SOME BASIC FACTS ABOUT HILBERT AND BANACH SPACES

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Abstract. This follows Gilbarg and Trudinger Chapter 5, along with some things from Folland's *Real and Functional Analysis*.

We assume familiarity with the definition of a Banach space, i.e. a complete vector space, and what a norm is. We use the notation $\|\cdot\|_V$ for the norm on the vector space V.

1. The Contraction Mapping Principle

Definition 1.1. Let T be a map from a normed vector space, V into itself. T is a contraction mapping if there exists a $\theta > 1$ so that

$$\|Tx - Ty\| \leq \theta \|x - y\| \qquad \text{for all } x, y \in V \tag{1.1}$$

Theorem 1.2. *A contraction mapping in a Banach space has a unique fixed point. That is, there exists a unique x so that $Tx = x$.*

Proof. Let x_0 be a point in our Banach space, B. We define a sequence $\{x_n\}$ in B in the following way:

$$x_n = T^n x_0.$$

Now, for n, m where $n \geq m$,

$$\begin{aligned} \|x_n - x_m\| &\leq \sum_{j=m+1}^{n} \|x_j - x_{j-1}\| \qquad \text{by the triangle inequality} \\ &\leq \sum_{j=m+1}^{n} \|T^{j-1}(Tx_0 - x_0)\| \\ &\leq \sum_{j=m+1}^{n} \theta^{j-1} \|Tx_0 - x_0\| \\ &\leq \frac{\|Tx_0 - x_0\|\, \theta^m}{1 - \theta} \end{aligned}$$

As $m \to \infty$, the above goes to 0. Hence, $\{x_n\}$ is a Cauchy sequence in our Banach space, B, and hence, must converge. Thus, let x be the limit of the the sequence. Then

$$Tx = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x = x.$$

Therefore, T must have a fixed point. If there were another point $y \in B$ so that $Ty = y$, then

$$\|x - y\| = \|Tx - Ty\| \leq \theta\|x - y\|.$$

But $\theta > 1$. Thus, we must have $x = y$, and the fixed point is unique.

□

Remark. The method of proof in the previous theorem is known as the *Method of Successive Approximations.*

2. Method of Continuity

Let V_1, V_2 be two normed linear vector spaces. A linear mapping $T : V_1 \to V_2$ is bounded if

$$\|T\| = \sup_{x \in V_1, x \neq 0} \frac{\|Tx\|}{\|x\|}$$

is finite.

A linear map is continuous if and only if it is bounded. We consider when a linear map will be invertible.

Theorem 2.1. *Let B be a Banach space, V any normed vector space, and let L_0, L_1 be bounded linear operators from B into V. Define a homotopy from L_0 to L_1 in the following way:*

$$L_t = (1 - t)L_0 + tL_1, \qquad \text{for each } t \in [0, 1]. \tag{2.1}$$

Suppose there is a constant $C > 0$ so that

$$\|x\| \leq C\|L_t x\| \tag{2.2}$$

for every $t \in [0, 1]$. The L_1 is onto if and only if L_0 is onto.

Proof. We only proof one direction since the other direction is identical. Suppose that L_0 is onto. L_0 is also one-to-one by (2.2), and thus, we have an inverse mapping $L_0^{-1} : V \to B$. Given a y in V, $t \in [0, 1]$ solving the equation $L_t x = y$ is equivalent to solving the equation

$$\begin{aligned} L_0(x) &= y + (L_0 - L_t)(x) \\ &= y + tL_0 x - tL_1 x. \end{aligned}$$

Applying L_0^{-1} to both sides, we have

$$x = L_0^{-1} y + tL_0^{-1}(L_0 - L_1)x. \tag{2.3}$$

Let T be the linear map

$$Tx = L_0^{-1} y + tL_0^{-1}(L_0 - L_1)x.$$

Note that since $\|x\| \leq C\|L_t x\|$, we have $\|L_0^{-1}\| \leq C\|y\|$ for all $y \in B$. Thus, for $x, z \in B$

$$\begin{aligned} \|Tx - Tz\| &\leq \|t(L_0^{-1}(L_0 - L_1)(x - z)\| \\ &\leq t\,C\|(L_0 - L_1)(x - z)\| \\ &\leq t\,C\|L_0 - L_1\|\|x - z\| \\ &\leq t\,C\left(\|L_0\| + \|L_1\|\right)\|x - z\|, \end{aligned}$$

and T is a contraction mapping if

$$t \leq \left(C\left(\|L_0\| + \|L_1\|\right)\right)^{-1} = \delta.$$

Thus, by the Contraction Mapping Principle, we have a unique solution to the equation (2.3). Hence, L_t is invertible, given that $t < \delta$.

Now, the above proof is not specific to L_0. We could just have easily supposed that L_s is invertible for some $s \in [0, 1]$. Hence, dividing $[0, 1]$ into finite number of balls of δ length, we have that L_t is invertible for all $t \in [0, 1]$, as long as there exists some $t \in [0, 1]$ for which it is onto, in particular, L_0, or L_1. Thus, L_1 must also be invertible, and therefore onto.

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