the classical situation, it is not true that $\Lambda_{\mathfrak{m}}$ is a discrete valuation ring for all but finitely many $\mathfrak{m}$. However, for each $M \in \text{fingen}(\Lambda)$ we have that $M_{\mathfrak{m}}$ is a free $\Lambda_{\mathfrak{m}}$-module for all but finitely many $\mathfrak{m}$. The difference between this and the classical situation is that the set of nontrivial (= nonfree) localizations changes from module to module.

Question (ii): additional information. This additional information is given by the ideal class groups of the Dedekind-domain direct summands of Γ (discussed below) and by the units in residue fields of Λ and Γ. We omit discussion of this complicated topic, except to say that the two ingredients in it are put together in a Mayer-Vietoris sequence. See (5.5.1) for slightly more detail.

Question (iii): direct-sum relations. To answer this, we introduce the *web of genus class groups.* This consists of a system of genus class groups, together with a system of homomorphisms — called ξ-maps — between certain pairs of these groups. The rest of the present section describes, roughly, how this works.

Definitions 5.3. For $M \in \operatorname{fingen}(\Lambda)$ let

$$\begin{aligned} \operatorname{genus}(M) &= \{N \in \operatorname{fingen}(\Lambda) \mid \hat{N}_{\mathfrak{m}} \cong \hat{M}_{\mathfrak{m}} \ \ \forall \mathfrak{m} \in \operatorname{maxspec}(\Lambda)\} \\ &= \{N \in \operatorname{fingen}(\Lambda) \mid N_{\mathfrak{m}} \cong M_{\mathfrak{m}} \ \ \forall \mathfrak{m} \in \operatorname{maxspec}(\Lambda)\} \end{aligned} \tag{5.3.1}$$

and let $[M]$ denote the isomorphism class of M. The idea conveyed by the term "genus" is that if one considers an isomorphism class to be the analog of the biological notion of "species", then the next more general class is analogous to a genus.

The next step is to define the abelian *genus class group* $\mathcal{G}(M)$. The elements of this group are the isomorphism classes of modules in genus(M). It can be shown that this collection of isomorphism classes can be made into an abelian group in such a way that $[M]$ is the zero element of this group, and:

$$\begin{gathered} \oplus_{i=1}^{m} M_i \cong \oplus_{i=1}^{n} N_i \qquad \big(M_i, N_i \in \operatorname{genus}(M)\big) \\ \Longleftrightarrow \\ m = n \quad \text{and} \quad \sum_i [M_i] = \sum_i [N_i] \quad \text{in } \mathcal{G}(M) \end{gathered} \tag{5.3.2}$$

In fact, this can be done in exactly one way, and the definition is implicitly contained in (5.3.2): Take $m = n = 2$ and $M_2 = M$. Since $[M] = 0$ in $\mathcal{G}(M)$, the second line of (5.3.2) becomes $[M_1] = [N_1] + [N_2]$ in $\mathcal{G}(M)$, while the first line states that this sum $[M_1]$ of $[N_1]$ and $[N_2]$ in $\mathcal{G}(M)$ must be the isomorphism class that satisfies the direct-sum condition $M_1 \oplus M \cong N_1 \oplus N_2$ (and a proof is required, to show that a unique such $[M_1]$ always exists for given N_1 and N_2).

These are well-known facts for torsionfree modules over the rings that occur in integral representation theory. In the present generality, the proof requires Serre's direct-summand theorem, Bass's cancellation theorem, and the following cancellation result of Guralnick and Levy [GL, 5.10] (extending an earlier result of Drozd). Let Ω be a reduced noetherian ring of Krull dimension 1, and suppose that $M \oplus X \cong N \oplus X$, with $M, N, X \in \operatorname{fingen}(\Omega)$ and $X \in \operatorname{genus}(M)$. Then $M \cong N$.

Remark on the zero element $[M]$ *of* $\mathcal{G}(M)$*.* Note that, in the above definition, $[M]$ was an arbitrarily chosen isomorphism class in its genus. We call it the *base-point* of genus(M), to emphasize this arbitrariness.

It is evident that (5.3.2) describes all direct-sum relations in genus(M), once one knows the group $\mathcal{G}(M)$. Actual computation of this group in specific instances is at least as difficult as computing ideal class groups of rings of integers in algebraic number fields. However, as in the classical situation of Dedekind domains, merely knowing that $\mathcal{G}(M)$ is an abelian group gives a lot of useful information (see below).

Remaining question: How do we adapt (5.3.2) to handle direct sums when the terms are not all one genus? Informally, we want to "add" elements in different groups. Formally, we use the web of class groups. The basic idea is simple, but since the details are complicated, we just sketch the main ideas.

Definition 5.4 (Web of genus class groups)**.** As a set, this consists of one genus class group $\mathcal{G}(M)$ — together with its arbitrarily selected base point $[M]$ — for each genus in $\operatorname{fingen}_\infty(\Lambda)$. Natural maps $\xi = \xi^{M,N}\colon \mathcal{G}(M) \to \mathcal{G}(N)$ are defined for certain pairs M, N; and the *web of class groups* consists of these ξ-maps together with the aforementioned groups $\mathcal{G}(M)$.

Let $M, N \in \text{fingen}_\infty(\Lambda)$. We say that $\xi^{M,N}$ *is defined* if, for each module $M' \in \text{genus}(M)$, there exists a module $N' \in \text{genus}(N)$, *unique up to isomorphism*, such that $M' \oplus N \cong M \oplus N'$. When this is the case, we set $\xi^{M,N}[M'] = [N']$. See [KL3, Definition 25.4ff] for more about this.

Instead of dwelling on the definition, we proceed to survey how these ξ-maps are used, to understand direct sums and the relation of the various genus class groups to each other.

We call these ξ-maps *loss of structure maps* because they can have nonzero kernels, indicating that information is lost when we pass from one genus to another by means of a ξ-map with nonzero kernel.

The normalization Γ is always a direct sum of Dedekind domains. We regard Γ as an approximation to Λ, and $\text{fingen}(\Gamma)$ as an approximation to $\text{fingen}(\Lambda)$. In particular, the (known) structure of $\text{fingen}(\Gamma)$ is a special case of what we describe here. In several places we comment on this, and on the way in which it serves as an approximation. Our starting point is:

Theorem 5.5. *Let $M, N \in \text{fingen}_\infty(\Lambda)$.*

- (i) *If M is faithful and ${}_\Lambda\Gamma$ is finitely generated, then $\xi^{M,\Gamma}$ is defined and surjective.* [KL3, Proposition 32.3]
- (ii) *If M, N are both faithful and $\xi^{M,N}$ is defined, then $\xi^{M,N}$ is surjective.* [KL3, Lemma 25.6]

To understand statement (i), assume that we are in the classical situation that ${}_\Lambda\Gamma$ is finitely generated. Then $\mathcal{G}({}_\Lambda\Gamma)$ is meaningful; and it is not difficult to see that this coincides with $\mathcal{G}({}_\Gamma\Gamma)$, the genus class group of Γ, considered as a Γ-module. Thus the fact that $\xi^{M,\Gamma}$ is surjective but can have nonzero kernel indicates that $\mathcal{G}(M)$ contains at least as much "global information" as $\mathcal{G}(\Gamma)$ contains. Thus $\mathcal{G}({}_\Gamma\Gamma)$ is a first approximation to $\mathcal{G}(M)$ for every faithful ${}_\Lambda M$.

We can be more explicit about this — *whether or not ${}_\Lambda\Gamma$ is finitely generated.* Since Γ is a direct sum of Dedekind domains, say $\Gamma = \oplus_h \Gamma_h$, the group $\mathcal{G}({}_\Gamma\Gamma)$ can be identified with the direct sum of the genus class groups $\mathcal{G}({}_{\Gamma_h}\Gamma_h)$ of the various Γ_h, more classically known as their *ideal class groups.* Thus the direct sum of these ideal class groups forms the first approximation to every $\mathcal{G}(M)$ when ${}_\Lambda M$ is faithful. In slightly more detail, there is a "Mayer-Vietoris" short exact sequence for every faithful ${}_\Lambda M$:

$$0 \to K \to \mathcal{G}(M) \xrightarrow{\xi} \mathcal{G}({}_\Gamma\Gamma) \to 0 \tag{5.5.1}$$

In the classical situation that ${}_\Lambda\Gamma$ is finitely generated, we can use $\xi = \xi^{M,\Gamma}$ here. In general, ξ denotes the composition of $[M] \to {}_\Gamma[\Gamma \otimes_\Lambda M]$ with the ξ-map of Γ-modules $\xi_\Gamma^{(\Gamma\otimes_\Lambda M),\Gamma}$. The kernel K is formed from subgroups of the groups of units of residue fields of Λ and Γ. Readers familiar with classical Mayer-Vietoris sequences might be interested to know that, when ${}_\Lambda\Gamma$ is not finitely generated — and hence there is no conductor ideal for Λ and Γ — we take advantage of the fact that local conductors always exist for Dedekind-like rings. See [KL3, Corollary 32.4 and Theorem 25.14] for details.

To help clarify the role of faithfulness in statement (ii) of Theorem 5.5, consider the trivial case $\Lambda = \Gamma$, and suppose that $\Gamma = \Gamma_1 \oplus \Gamma_2$, the direct sum of two

Dedekind-domains. Then $\mathcal{G}(\Gamma) = \mathcal{G}(\Gamma_1) \oplus \mathcal{G}(\Gamma_2)$, and $\xi^{\Gamma_1,\Gamma}$ can be identified with the first coordinate projection in this direct sum. This is never a surjection unless $\mathcal{G}(\Gamma_2) = \{0\}$. In greater generality, lack of surjectivity of $\xi^{M,N}$ occurs precisely when the torsionfree part of the Γ-module $\Gamma \otimes_\Lambda N$ involves coordinate rings Γ_h of Γ that have nontrivial ideal class groups but the torsionfree part of the Γ-module $\Gamma \otimes_\Lambda M$ does not involve these coordinate rings.

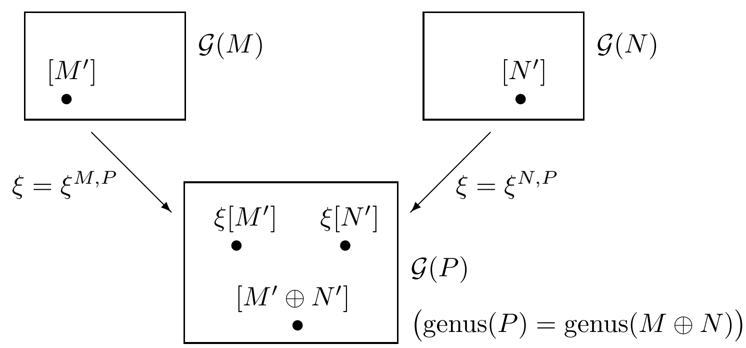
The next result gives one basic situation in which the ξ-map is defined [KL3, Corollary 25.9 and Lemma 25.6].

Theorem 5.6. *Let M, N be arbitrary in $\operatorname{fingen}_\infty(\Lambda)$, and $P \in \operatorname{genus}(M \oplus N)$. Then $\xi^{M,P}$ and $\xi^{N,P}$ are defined. If $\operatorname{ann}(M) = \operatorname{ann}(N)$, then $\xi^{M,P}$ and $\xi^{N,P}$ are surjective. ("*ann*" denotes "annihilator".)*

We can finally explain how the ξ-maps describe direct-sum behavior.

Notation 5.7 (Direct sums)**.** Let $M', N' \in \operatorname{fingen}_\infty(\Lambda)$ be given, and let M, N be the base-points in their respective genera. Also, let P be the base-point in $\operatorname{genus}(M \oplus N)$. The three boxes in diagram (5.7.1) show the three corresponding genus class groups, the given $[M'], [N']$, their natural images in $\mathcal{G}(P)$, and the element $[M' \oplus N']$ of $\mathcal{G}(P)$.

(5.7.1)



Note that the equation $\operatorname{genus}(P) = \operatorname{genus}(M \oplus N)$, at the bottom of the diagram, cannot be replaced by $\mathcal{G}(P) = \mathcal{G}(M \oplus N)$ because $M \oplus N$ might not be the base-point of its genus.

Theorem 5.8. *Keep the above notation. Then:*

$$[M' \oplus N'] = \xi[M'] + \xi[N'] + [P_0] \quad \text{in } \mathcal{G}(P) \tag{5.8.1}$$

where the "correction term" $[P_0]$ depends only on the (arbitrarily selected) base points M, N, P, and not on M' or N'. [KL3, Theorem 29.1]

Informally, the theorem says that the "sum" $[M'] + [N']$ is obtained by adding the natural images $\xi[M']$ and $\xi[N']$ in $\mathcal{G}(P)$, and then adding an (annoying) correction term whose presence is due to the completely arbitrary choice of base-points M, N, P in the three genera. This correction term seems unavoidable. For example, if we happen to have chosen $P = M \oplus N$, then the correction term becomes zero, and we get the more pleasant formula [KL3, Corollary 29.2]:

$$[M' \oplus N'] = \xi[M'] + \xi[N'] \quad \text{in } \mathcal{G}(P) \tag{5.8.2}$$

Nevertheless, Theorem 5.8 shows that *all non-locally-determined aspects of direct-sum behavior are determined by abelian groups and group homomorphisms.*

The reason for only saying *seems* unavoidable, above (5.8.2), is that we do not know whether it is possible to pick the set of base-points in such a way that it is closed under direct sums [KL3, §37 Problem 6].

As an application of this we get a definitive answer to the question: How can direct-sum cancellation fail? First we review a well-known result [E].

Lemma 5.9 (Evans)**.** *The implication* $M' \oplus N \cong M'' \oplus N \implies M'' \in \mathrm{genus}(M')$ *holds for finitely generated modules over any noetherian ring.*

Theorem 5.10. *Let* $M', N' \in \mathrm{fingen}_\infty(\Lambda)$*, let* M, N, P *be the base-points in the genera of* $M', N', (M' \oplus N')$ *respectively, and let* $M'' \in \mathrm{fingen}_\infty(\Lambda)$*. Then:*

$$M' \oplus N' \cong M'' \oplus N' \iff \quad M'' \in \mathrm{genus}(M) \quad \textit{and} \quad [M'] - [M''] \in \ker(\xi^{M,P}) \quad \textit{in } \mathcal{G}(M) \tag{5.10.1}$$

Proof. Suppose that the isomorphism in (5.10.1) holds. Then, by Evans's lemma, M' and M'' are both in the genus having base-point $[M]$. Since $[M' \oplus N'] = [M'' \oplus N']$, two applications of (5.8.1) yield:

$$\xi^{M,P}[M'] + \xi^{N,P}[N'] + [P_0] = \xi^{M,P}[M''] + \xi^{N,P}[N'] + [P_0] \qquad \text{in } \mathcal{G}(P)$$

Since $\mathcal{G}(P)$ is an abelian group and $\xi^{M,P}$ is a homomorphism, we deduce that $[M'] - [M''] \in \ker(\xi^{M,P})$.

Reversing the reasoning in the previous paragraph proves the converse part of the theorem. □

Remarks 5.11. (i) *Note the uniformity of the cancellation result in Theorem 5.10:* It is independent of the choice of N' in genus(N) and independent of which M' in genus(M) one starts with.

(ii) It is very common that cancellation actually fails. For example, it can fail for rings of all of the types (i)–(iv) enumerated in Examples 4.9 [but not for types (v) and (vi)].

(iii) Failure of cancellation, when it occurs, is very different from the failure of cancellation that occurs in K-theory because of instability. The latter type of failure occurs when the stable isomorphism classes that are elements of the appropriate K_0-group are not actual isomorphism classes. However, the elements of our genus class groups are always actual isomorphism classes.

Other direct-sum behavior. Dedekind-like rings exhibit a very rich variety of describable direct-sum behavior (in addition to failure of direct-sum cancellation).

Example 5.12. Choose $n \geq 2$. Then there exist a Dedekind-like ring Λ and $M \in \mathrm{fingen}(\Lambda)$ such that M is the direct sum of s indecomposables for every s in interval $2 \leq s \leq n$. (Since Λ and M are noetherian, we can never find a single M such that this decomposition property holds for all $s \geq 2$.)

In fact, Λ can be chosen to be a group ring $\mathbb{Z}G_m$ (G_m cyclic of squarefree order m) [L3, (0.1)]; and can also be chosen to be a subring of squarefree index in $\mathbb{Z} \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$ [L1].

Coherence of the web of class groups. The set of all genus class groups of faithful modules in $\mathrm{fingen}_\infty(\Lambda)$ has an inverse limit s$\mathcal{G}$, with respect to the ξ-maps. (Recall that these maps are surjections when they are defined [Theorem 5.5]).

If ${}_{\Lambda}\Gamma$ is finitely generated, then this “super genus class group” s$\mathcal{G}$ is an actual genus class group $\mathcal{G}(P)$, where P has torsionfree rank ≤ 2 [KL3, Theorem 27.12].

Suppose that Λ is an integral domain. Then there are situations in which P cannot be chosen to have torsionfree rank 1 [KL3, Example 34.1]. In such situations, the resulting P of rank 2 *cannot be tosionfree!* [KL3, Theorem 27.12] This is a counterexample to the generally held belief that torsionfree modules can hold more “global information” than any other modules.

6. mod-Γ as approximation to mod-Λ

In this section Λ is again a non-exceptional Dedekind-like ring with normalization Γ. We have already discussed how the isomorphism classes in $\mathrm{fingen}_{\infty}(\Lambda)$ are approximated by those in fingen(Γ) — via Mayer-Vietoris sequences. In the present section we disuss how homomorphisms in fingen(Λ) are approximated by those in fingen(Γ). This approximation, called a “separated cover”, is the basic tool used in this series of papers to obtain the structure of Λ-modules from the known theory of Γ-modules.

We have a basic difficulty that does not occur in the study of torsionfree modules: *It is not true that every finitely generated Λ-module is contained in some Γ-module.* The following two definitions are designed to deal with this difficulty.

Definitions 6.1. We define a Γ-*separated* Λ-*module* to be any Λ-submodule of any Γ-module.

We define a Γ-*separated cover* $\phi\colon S \twoheadrightarrow M$ of $M \in$ fingen(Λ) to be a Λ-module homomorphism such that:

- S is a Γ-separated Λ-module; and
- S is “as close as possible” to M, in the sense that in all (surjective) factorizations

$$\phi\colon S \overset{\theta}{\twoheadrightarrow} S' \twoheadrightarrow M \quad \text{(with } S' \ \Gamma\text{-separated)} \tag{6.1.1}$$

θ must be an isomorphism. In other words, in any factorization of the form (6.1.1), S' is no closer to M than S is.

Since free Λ-modules are Γ-separated, and since all of the modules we are dealing with are noetherian, it is a triviality that *every $M \in$ fingen(Λ) has a Γ-separated cover such that the separated covering module S is again in* fingen(Λ).

We think of S as the “best approximation” to M by a Λ-submodule of a Γ-module. The reason that separated covers are useful is [KL3, Theorem 18.10]:

Theorem 6.2 (Almost functorial property)**.** *Let $f\colon N \to M$ be a homomorphism in* fingen(Λ)*, and let ϕ', ϕ be Γ-separated covers. Then f can be lifted to a Λ-homomorphism θ such that the following diagram commutes.*

$$\begin{array}{ccc} S' & \overset{\theta}{\dashrightarrow} & S \\ \Big\downarrow{\scriptstyle \phi'} & & \Big\downarrow{\scriptstyle \phi} \\ N & \overset{f}{\longrightarrow} & M \end{array} \tag{6.2.1}$$

If f is one-to-one or onto then θ has the same property.

Corollary 6.3 (Uniqueness of separated covers)**.** *For any $M \in$* fingen(Λ)*, the Γ-separated cover $\phi\colon S \twoheadrightarrow M$ of M is unique up to isomorphism over M.*

Proof. Let ϕ' be another Γ-separated cover of M, and apply Theorem 6.2 with f equal to the identity map. □

See Theorem 8.5 for an explicit display of the Γ-separated covers of all indecomposable modules in fingen(Λ), for the ring $\Lambda = \mathbb{R} + x\mathbb{C}[[x]]$.

For an arbitrary pair of rings $\Upsilon \supseteq \Delta$, we define an *$\Upsilon$-separated Δ-module* and an *Υ-separated cover of a Δ-module* by replacing Γ and Λ with Υ and Δ respectively, in Definitions 6.1. This additional flexibility is needed in the next result which states, informally, that separated covers are stable with respect to localization and completion [KL3, Theorem 18.13].

Recall that if Λ is a Dedekind-like ring with normalization Γ, then $\big(\forall \mathfrak{m} \in \operatorname{maxspec}(\Lambda)\big)$ $\Lambda_\mathfrak{m}$ and $\hat{\Lambda}_\mathfrak{m}$ are again Dedekind-like with normalization $\Gamma_\mathfrak{m}$ and $\hat{\Gamma}_\mathfrak{m}$, respectively [Proposition 5.1].

Corollary 6.4. *Consider a surjective Λ-homomorphism ϕ: $S \twoheadrightarrow M$ in* fingen(Λ)*, where S is Γ-separated.*

(i) *ϕ is a Γ-separated cover if and only if its localization $\phi_\mathfrak{m}$: $S_\mathfrak{m} \twoheadrightarrow M_\mathfrak{m}$ is a $\Gamma_\mathfrak{m}$-separated cover* $\big(\forall \mathfrak{m} \in \operatorname{maxspec}(\Lambda)\big)$.

(ii) *ϕ is a Γ-separated cover if and only if its $\mathfrak{m}$-adic completion $\hat{\phi}_\mathfrak{m}$: $\hat{S}_\mathfrak{m} \twoheadrightarrow \hat{M}_\mathfrak{m}$ is a $\hat{\Gamma}_\mathfrak{m}$-separated cover* $\big(\forall \mathfrak{m} \in \operatorname{maxspec}(\Lambda)\big)$.

7. Module Structure: Complete Local Case

We are separating this discussion from the earlier "non-local" parts of this survey for two reasons. (i) When Λ is a complete local ring, fingen(Λ) is a Krull-Schmidt category; and therefore describing the indecomposable modules becomes much more important. The full description is given in [KL2, §§2,3] and is quite long. Here we give just enough to establish its flavor and the flavor of the method used to obtain the description. (ii) All indecomposable artinian rings are complete local rings.

As our starting point we give the promised explicit description of the structure of complete local Dedekind-like rings. For a first reading, it is probably better to simply assume that all rings in this section are complete local rings. However, we invoke the completeness hypothesis only when it is needed.

7.1. **Strictly Split local Dedekind-like rings.** Let $(\Gamma_1, \mathfrak{m}_1, k)$ and $(\Gamma_2, \mathfrak{m}_2, k)$ be DVRs with the same residue field k, and ρ_i: $\Gamma_i \twoheadrightarrow k$ $(i = 1, 2)$ the natural homomorphism. Let

$$\Lambda = \{(x_1, x_2) \in \Gamma = \Gamma_1 \oplus \Gamma_2 \mid \rho_1(x_1) = \rho_2(x_2)\} \tag{7.1.1}$$

Then $(\Lambda, \mathfrak{m}_1 \oplus \mathfrak{m}_2, k)$ *(maximal ideal $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$) is a local Dedekind-like ring with normalization* Γ. [KL1, Lemma 2.15] We call such a local Dedekind-like ring *strictly split*. These are the Dedekind-like rings studied by Nazarova and Roiter in [NR]. The simplest example here (though not the example that interested Nazarova and Roiter) is $\Lambda = k[[x, y]]/(xy)$ [KL1, 2.17].

7.2. **Unsplit local Dedekind-like rings.** Let $(\Gamma, \mathfrak{m}, F)$ be a DVR whose residue field F is a 2-dimensional extension of a subfield k, and let ρ: $\Gamma \twoheadrightarrow F$ be the natural homomorphism. Let

$$\Lambda = \{x \in \Gamma \mid \rho(x) \in k\} \tag{7.2.1}$$

Then $(\Lambda, \mathfrak{m}, k)$ *is a local Dedekind-like ring with normalization* Γ. [KL1, Lemma 2.16] We call such a local Dedekind-like ring *unsplit.* Moreover, Λ *is exceptional if and only if* F *is an inseparable extension of* k (necessarily of characteristic 2, since the dimension is 2).

The simplest example here is the power-series ring $\Lambda = k + xF[[x]]$, and is exceptional if and only if k has characteristic 2 and its dimension-2 extension F is inseparable [see Definition 4.3].

There are two other kinds of local Dedekind-like rings. One is called "nonstrictly split" [KL1, Definition 2.1] and cannot occur in the complete local case, the other is the familiar DVR.

We have [KL1, Definition 2.5, Lemmas 2.15, 2.16]:

Theorem 7.3. *Every complete local Dedekind-like ring is either strictly split, unsplit, or a DVR. (The one or two DVRs whose direct sum is the normalization of* Λ *are complete DVRs.)*

When Λ is complete local, Γ-separated covers [Definitions 6.1] are the basic tool used in [KL2] to find the explicit structure of isomorphism classes in fingen(Λ). Separated covers are used to convert the isomorphism question for Λ-modules to a matrix problem over the residue fields of Λ and Γ. The module-structure problem then separates naturally into three cases: strictly split, unsplit, and DVR. We ignore DVRs, since we have nothing new to say about them. We discuss the remaining two cases separately, below. An interesting curiosity is that we do not need to assume completeness, because fingen(Λ) is a Krull-Schmidt category in the strictly split and unsplit cases [KL1, Lemma 1.3].

7.4. **Strictly Split Case.** Here Γ is the direct sum of two DVRs, each with the same residue field as Λ. Say $\Gamma = \Gamma_1 \oplus \Gamma_2$. The solution of the associated matrix problem makes use of the "sweeping similarity" results in [KL0]. Matrix problems of this type are well-known in the finitely generated algebra community, whose methods provide an alternative solution of the matrix problem.

However, even for finite-length Λ-modules, our methods provide a description of the indecomposable Λ-modules that is slightly different from the description usually given for finite dimensional algebras. We call the two types of indecomposable modules "deleted cycle" and "block cycle" modules. These are known to the finite dimensional algebra community as "string" and "band" modules, respectively.

Our description gives a bit of new insight into the reason for their biserial nature: Consistent with our point-of-view that fingen(Γ) is an approximation to fingen(Λ), we begin with a direct sum $X = \oplus_i X_i$ of indecomposable modules in fingen(Γ). Each X_i is uniserial as a Γ-module, since Γ is the direct sum of two DVRs. Pairs of these modules X_i can then be glued together at the top and — when they have finite length — at the bottom. When this gluing is done appropriately, it results in the deleted cycle (string) and block cycle (band) modules.

When k is not algebraically closed, this gluing essentially amounts to another way of viewing the original Nazarova-Roiter results. We omit further details of the gluing here, since they are similar to the details in the unsplit case, which we discuss more fully below. The full details can be found in [KL2, §3].

7.5. **1-parameter families.** Readers familiar with representations of finite dimensional algebras are probably wondering about the role of 1-parameter families here. The block cycle (band) modules are indeed organized into 1-parameter families, as expected. But since the residue field k of Λ is not algebraically closed (and hence the Jordan canonical form is no longer available), the parameter is no longer an element of the residue field k of Λ. Instead, the parameter becomes a class of matrices that is indecomposable under similarity. Alternatively, by using the rational canonical form, the parameter can be taken to be a power of an irreducible polynomial in $k[x, x^{-1}]$.

However, all indecomposable modules of infinite length in fingen(Λ) are deleted cycle (string) modules, and therefore never have a corresponding "parameter". Thus, after separating fingen(Λ) into the two categories finlen(Λ) and $\text{fingen}_\infty(\Lambda)$, and noting that finlen(Λ) is really part of the complete local case, these 1-parameter families play no further role in the nonlocal situation studied in [KL3].

7.6. **Unsplit Case.** Here, Λ and Γ are both integral domains, and hence the normalization Γ of Λ is a DVR. The residue field F of Γ is a 2-dimensional separable extension of the residue field k of Λ (in the non-exceptional case). This situation, of course, does not occur when k is algebraically closed. It might be that our results are new here, even for finite dimensional Λ-modules when Λ is a k-algebra. The prototypical example is $k = \mathbb{R}$ and $F = \mathbb{C}$, and $\Lambda = \mathbb{R} + x\mathbb{C}[[x]]$. We give a detailed description of the resulting indecomposable modules in fingen(Λ) in this example in Section 8 below. The results in the general unsplit case are only slightly more complicated.

Again we obtain our indecomposable Λ-modules as certain combinations of indecomposable Γ-modules, the latter being uniserial because Γ is a DVR.

7.7. **Klein Rings.** To complete the complete local picture, we need to comment on Klein rings. Let $(\Omega, \mathfrak{n}, k)$ be a Klein ring. Then [KL2, Theorem 11.2]:

(i) Ω is a quasi-Frobenius ring with simple socle $\mathfrak{n}^2$ and k has characteristic 2.

(ii) $\Upsilon = \Omega/\mathfrak{n}^2$ is a homomorphic image of a stricIty split (therefore non-exceptional, and hence fingen-tame) complete local Dedekind-like ring Λ.

(iii) Every Ω-module is the direct sum of a free module and an Υ-module.

Perhaps the most tantalizing fact about Klein rings is the following [KL1, Theorems 5.2, 5.1]. *A Klein ring $(\Omega, \mathfrak{n}, k)$ is a homomorphic image of some Dedekind-like ring (say) Λ if and only the residue field k is an imperfect field. When the condition holds, the Dedekind-like ring Λ must be exceptional* (in which case we do not know whether Λ is tame, wild, or neither, although Ω is necessarily tame!).

7.8. **Link to Non-local Case.** The reduction to the complete local case [KL3] is obtained by intensive use of Corollary 6.4, and the resulting arguments are unfortunately quite long, as is the complete local case itself, in [KL2]. Perhaps someone with a fresh approach to the subject can find a simpler path to our results.

8. Module structure, one special (complete local) case

Notation 8.1. In this section $(\Lambda, \mathfrak{m}, k)$ denotes the complete local unsplit Dedekind-like ring displayed in (8.1.1), together with its normalization Γ and some other details.

$$\begin{aligned} \Lambda &= \mathbb{R} + x\mathbb{C}[[x]], \qquad \Gamma = \mathbb{C}[[x]] \\ \mathfrak{m} &= x\mathbb{C}[[x]] = \Lambda x + \Lambda ix, \qquad \Lambda/\mathfrak{m} = \mathbb{R}, \quad \Gamma/\mathfrak{m} = \mathbb{C} \end{aligned} \tag{8.1.1}$$

Since *the Krull-Schmidt theorem holds for finitely generated modules* in the complete local case, we may limit our description of module structure to the structure of indecomposable modules in fingen(Λ). We describe these modules in four steps, starting with the simple, well-known structure of Γ-modules. See [KL2, §2] for more generality (beyond this example), as well as more complete details.

8.2. **Step 1: Γ-modules.** Since Γ is a DVR, all indecomposable modules in fingen(Γ) are uniserial and their isomorphism class is determined by their length: a positive integer or ∞. In more detail, the indecomposable Γ-module of any length d is

$$\tilde{\Gamma}_d := \Gamma/\mathfrak{m}^d \tag{8.2.1}$$

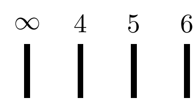
where we make the convention that $\mathfrak{m}^\infty = 0$ and hence $\tilde{\Gamma}_\infty = \Gamma$. When d is finite we can conveniently visualize $\tilde{\Gamma}_d$ by setting $x^d = 0$. More formally:

$$\tilde{\Gamma}_d = \mathbb{C} + \mathbb{C}\tilde{x}_d + \mathbb{C}\tilde{x}_d{}^2 + \cdots + \mathbb{C}\tilde{x}_d{}^{d-1} \qquad (\tilde{x}_d{}^d = 0,\ d < \infty) \tag{8.2.2}$$

where $\tilde{x}_d$ is the coset $x + \mathfrak{m}^d$. We refer to elements of $\tilde{\Gamma}_d$ as *truncated power series* when $d \neq \infty$.

Start with a direct sum X of indecomposable Γ-modules, for example the Γ-module X shown in (8.2.3) where each vertical bar represents one of the given indecomposable summands of X.

(8.2.3) $\qquad X = \tilde{\Gamma}_\infty \oplus \tilde{\Gamma}_4 \oplus \tilde{\Gamma}_5 \oplus \tilde{\Gamma}_6$:



We call the number over each vertical bar its *length label.*

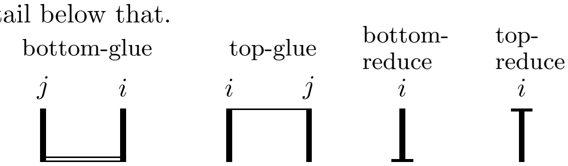
8.3. **Step 2: gluing and reduction.** The critical fact is that the uniserial Γ-modules $\tilde{\Gamma}_d$ are never uniserial Λ-modules. We repeatedly use the facts that the Γ-top and (when $d < \infty$) Γ-bottom of $\tilde{\Gamma}_d$ have the following form.

$$\begin{array}{ll} (\text{Top } \mathbb{C} \text{ of } \tilde{\Gamma}_d\text{:}) & \Gamma/\mathfrak{m} = \mathbb{C} \\ (\text{Bottom } \mathbb{C} \text{ of } \tilde{\Gamma}_d,\ d < \infty\text{:}) & \mathfrak{m}^{d-1}/\mathfrak{m}^d = \mathbb{C}\tilde{x}_d{}^{d-1} \cong \mathbb{C} \end{array} \tag{8.3.1}$$

where the "equality" in the first line is the identification induced by mapping each power series or truncated power series to its constant term, and the isomorphism at the end of the second line is the map $\alpha\cdot\tilde{x}_d{}^{d-1} \to \alpha$ ($\alpha \in \mathbb{C}$). Thus each of these (simple) Γ-tops and Γ-bottoms has dimension 2 as an $\mathbb{R}$-vector space, and hence is the direct sum of two isomorphic simple Λ-modules.

We use these facts to build new Λ-modules in four ways, as shown in (8.3.2) and described in detail below that.

(8.3.2)



We denote the complex conjugate of an element $\alpha \in \mathbb{C}$ by $\bar{\alpha}$.

Bottom-glue: (when $i, j < \infty$) This is the Λ-module $(\tilde{\Gamma}_j \oplus \tilde{\Gamma}_i)/K$ where K is the set of ordered pairs $(\alpha\cdot\tilde{x}_j{}^{j-1}, \bar{\alpha}\cdot\tilde{x}_i{}^{i-1})$ with $\alpha \in \mathbb{C}$. Less formally, this amalgamates the bottom $\mathbb{C}$ of $\tilde{\Gamma}_j$ with that of $\tilde{\Gamma}_i$ by identifying $\alpha\cdot\tilde{x}_j{}^{j-1}$ with $-\bar{\alpha}\cdot\tilde{x}_i{}^{i-1}$. This amalgamation is $\mathbb{R}$-linear but not $\mathbb{C}$-linear, and hence is Λ-linear but not Γ-linear. Hence it defines a Λ-module but not a Γ-module.

Top-glue: This is the Λ-submodule of $\tilde{\Gamma}_i \oplus \tilde{\Gamma}_j$ consisting of all pairs $\big(f(\tilde{x}_i), g(\tilde{x}_j)\big) \in \tilde{\Gamma}_i \oplus \tilde{\Gamma}_j$ such that the constant term of $f(\tilde{x}_i)$ equals the complex conjugate of the constant term of $g(\tilde{x}_j)$. As before, this Λ-module is not a Γ-module.

Bottom-reduce: (when $i < \infty$) This diagram represents the Λ-module $\tilde{\Gamma}_i/\mathbb{R}\tilde{x}_i{}^{i-1}$, which makes sense since the Γ-socle of $\tilde{\Gamma}_i$ equals $\mathbb{C}x_i^{i-1}$, as is evident from (8.3.1). Again, this is not a Γ-module.

Top-reduce: This diagram represents the Λ-submodule of $\tilde{\Gamma}_i$ consisting of all $f(\tilde{x}_i) \in \tilde{\Gamma}_i$ whose constant term is real.

The two gluing operations look very much like the familiar string modules. The top and bottom reductions, however, are a new complication not present in the strictly split case.

8.4. **Steps 3 and 4: Combine these gluings and reductions,** building the indecomposables described in steps 3 and 4. In each case we obtain the desired Λ-module $M(\mathcal{D})$ in the form $M(\mathcal{D}) = S/K$ from a displayed diagram $\mathcal{D}$ and Γ-module X. In step 3, the numerator S is the direct sum of Λ-submodules of X consisting of one term for each top-gluing and one for each top-reduction displayed in $\mathcal{D}$. The denominator K is the direct sum of one Λ-submodule of S for each bottom-gluing and one for each bottom-reduction displayed in $\mathcal{D}$. The same basic idea applies to step 4, but this is done in a more complicated "block form".

Before proceeding to the details of these steps, we note the following fact, which is proved at the beginning of [KL2, subsection 9.6].

Theorem 8.5. *In steps 3 and 4, the natural map $S \twoheadrightarrow S/K = M(\mathcal{D})$ is always a Γ-separated cover of $M(\mathcal{D})$.*

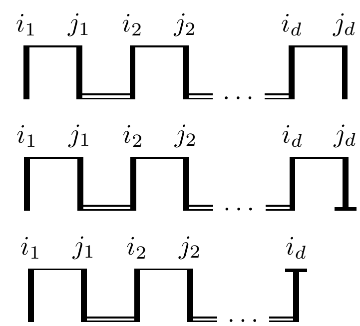
8.6. **Step 3: First series of combinations.** As shown in (8.6.1), begin with a direct sum $X = \tilde{\Gamma}_{i_1} \oplus \tilde{\Gamma}_{j_1} \oplus \tilde{\Gamma}_{i_2} \ldots$, and combine the terms by alternately top and bottom-gluing. Then either stop, or do a top- or bottom-reduction *at the right-hand end.*

(8.6.1)

$\mathcal{D}_{\mathrm{Nrd}}$: ("Nonreduced")

$\mathcal{D}_{\mathrm{Brd}}$: ("bottom-reduced")

$\mathcal{D}_{\mathrm{Trd}}$: ("top-reduced")



Each top-gluing results in replacing a direct sum $\tilde{\Gamma}_{i_c} \oplus \tilde{\Gamma}_{j_c}$ by a Λ-submodule S_c, and each top-reduction results in replacing some $\tilde{\Gamma}_{i_c}$ by a Λ-submodule S_c. (The top-reduction occurs in only one of these diagrams.) Let $S = \oplus_c S_c$.

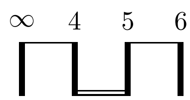
Each bottom-gluing is given by a Λ-submodule K_c of the bottom $\mathbb{C} \oplus \mathbb{C}$ of some $\tilde{\Gamma}_{j_c} \oplus \tilde{\Gamma}_{i_c}$, and each bottom-reduction is given by a Λ-submodule K_c of some $\tilde{\Gamma}_{j_c}$. (The bottom-reduction occurs in only one of these diagrams.) Let $K = \oplus_c K_c$.

If the conditions in the next remark are satisfied, we call $\mathcal{D}$ a *standard diagram.* We set $M(\mathcal{D}) = S/K$.

Remarks 8.7 (Restrictions in (8.6.1))**.** (i) In order for the above combinations to make sense: *The lengths of all uniserial* Γ*-modules in these diagrams must be finite except possibly for* i_1 *and, in the nonreduced case,* j_d. This holds because a Γ-module $\tilde{\Gamma}_e$ can be involved in bottom-gluing or bottom-reduction only if its length e is finite.

(ii) *The only situations in which we allow a uniserial* Γ*-module* $\tilde{\Gamma}_i$ *to equal the length-1 module* $\tilde{\Gamma}_1$ *in these diagrams is* i_1 *and, in the nonreduced case,* j_d*.* Ignoring this restriction results in construction of Λ-modules that can be constructed by not ignoring the restriction. [See (8.9.1) for a nonstandard example.] Thus this restriction is included only to simplify the statement of our uniqueness Theorem 8.12.

Examples 8.8. (i) We start with the Γ-module X shown in (8.2.3), and build the Λ-module $M = M(\mathcal{D})$ determined by the nonreduced diagram in (8.8.1).

(8.8.1) $\mathcal{D}$: 

Here, X is the set of 4-tuples:

(8.8.1) $$(q,r,s,t) = \big(q(\tilde{x}_\infty), r(\tilde{x}_4), s(\tilde{x}_5), t(\tilde{x}_6)\big) \in \ X = \tilde{\Gamma}_\infty \oplus \tilde{\Gamma}_4 \oplus \tilde{\Gamma}_5 \oplus \tilde{\Gamma}_6$$

Note that $q = q(\tilde{x}_\infty)$ is an actual power series, and so we could have written $q = q(x)$. But r, s, t are truncated power series.

We have $S = S_1 \oplus S_2$ where S_1 is the set of elements (q, r) of $\tilde{\Gamma}_\infty \oplus \tilde{\Gamma}_4$ such that the constant term of q equals the conjugate of that of r, and S_2 is the set of elements of (s,t) of $\tilde{\Gamma}_5 \oplus \tilde{\Gamma}_6$ such that the constant term of s equals the conjugate of that of t.

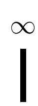
$K = K_1$, a single term since $\mathcal{D}$ has only one bottom-gluing and no bottom-reduction. In fact, K_1 is the set of elements $(0, \alpha{\cdot}\tilde{x}_4{}^3, \bar{\alpha}{\cdot}\tilde{x}_5{}^4, 0)$ where α ranges through the complex numbers.

Thus we have defined $M(\mathcal{D}) = S/K$.

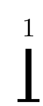
(ii) Suppose that, in addition to the gluing in part (i), we want a bottom-reduction at the right-hand end of the diagram. Then S is the same as in part (i). But $K = K_1 \oplus K_2$ where K_1 is as in part (i) and K_2 is the Λ-submodule $\mathbb{R}\tilde{x}_6{}^5)$ of $\tilde{\Gamma}_6$.

Examples 8.9. Before proceeding, we explicitly display the unique *standard diagrams* — that is, those of the form (8.6.1) — corresponding to the Λ-modules Λ, Γ, and $\mathbb{R} = \Lambda/\mathfrak{m}$. We also include one nonstandard diagram, in order to illustrate the kind of duplication that can occur if nonstandard diagrams are allowed.

(8.9.1) Λ : 

Γ : 

$\mathbb{R}$: 

$\mathbb{R}$ (nonstandard) : 

Theorem 8.10. *Consider the* Λ*-modules of types* (8.6.1)*.*

(i) *A nonreduced module is indecomposable if and only if the corresponding diagram* $\mathcal{D}_{\mathrm{Nrd}}$ *does not equal its left-right mirror image.*
(ii) *The reduced modules are all indecomposable.*

(iii) *Every indecomposable module of infinite length in* fingen(Λ) *is of exactly one of these types.*

This theorem is part of [KL2, Theorem 2.7], proved in [KL2, §9].

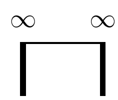
In connection with statement (iii) above, note that indecomposable modules of the types shown in (8.6.1) can also have finite length. This happens if and only if no uniserial Γ-module of length ∞ occurs in the diagram.

For examples that illustrate Theorem 8.10, note that the modules M constructed in Examples 8.8 are indecomposable, but the Λ-module determined by (8.11.1) is decomposable because it equals its left-right mirror image.

Corollary 8.11. *Every indecomposable module of infinite length in* fingen(Λ) *has torsionfree rank* ≤ 2. *If the rank equals 2, then the module cannot be torsionfree.*

Proof. The torsionfree rank of the module is easily seen to be the number of indices i_k and j_k that equal ∞. That this number is ≤ 2 is an immediate consequence of Theorem 8.10 and Remarks 8.7. The reason for the second (surprising!) statement of the corollary is that rank 2 can occur only in the nonreduced diagram $\mathcal{D}_{\mathrm{Nrd}}$ with $i_1 = j_d = \infty$. In this situation, if the Λ-module were torsionfree, then no $\tilde{\Gamma}_e$ could have finite length, so the diagram would have the form

(8.11.1)



and hence equal its left-right reflection. Therefore, by Theorem 8.10, the module would be decomposable. □

Theorem 8.12. *Two Λ-modules constructed from standard diagrams $\mathcal{D}$ and $\mathcal{D}'$ of the form* (8.6.1) *are isomorphic if and only if either:*

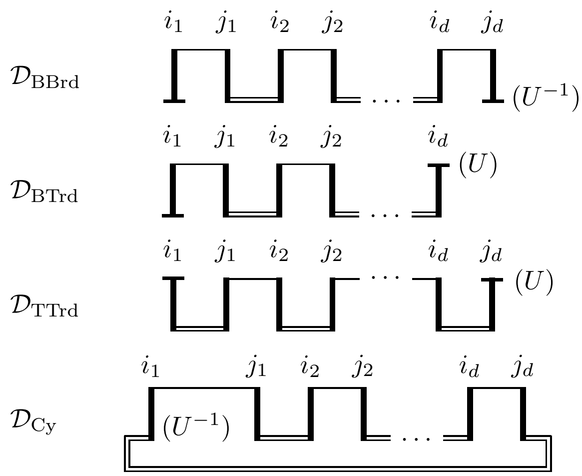
(i) *$\mathcal{D} = \mathcal{D}'$; or*
(ii) *$\mathcal{D}$ and $\mathcal{D}'$ are nonreduced and $\mathcal{D}'$ is the left-right mirror image of $\mathcal{D}$.*

This theorem is part of [KL2, Theorem 2.8], proved in [KL2, §9.3].

8.13. **Step 4: Second series of combinations.** Each Λ-module $M(\mathcal{D}) = S/K$ in this series is determined by one of the diagrams in (8.13.1). These diagrams display gluings, reductions, and an $m \times m$ invertible matrix U over $\mathbb{C}$, called the *blocking matrix.* We call m the *block size.* Each gluing and reduction in this series involves $2m$ or m uniserial Γ-modules, respectively; and we require every length label i_c and j_c to be both $\neq \infty$ and $\neq 1$, in order for the construction of $M(\mathcal{D})$ to make sense. In particular, the modules $M(\mathcal{D})$ in this series all have finite length. As suggested by their labels, we call these diagrams *Bottom-bottom-reduced, Bottom-Top-reduced, Top-Top-reduced, and Cycle,* respectively.

If the restrictions described in Remarks 8.16 are satisfied, we call $\mathcal{D}$ a *standard diagram.* These somewhat technical conditions must be imposed in order to insure indecomposability. (But the construction of $M(\mathcal{D})$ makes sense without these additional restrictions.)

(8.13.1)



Construction of $M(\mathcal{D})$. Let $\mathcal{D}$ and U (of size $m \times m$) be given. The Γ-module from which we build $M(\mathcal{D}) = S/K$ is:

(8.13.2) $$X = \tilde{\Gamma}_{i_1}^{(m)} \oplus \tilde{\Gamma}_{j_1}^{(m)} \oplus \ldots \oplus \tilde{\Gamma}_{i_d}^{(m)} \oplus \tilde{\Gamma}_{j_d}^{(m)}$$

Let $S = \oplus_c S_c$ where each S_c is (arbitrarily) associated with one of the top-gluings or top-reductions in $\mathcal{D}$, and is as defined below.

Top-glue. Suppose that a top-gluing edge connects $\tilde{\Gamma}_h$ to $\tilde{\Gamma}_k$ in $\mathcal{D}$. Then the corresponding S_c is the Λ-submodule of $\tilde{\Gamma}_h^{(m)} \oplus \tilde{\Gamma}_k^{(m)}$ consisting of all $2m$-tuples $(p_1, \ldots, p_m; q_1, \ldots, q_m)$ such that the constant term of each p_a is the complex conjugate of that of q_a.

Top-reduce. Suppose that a top-reduction occurs in $\mathcal{D}$ at some $\tilde{\Gamma}_h$ with U at its top. Then the corresponding S_c is the Λ-submodule of $\tilde{\Gamma}_h^{(m)}$ consisting of of all m-tuples $(p_1, \ldots, p_m)$ such that:

(8.13.3) $$\big(p_1(0), p_2(0), \ldots, p_m(0)\big) \in \mathbb{R}^{(m)}U \quad (\mathbb{R}\text{-row space of } U)$$

If, on the other hand, no matrix is attached to the top of $\tilde{\Gamma}_h$ in $\mathcal{D}$, then modify the corresponding S_c by replacing U by the identity matrix in (8.13.3) (thus getting the direct sum of m ordinary top-reductions) . Let $K = \oplus_c K_c$ where each K_c is (arbitrarily) associated with one of the bottom-gluings or bottom-reductions in $\mathcal{D}$, as defined below.

Bottom-glue. Suppose that a bottom-gluing edge connects $\tilde{\Gamma}_h$ to $\tilde{\Gamma}_k$ in $\mathcal{D}$, and U^{-1} is attached to the bottom of $\tilde{\Gamma}_h$. Then the corresponding K_c consists of all ordered pairs (each entry of which is an m-tuple) of the form:

(8.13.4) $$(\boldsymbol{\alpha} U^{-1} \tilde{x}_h{}^{h-1}, \bar{\boldsymbol{\alpha}} \tilde{x}_k{}^{k-1}) \in \mathbb{C}^{(m)} \tilde{x}_h{}^{h-1} \oplus \mathbb{C}^{(m)} \tilde{x}_k{}^{k-1} \subseteq \tilde{\Gamma}_h^{(m)} \oplus \tilde{\Gamma}_k^{(m)}$$

If, on the other hand, the bottom of $\tilde{\Gamma}_h$ is not attached to a matrix, modify the corresponding K_c by replacing U by the identity matrix in (8.13.4) (and getting the direct sum of m ordinary bottom-gluings).

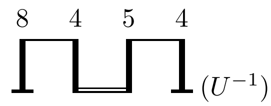
Bottom-reduce. Suppose that a bottom-reduction occurs at some $\tilde{\Gamma}_h$ in $\mathcal{D}$ with U^{-1} at its bottom. Then the corresponding K_c equals the Λ-submodule $\mathbb{R}^{(m)} U^{-1} \tilde{x}_h{}^{h-1}$ of $\tilde{\Gamma}_h^{(m)}$.

If no matrix is attached to the bottom of $\tilde{\Gamma}_h$, then the corresponding K_c is the Λ-submodule $\mathbb{R}^{(m)} \tilde{x}_h{}^{h-1}$ of $\tilde{\Gamma}_h^{(m)}$ (the direct sum of m ordinary bottom-reductions).

Abuse of notation. We do not require all of the numbers i_a and j_b in (8.13.1) to be distinct. Consequently, when repetition occurs, it is necessary to refer to the diagram to identify which summand of X the notation $\tilde{\Gamma}_h$ refers to. See Examples 8.14 for an instance of this.

Examples 8.14. (i) We construct the bottom-bottom reduced Λ-module $M = M(\mathcal{D}) = S/K$ determined by diagram (8.14.1) and a 3×3 blocking matrix U.

(8.14.1) $\mathcal{D}_{\text{BBrd}}$:



The Γ-module that we start with is:

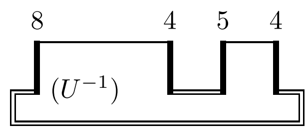
$$X = \tilde{\Gamma}_8^{(3)} \oplus \tilde{\Gamma}_4^{(3)} \oplus \tilde{\Gamma}_5^{(3)} \oplus \tilde{\Gamma}_4^{(3)} \tag{8.14.2}$$

We have $S = S_1 \oplus S_2$ where S_1 and S_2 result from the first and second displayed top-gluings respectively. Thus, for example, S_1 is the set of 6-tuples $(p_1, p_2, p_3; q_1, q_2, q_3)$ in $\tilde{\Gamma}_8^{(3)} \oplus \tilde{\Gamma}_4^{(3)}$ such that the constant term of each p_i is the conjugate of that of q_i. (Note that one must look at the diagram to determine which of the two copies of $\tilde{\Gamma}_4^{(3)}$ we are refering to .)

We have $K = K_1 \oplus K_2 \oplus K_3$, where successive terms K_i refer to the three displayed bottom operations. In particular, K_3 is the Λ-submodule $\mathbb{R}^{(3)} U^{-1} \tilde{x}_4{}^3$ of the second $\tilde{\Gamma}_4^{(3)}$ displayed in (8.14.2).

(ii) We construct the cycle module $M = M(\mathcal{D})$ determined by diagram (8.14.3) and a 3×3 blocking matrix U.

(8.14.3) $\mathcal{D}_{\text{Cy}}$:



We have $S = S_1 \oplus S_2$, corresponding to the two top-gluings, and $K = K_1 \oplus K_2$ corresponding to the two bottom-gluings. In fact, S is the same as in part (i). We explicitly display the term K_2 corresponding to the bottom-gluing of $\tilde{\Gamma}_8^{(3)}$ and (the second displayed) $\Gamma_4^{(3)}$. By (8.13.4), K_2 is the set of elements $(\boldsymbol{\alpha} U^{-1} \tilde{x}_8{}^7, \bar{\boldsymbol{\alpha}} \tilde{x}_4{}^3)$ in $\tilde{\Gamma}_8^{(3)} \oplus \tilde{\Gamma}_4^{(3)}$ where $\boldsymbol{\alpha}$ ranges through $\mathbb{C}^{(3)}$.

Remark 8.15 (Moving U)**.** The placement of U in our standard diagrams is somewhat arbitrary. It is possible to define $M(\mathcal{D})$ with its matrix attached to the top or bottom of any vertical bar in $\mathcal{D}$. Then, for example, U can be moved from the top of any vertical bar to the bottom (or vice-versa), if we replace it by U^{-1}. This explains our use of U at the top and U^{-1} at the bottom in (8.13.1). For the complete rules about moving U, see [KL2, Proposition 2.9].

Remarks 8.16 (Restrictions in (8.13.1))**.** We have already stated that every length label must be $\neq \infty$ and $\neq 1$. We call $\mathcal{D}$ *standard* if it satisfies the following additional conditions.

Except for cycle diagrams we require:

(i) $U\bar{U}^{-1}$ is indecomposable under similarity; and
(ii) The sequence $\{i_1, i_2, i_3, \ldots, j_3, j_2, j_1\}$ does not equal some strictly shorter sequence repeated some number of times.

Note that condition (i), above, is stronger than indecomposability of U itself. For example, if U is any invertible matrix with entries in $\mathbb{R}$, then $U\bar{U}^{-1} = I$. For a proof that matrices of all sizes exist satisfying condition (i). see Remark 8.17.

For cycle diagrams we require all of the following three conditions.

(i) U is indecomposable under similarity.
(ii) Either:
  (a) U is not similar to $\bar{U}^{-1}$, or
  (b) The sequence $\{j_1, j_2, \ldots, j_d\}$ does not equal a cyclic shift of the sequence $\{i_d, \ldots, i_2, i_1\}$ (i.e. relocation of some subsequence from the beginning to the end).
(iii) The sequence of pairs $(i_1, j_1), \ldots, (i_d, j_d)$ does not equal some strictly shorter sequence repeated some number of times.

Remark 8.17 (Indecomposable matrices $U\bar{U}^{-1}$)**.** *For every m there is an $m \times m$ matrix U over $\mathbb{C}$ such that $U\bar{U}^{-1}$ is indecomposable under similarity.*

Proof. Let W be the companion matrix of the polynomial $(x+1)^m$. Then W is indecomposable under similarity. With the help of the Cayley-Hamilton theorem, one can show that W is similar to $\bar{W}^{-1}$. See [KL2, (2.12.1)] for some details. A "Hilbert Theorem 90 for matrices" due to Ballantine [Ba, Lemma 8.11] states that this last similarity condition for a matrix W is equivalent to the existence of a matrix U such that $U\bar{U}^{-1} = W$. □

We are gateful to Robert Guralnick for showing us this theorem of Ballantine.

The remaining theorems in this section complete our description of the indecomposable modules in fingen(Λ), for the ring $\Lambda = \mathbb{R} + x\mathbb{C}[[x]]$ in (8.1.1). These results are special cases of [KL2, Theorem 2.8], proved in [KL2, Theorem 8.18].

Theorem 8.18. *Let $\mathcal{D}$ be any of the diagrams in* (8.13.1) *(i.e. involving a blocking matrix). If $\mathcal{D}$ is standard [i.e. satisfies the conditions in Remarks 8.16] then $M(\mathcal{D})$ is indecomposable.*

Theorem 8.19. *Every indecomposable module in* fingen(Λ) *is isomorphic to $M(\mathcal{D})$ for some unique type of standard diagram (i.e. bottom-reduced, bottom-bottom reduced, ...) in one of the series* (8.6.1) *and* (8.13.1).

Theorem 8.20 (Isomorphism: except cycle diagrams)**.** *Let $\mathcal{D}$ be one of the standard diagrams $\mathcal{D}_{\mathrm{BBrd}}$, $\mathcal{D}_{\mathrm{BTrd}}$, $\mathcal{D}_{\mathrm{TTrd}}$, and U its blocking matrix. The isomorphism invariants of $M(\mathcal{D})$ are:*

(i) *The similarity class of $U\bar{U}^{-1}$; and*
(ii) (a) *(*BTrd *diagrams:) The sequences $\{i_k\}$ and $\{j_k\}$.*
  (b) *(Other two diagrams:) The sequences $\{i_k\}$ and $\{j_k\}$ modulo left-right reflection of the diagram. That is, interchanging the i-sequence with the j-sequence and then reversing each of them does not change the isomorphism class of $M(\mathcal{D})$.*

In the next theorem, $\mu(\ldots)$ ("mirror image") denotes the reversal of any finite sequence, and let $\nu(\ldots)$ denotes a cyclic-shift of a finite sequence (i.e. move some subsequence from the beginning to the end).

Theorem 8.21 (Isomorphism: cycle diagrams)**.** *Let $\mathcal{D}, \mathcal{D}'$ be cycle diagrams with blocking matrices U, V respectively. Call the two associated length-label sequences of the first diagram I, J, and those of the second diagram I', J'.*

Then $M(\mathcal{D}) \cong M(\mathcal{D}')$ if and only if:.

(i) *V is similar to U and $I' = \nu(I)$ and $J' = \nu(J)$ for some ν; or*
(ii) *V is similar to $\bar{U}^{-1}$, and $I' = \nu\mu(J)$ and $J' = \nu\mu(I)$ for some ν.*

Note that the sequence manipulations in Theorem 8.21(i) and (ii) correspond to obvious symmetries of the diagram (rotation and left-right reflection).

9. Epilog on the Concept "Dedekind-like"

We are grateful to one of the referees for asking why we use the term "Dedekind-like" to refer to rings that seem to have little to do with Dedekind domains. As he (correctly) observes: Their ideal structure, their homological properties, and their representation theory are all significantly more elaborate. Moreover, Chapter 5 of McConnell and Robson's well-known textbook *Noncommutative Noetherian Rings* [MR] is entitled "Some Dedekind-like Rings", terminology that seems to conflict with the present usage since the rings dealt with in that chapter are mostly hereditary noetherian prime rings, the most natural noncommutative noetherian generalization of commutative Dedekind domains.

We think of Dedekind-like behavior as direct-sum behavior that is controlled by two types of invariants: *counting*-invariants, and *group* invariants. In the case of Dedekind-like rings Λ, the counting-invariants come from the fact that, for each maximal ideal of Λ, the category fingen$(\hat{\Lambda}_{\mathfrak{m}})$ is a Krull-Schmidt category. The group invariants come from the web of genus class groups [Definition 5.4].

These ideas come from Steinitz's original papers on modules over Dedekind domains [S], although it was not immediately evident that his invariants could be expressed in the form in which we view them. Let Λ be a Dedekind domain, a special case of Dedekind-like rings. Then every module in fingen(Λ) has a decomposition $P \oplus T$ where P is projective and T ("torsion") has finite length, and the two terms are unique up to isomorphism. Since finlen(Λ) is a Krull-Schmidt category, only counting-invariants occur in the description of T. Steinitz's really new idea involved the projective term P. There is a decomposition $P \cong \oplus_{i=1}^{n} M_i$ where each M_i is a nonzero ideal of Λ. Moreover, for any two finite families of nonzero ideals M_i, N_i of Λ we have

$$\oplus_{i=1}^{m} M_i \cong \oplus_{i=1}^{n} N_i \quad\Longleftrightarrow\quad m = n \quad \text{and} \quad \sum_i [M_i] = \sum_i [N_i] \quad \text{in } \mathcal{G}(\Lambda) \tag{9.0.1}$$

This becomes a special case of (5.3.2) as soon as one realizes that all nonzero ideals of Λ are in genus(Λ), because all localizations and completions of Λ at maximal ideals are principal ideal domains. Thus, in Steinitz's situation, our whole web of class groups collapses to the genus class group of the ring, more classically called the "ideal class group". More precisely, we can ignore all genus class groups except for $\mathcal{G}(\Lambda)$, because of (9.0.1).

The idea of genus class group occurs elsewhere in the literature, as mentioned earlier. One of the main new ideas in the present series of papers is that one can "add" elements of different class groups, by making use of the web of class groups — in order to describe all direct-sum relations over an arbitrary Dedekind-like ring.

Now consider (noncommutative!) HNP (hereditary noetherian prime) rings Λ. Most of the rings in McConnell-Robson's Chapter 5, "Some Dedekind-like rings" are of this type. The name refers to the fact that, as in the commutative case, every module in fingen(Λ) has a unique decomposition $P \oplus T$, up to isomorphism, where P is projective and T has finite length. Moreover, there is a decomposition $P \cong \oplus_{i=1}^{n} M_i$ where each M_i is a *uniform right ideal* of Λ (i.e. the intersection of any two nonzero submodules is again nonzero).

But what no-one seemed to suspect at the time is that Dedekind-like behavior — in the sense of the present paper — occurs here too, in the following form. It is possible to define a (nonunique) *normalization* Γ of Λ, a "genus class group" $\mathcal{G}(\Gamma)$, a somewhat "natural image" $[M]$ of the isomorphism class of every uniform right ideal M, in $\mathcal{G}(\Gamma)$, and *ranks* $\rho_{\mathfrak{m}}$ *at nonzero maximal ideals* $\mathfrak{m}$, in such a way that, for any two finite families, each consisting of *two or more* uniform right ideals of Λ, we have $\oplus_{i=1}^{m} M_i \cong \oplus_{i=1}^{n} N_i$ if and only if the following conditions hold.

(9.0.2) (i) $m = n$;
(ii) $\sum_i \rho_{\mathfrak{m}}(M_i) = \sum_i \rho_{\mathfrak{m}}(N_i)$ for every $\mathfrak{m}$; and
(iii) $\sum_i [M_i] = \sum_i [N_i]$ in $\mathcal{G}(\Gamma)$

Well-known counterexamples, dealing with "stable isomorphism" versus actual isomorphism show that the theorem fails unless the direct sums contain at least two terms. For details see Levy and Robson's papers [LR1, LR2]. HNP rings seem to be the only noncommutative noetherian rings whose projective modules exhibit nontrivial direct-sum behavior and possess a theorem describing that behavior.

A subsequent paper of Levy and Robson [LR3] gives a structure theorem for infinitely generated projective modules over HNP rings, again possibly the only noncommutative noetherian rings whose projective modules exhibit nontrivial direct-sum behavior and have a structure theorem describing that behavior. The main difference between this and the finitely generated case is that only conditions (i) and (ii) apply; that is, class groups disappear, and direct-sum behavior is determined by counting-invariants alone. The fact that class groups do not occur in the infinitely generated situation had been noted long ago, for commutative Dedekind domains, by Kaplansky [Ka]. He showed that, for commutative Dedekind domains, all nonfinitely generated projectives are free. Thus nontrivial direct-sum behavior does not occur until one considers noncommutative HNP rings.

There is also an interesting relationship between HNP rings and the tame-wild phenomenon. The category finlen(Λ) over an HNP ring Λ is a Krull-Schmidt category, and hence no nontrivial direct-sum behavior occurs. However, no description has been given of the indecomposable modules in finlen(Λ). Klingler and Levy [KL4] explain this by giving an example of an HNP ring Λ such that the category finlen(Λ) has wild representation type — strictly wild in this case. (Λ is the Weyl algebra $A_1(k)$ over an arbitrary field of characteristic 0.)

Thus it seems fitting to close this paper on tameness versus wildness for noetherian rings with some problems, starting with: *Is there a tame-wild theorem for modules of finite length over HNP rings?* There are probably other interesting connections between what representation theorists and other noetherian ring theorists study. For example, what happens to the main results of the present survey in the noncommutative noetherian case? Klingler made a beginning in his description of fingen(Λ) where $\Lambda = \mathbb{Z}G$, the integral group ring of a nonabelian group of order pq [K]. But a full tame-wild theorem for finitely generated modules (if one exists) seems to be a significant challenge, even in the more limited context of the rings that occur in integral representation theory.

References

[Ba] C. S. Ballantine, "Products of complic cosquares and pseudo-involutory matrices", *Lin. and Multilin. Alg.* **8** (1979), 73–78.

[B] S. Brenner, "Decomposition properties of some small diagrams of modules," *Symposia Mathematica* **13** (1974), 127–141.

[D1] Yu. A. Drozd, "Representations of commutative algebras" (Russian), *Funkstional'nyi Analiz i Ego Prilozheniya* **6** (1972), 41–43. English Translation in *Functional Analysis and its Applications* **6** (1972)

[D2] Yu. A. Drozd, "Finite modules over pure noetherian algebras", *Proc. Steklov Institute of Math* **4** (1991), 97–108.

[E] E. G. Evans, Jr., "Krull-Schmidt and cancellation over local rings," *Pac. J. Math.*, **46** (1973), 115–121.

[GL] R. M. Guralnick, and L. S. Levy,"Cancellation and direct summands in dimension 1," *J. Alg.*, **142** (1991), 310–347.

[GLW] R. M. Guralnick, L. S. Levy and R.B. Warfield, Jr., "Cancellation counterexamples in Krull Dimension 1", *Proceedings Amer. Math. Soc.* **109** (1990), 323-326.

[HL] W. J. Heinzer and L. S. Levy, "Domains of dimension 1 with infinitely many singular maximal ideals," (preprint).

[Ka] I. Kaplansky, "Modules over Dedekind rings and valuation rings", *Trans. Amer. Math. Soc.* **72** (1952), pp. 327–340.

[K] L. Klingler, "Modules over the integral group ring of a nonabelian group of order pq", *Mem. Amer. Math. Soc.* 59 (1986).

[KL0] L. Klingler and L. S. Levy, "Sweeping-similarity of matrices", *Lin. Alg. Appl.* **75** (1986), 67–104.

[KL1] L. Klingler and L. S. Levy, "Representation type of commutative noetherian rings I: local wildness," *Pacific J. Math.*, **200** (2001), 345–386.

[KL2] L. Klingler and L. S. Levy, "Representation type of commutative noetherian rings II: local tameness," *Pacific J. Math.*, **200** (2001), 387–483.

[KL3] L. Klingler and L. S. Levy, "Representation Type of Commutative Noetherian Rings III: Global Wildness and Tameness," *Mem. Amer. Math. Soc.* (to appear)

[KL4] L. Klingler and L. S. Levy, "Wild torsion modules over Weyl algebras, and general torsion modules over HNPs", *J. Algebra* **172** (1995)

[L1] L. S. Levy, "Krull-Schmidt uniqueness fails dramatically over subrings of $\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}$" *Rocky Mountain J. Math.* **13** (1983), 659–678.

[L2] L. S. Levy, "Modules over Dedekind-like rings," *J. Alg.*, **93** (1985), 1–116.

[L3] L. S. Levy, "$\mathbb{Z}G_n$-modules, G_n cyclic of square-free order n," *J. Alg.*, **93** (1985), 354–375.

[LO] L. S. Levy and C. J. Odenthal, "Package deal theorems and splitting orders, in dimension 1," *Trans. Amer. Math. Soc.*, **348** (1996), 3457–3503.

[LR1] L. S. Levy and J. C. Robson, "Hereditary noetherian prime rings 1: Integrality and simple modules", *J. Algebra* **218** (1999), pp. 307–337.

[LR2] L. S. Levy and J. C. Robson, "Hereditary noetherian prime rings 2: Finitely generated projective modules", *J. Algebra* **218** (1999), pp. 338–372.

[LR3] L. S. Levy and J. C. Robson, "Hereditary noetherian prime rings 3: Infinitely generated projective modules", *J. Algebra* **225** (2000), pp. 275–298.

[MR] J. C. McConnell and J.C. Robson, *Noncommutative Noetherian Rings*, Graduate Studies in Mathematics **30**, American Mathematical Society (1987, 2001).

[NR] L. A. Nazarova and A. V. Roiter, "Finitely generated modules over a dyad of two local rings and finite groups with an abelian normal divisor of Index p," *Zap. Nauch. Sem. Leningrad Odtel. Mat. Inst. Steklov (LOM1) Izv. Akad. Nauk SSSR,* Ser Mat. **33**, No. 1 (1969). English transl: *Math. USSR Izvestija* **3**, No. 1 (1969).

[NRSB] L. A. Nazarova, A. V. Roiter, V. V. Sergeichuk, and V. M. Bondarenko, "Application of modules over a dyad for the classification of finite p-groups that have an abelian subgroup of index p and of pairs of mutually annihilating operators" (Russian), *Zap. Nauch. Sem. Leningrad Odtel. Mat. Inst. Steklov (LOM1)* **28** (1972), 69–92. English transl: *J. Soviet Math.* **3** (1975), 636–653.

[Nz] L. A. Nazarova, "Representations of quivers of infinite type," *Izv. Akad. Nauk SSSR,* Ser Mat. **37**, No. 4 (1973). English transl: *Math. USSR Izvestija* **7**, No. 4 (1973) 749–792.

[R] K. M. Ringel, "The representation type of local algebras," *Springer Lecture Notes in Mathematics* **488** (1975), 282–305.

[S] E. Steinitz, "Rechteckige Systeme und Moduln in Algebraischer Zahlkörper I,II," *Math. Ann.*, **71** (1911), 328–354, **72** (1912), 297–345.

Lee Klingler, Mathematics Department, Florida Atlantic University, Boca Raton, Florida 33431–0991, USA

E-mail address: klingler@fau.edu

L. S. Levy, Mathematics Department, University of Nebraska, Lincoln, NE 68588–0323 USA, Mailing address: 2528 Van Hise Ave., Madison, WI 53705-3850 USA

E-mail address: levy@math.wisc.edu